Particulars of other books in this Series are given on page 521.

The Directly-Useful **D.U.** Technical Series

Founded by the late WILFRID J. LINEHAM, B.Sc., M.Inst.C.E.
General Editor: JOHN L. BALE.

Mathematics for Engineers

PART I

INCLUDING
ELEMENTARY AND HIGHER ALGEBRA,
MENSURATION AND GRAPHS, AND
PLANE TRIGONOMETRY

BY

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FIFTH EDITION

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FOUNDER'S NOTE

The Directly Useful Technical Series requires a few words by way of introduction. Technical books of the past have arranged themselves largely under two sections: the Theoretical and the Practical. Theoretical books have been written more for the training of college students than for the supply of information to men in practice, and have been greatly filled with problems of an academic character. Practical books have often sought the other extreme, omitting the scientific basis upon which all good practice is built, whether discernible or not. The present series is intended to occupy a midway position. The information, the problems, and the exercises are to be of a directly useful character, but must at the same time be wedded to that proper amount of scientific explanation which alone will satisfy the inquiring mind. We shall thus appeal to all technical people throughout the land, either students or those in actual practice.

AUTHOR'S PREFACE

An endeavour has here been made to produce a treatise so thorough and complete that it shall embrace all the mathematical work needed by engineers in their practice, and by students in all branches of engineering science. It is also hoped that it will prove of special value for private study, and as a work of reference.

Owing to the vast amount of ground to be covered, it has been found impossible to include everything in one volume: and accordingly the subject matter has been divided into two portions, with the first of which the present volume deals. Stated briefly, Part I treats fully of the fundamental rules and processes of Algebra, Plane Trigonometry, Mensuration, and Graphs, the work being carefully graded from an elementary to an advanced stage; while Part II is devoted to the Calculus and its applications, Harmonic Analysis, Spherical Trigonometry, etc.

It is felt that the majority of books on Practical Mathematics, in the endeavour to depart from a theoretical treatment of the subject, neglect many essential algebraic operations, and, in addition, limit the usefulness of the rules given by the omission of the proofs thereof. Throughout the book great attention has been paid to the systematic development of the subject, and, wherever possible, proofs of rules are given. Practical applications are added in the greater number of instances, the majority of the exercises, both worked and set, having a *direct bearing on engineering practice,* thus fulfilling the main purpose of the book: and strictly academic examples are only introduced to emphasise mathematical processes needful in the development of the higher stages.

In order to make the work of the greatest use to the engineer as a means of reference, many practical features have been intro-

at all stages of the progress of the book the Author desires to tender his sincere thanks to MESSRS. W. J. LINEHAM, B.Sc., M.I.C.E., J. L. BALE, C. B. CLAPHAM, B.Sc., and G. T. WHITE, B.Sc.

While it is hoped that the book is free from errors, it is possible that some may have been overlooked; and notification of such will be esteemed a great favour.

W. N. ROSE.

Goldsmiths' College,
New Cross, S.E.,
January, 1918.

NOTE TO THIRD EDITION

THE favourable reception accorded the first and second editions inspires the hope of similar appreciation of the third edition.

In this edition the need for the inclusion of some explanation of the determinant mode of expression employed in treatises on aerodynamics has been recognised by the addition of a section dealing with determinants.

The work has been subjected to thorough revision, corrections being made where necessary, and further exercises have been added.

December, 1921.

NOTE TO FOURTH EDITION

THE work has been carefully revised and corrections and minor improvements have been effected where necessary. An appendix has been added on the "Fuller Slide Rule."

August, 1923.

NOTE TO FIFTH EDITION

A GENERAL revision has been carried out, and the appendix has been enlarged by the inclusion of miscellaneous exercises, some of which are more particularly related to aeronautical problems.

June, 1924.

CHAPTER XI

CHAPTER XII

MATHEMATICS FOR ENGINEERS

INTRODUCTORY

Previous Knowledge.—While this work is intended to supply all the mathematical rules and processes used by the engineer, certain elementary branches of the subject have necessarily been omitted. It is assumed that the reader has a sound working knowledge of arithmetic, and also is acquainted with the four simple rules of algebra, viz. addition, subtraction, multiplication and division. Thus the meaning of the following algebraic processes should be known—

$$2a = 2\times a;\quad a^3 = a\times a\times a;\quad (x^2)^9 = x^{18};\quad \frac{x^{45}}{x^5} = x^{40};$$

$$6(5a-7b+12c) = 30a-42b+72c;\quad \frac{2x-5y}{4} = \cdot 5x-1\cdot 25y;$$

$$(4a-7b)(a+9b) = 4a^2+29ab-63b^2.$$

Again, the use of the 10-inch slide rule is not explained in detail as regards multiplying, dividing, involution and evolution; but the special application of the slide rule is dealt with as occasion arises

Definitions and Abbreviations. — An **expression** is any mathematical statement containing numbers, letters and signs.

Terms of an expression are connected one with another by $+$ or $-$ signs.

The **factors** of an expression are those quantities, numerical or literal, which when multiplied together give the expression.

Thus considering the expression—

$$15a^2b-29a^3xb^2+108ay^6$$

$15a^2b$, $29a^3xb^2$ and $108ay^6$ are *terms*; and each of these terms can be broken up into a number of *factors*; e. g.—

$$15a^2b = 15\times a\times a\times b.$$

Again $(9a-4b)(5a+7b) = 45a^2+43ab-28b^2$; and $(9a-4b)$ and $(5a+7b)$ are the *factors* of $45a^2+43ab-28b^2$.

When an expression depends for its value on that given to one of the quantities occurring in it, the expression is said to be a **function** of that quantity. Thus $9x^3-\cdot 7x^2+5$ is a function of x; and this relation would be written in the shorter form—

$$9x^3-\cdot 7x^2+5=f(x).$$

If a letter or number is raised to a power, the figure which denotes the magnitude of that power is called the **exponent.**

An **obtuse angle** is one which is greater than a right angle.

An **acute angle** is one which is less than a right angle.

A **scalene triangle** has three unequal sides.

The **locus** of a point is the path traced by the point when its position is ordered according to some law.

The **abbreviations** detailed below will be adopted throughout.

$=$	stands for	"equals" or "is equal to."
$+$	,, ,,	"plus."
$-$	,, ,,	"minus."
$\times$	,, ,,	"multiplied by."
$\div$	,, ,,	"divided by."
$\therefore$	,, ,,	"therefore."
$\pm$	,, ,,	"plus or minus."
$>$	,, ,,	"greater than."
$<$	,, ,,	"less than."
$\odot$	,, ,,	"circle."
$\odot$ce	,, ,,	"circumference."
$\propto$	,, ,,	"varies as."
∞	,, ,,	"infinity."
$\angle$	,, ,,	"angle."
$\triangle$	,, ,,	"triangle" or "area of triangle."
$\lfloor 4$ or $4!$	,, ,,	"factorial four"; the value being that of the product 1.2.3.4 or 24.
${}^nP_2$	,, ,,	"the number of permutations of n things taken two at a time."
${}^nC_2$	,, ,,	"the number of combinations of n things taken two at a time."
n_3	,, ,,	$n(n-1)(n-2)$.
η	,, ,,	"efficiency."
a	,, ,,	"angle in degrees."
θ	,, ,,	"angle in radians."
I.H.P.	,, ,,	"indicated horse-power."
B.H.P.	,, ,,	"brake horse-power."
m.p.h.	,, ,,	"miles per hour."
r.p.m.	,, ,,	"revolutions per minute."
r.p.s.	,, ,,	"revolutions per second."
I.V.	,, ,,	"independent variable."

F°	stands for	"degrees Fahrenheit."
C°	,, ,,	"degrees Centigrade."
L.C.D.	,, ,,	"lowest common denominator."
E.M.F.	,, ,,	"electro-motive force."
I	,, ,,	"moment of inertia."
E	,, ,,	"Young's modulus of elasticity."
S_n	,, ,,	"the sum to n terms."
S_∞	,, ,,	"the sum to infinity (of terms)."
Σ	,, ,,	"sum of."
B.T.U.	,, ,,	"Board of Trade unit."
B.Th.U.	,, ,,	"British thermal unit."
τ	,, ,,	"absolute temperature."
μ	,, ,,	"coefficient of friction."
$\sin^{-1} x$	,, ,,	"the angle whose sine is x."
e	,, ,,	"the base of Napierian logarithms."
g	,, ,,	"the acceleration due to the force of gravity."
cms.	,, ,,	"centimetres."
grms.	,, ,,	"grammes."
$\underset{x \to a}{\mathrm{L}} y$	,, ,,	"limit to which y approaches as x approaches the value a."

Tables of Weights and Measures.

British Table of Length

12 inches (ins.) = 1 foot
3 feet (ft.) = 1 yard
$5\frac{1}{2}$ yards (yds.) = 1 pole
40 poles (po.) = 1 furlong
8 furlongs (fur.) = 1 mile.

1 nautical mile = 6080 feet
1 knot = 1 nautical mile per hour
1 fathom = 6 feet.

Square Measure

144 square inches (sq. ins.) = 1 square foot
9 square feet (sq. ft.) = 1 square yard
$30\frac{1}{4}$ square yards (sq. yds.) = 1 square pole
40 square poles = 1 rood
4 roods or 4840 sq. yds. = 1 acre
640 acres = 1 square mile.

Cubic Measure

1728 cubic inches (cu. ins.) = 1 cubic foot
27 cubic feet (cu. ft.) = 1 cubic yard.

Weight of 1 gallon of water = 10 lbs.
Weight of 1 cu. ft. of water = 62·4 lbs.
1 cu. ft. = 6·24 gallons.

Metric Table of Length

1 kilometre (Km.)	= 1000 metres	
1 hectometre (Hm.)	= 100 ,,	
1 dekametre (Dm.)	= 10 ,,	
metre (m.)		= 39·37″
1 decimetre (dm.)	= ·1 metre	
1 centimetre (cm.)	= ·01 ,,	(2·54 cms. = 1″.)
1 millimetre (mm.)	= ·001 ,,	

Land Measure

100 links = 1 chain
1 chain = 66 feet
10 chains = 1 furlong
80 chains = 1 mile

10 square chains = 1 acre.

Useful Constants.

$e = 2{\cdot}71828$ $\qquad \pi = 3{\cdot}14159$
$\log_e 10 = 2{\cdot}3026$ $\qquad \log_{10} e = {\cdot}4343$
$\log_{10} N = \log_e N \times {\cdot}4343$ $\qquad \log_e N = \log_{10} N \times 2{\cdot}303$
$g = 32{\cdot}18$ ft. per sec. per sec.

1 horse-power = 33000 foot lbs. per min. = 746 watts.
Absolute temperature $\tau = t^\circ\text{C.} + 273$ or $t^\circ\text{F.} + 461$.
1 radian = 57·3 degrees.
pressure of one atmosphere = 14·7 lbs. per sq. in.
1 inch = 2·54 centimetres. 1 sq. in. = 6·45 sq. cms.
1 kilometre = ·6213 mile. 1 kilogram = 2·205 lbs.
1 lb. = 453·6 grms.

The following are the statements of the propositions in Euclid, to which reference is made in the text—

Euc. I. 47. In any right-angled triangle, the square which is

described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Euc. III. 35. If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other.

Euc. III. 36. Corollary. If from any point without a circle there be drawn two straight lines cutting it, the rectangles contained by the whole lines and the parts of them without the circle equal one another.

Euc. VI. 4. The sides about the equal angles of triangles which are equiangular to one another are proportionals.

Euc. VI. 19. Similar triangles are to one another in the duplicate ratio of their homologous (*i. e.*, corresponding) sides.

Euc. VI. 20. Similar polygons have to one another the duplicate ratio of that which their homologous sides have. [From this statement it follows that corresponding areas or surfaces are proportional to the squares of their linear dimensions.]

CHAPTER I

AIDS TO CALCULATION

Approximation for Products and Quotients.—Whatever may be the calculations in which the engineer is involved, it is always desirable, and even necessary, to obtain some approximate result to serve as a check on that obtained by the use of the slide rule or logarithms; only in this way is confidence in one's working assured.

Speed in approximation is as important as reasonable accuracy, and the following method, it is hoped, will greatly assist in such acceleration, especially in the cases of products and quotients. The great trouble in the evaluation of such an expression as $\frac{47\cdot83\times3\cdot142\times9\cdot41\times\cdot0076}{33000}$ is the fixing of the position of the decimal point. The rules usually given in handbooks on the manipulation of the slide rule may enable this to be done, but they certainly give no ideas as to the actual figures to be expected.

The **method suggested for approximation** may be thus stated

> Reduce each number to a simple integer, *i. e.*, one of the whole numbers 1, 2, 3, etc., if possible choosing the numbers so that cancelling may be performed; this reduction involving the omission of multiples or sub-multiples of 10. To allow for this, for every "multiplying 10" omitted place one stroke in the corresponding line of a fraction spoken of as a *point* fraction, and for every "dividing 10" place one stroke in the other line of this fraction. Thus two fractions are obtained, the *number* fraction, giving a rough idea of the actual figures in the result, and the *point* fraction from which the position of the decimal point in the result is fixed. Accordingly, by combining these two fractions the required approximate result is obtained.

To illustrate the application of the method consider the following—

Example 1.—Find an approximate value of the quotient $\frac{\cdot 05}{4\cdot 81}$

The whole fraction may be written approximately as

$\frac{5}{5}$ (the number fraction) and $\frac{\ }{11}$ (the point fraction);

that is, we state 4·81 as 5 (working to the nearest integer). By so doing we are not multiplying or dividing by any power of ten, so that there would be nothing to write in the point fraction due to this change: but by writing 5 in place of ·05, we are omitting two "dividing tens"; therefore, since 5 is in the numerator of the number fraction, two strokes appear in the denominator of the point fraction. The number fraction reduces to 1; and the point fraction indicates that the result of the number fraction is to be divided by 100, since two strokes, corresponding to two tens multiplied together, appear in the denominator. Hence, a combination of the two fractions gives the approximate result as 1 ÷ 100 or ·01.

It may be easier to effect the combination of the two fractions according to the following plan—

The result of the number fraction being 1·; shift the decimal point two places to the left, because of the presence of the two strokes in the denominator of the point fraction, thus—

·01·

Example 2.—Determine the approximate value of $\frac{9764 \times \cdot 0213}{28\cdot 4 \times \cdot 00074}$

To apply the method to this example—

State 9764 as 10,000, *i. e.*, write 1 in the numerator of the number fraction and four strokes in the numerator of the point fraction.

For ·0213 write 2 in the numerator of the number fraction and two strokes in the *denominator* of the point fraction. The strokes are placed in the denominator because in substituting ·02 for 2 we are multiplying by 100, and therefore, to preserve the balance, we must divide the result by 100.

For 28·4 we should write 3 with 1 stroke in the denominator, and for ·00074 we should write 7 with 4 strokes in the numerator.

Thus—

Number fraction.		*Point fraction.*
$\frac{1 \times 2}{3 \times 7}$		$\frac{1111 \quad \cancel{1111}}{\cancel{11} \quad \cancel{1}}$
i. e., ·1	and	$\frac{11111}{\ }$ by cancelling.

Hence the approximate result is $\cdot 1 \times 10^5$, *i. e.*, 10,000; or, alternatively, the shifting of the decimal point would be effected thus—

·10000·

It will be seen from this method that it is often an advantage to express a very large or very small number as an equivalent simpler number multiplied by some power of ten. Not only is a saving of time obtained, but the method tends to greater accuracy. Thus 2,000,000 may be written as 2×10^6, a very compact form; also it is far more likely that an error of a nought may be made in the extended than in the shorter form. "Young's modulus" for steel is often written as 29×10^6 lbs. per sq. in., rather than 29,000,000 lbs. per sq. in.

Example 3.—Find the approximation for—

$$\frac{47{\cdot}83 \times 3{\cdot}142 \times 9{\cdot}41 \times {\cdot}0076}{33000}$$

The method will be understood from the explanation given in the previous examples; and for clearness the strokes are separated in the point fraction.

The approximation is—

$$\frac{5 \times 3 \times 1 \times 8}{3} \qquad \frac{\;\;\;}{111} \qquad \frac{\;\;\;}{1111}$$

which reduces to

$$40 \qquad\qquad 11111$$

$$i.e., \quad 40 \div 10^5 = \underline{{\cdot}0004}.$$

The change in the position of the decimal point would be ·00040·

Further examples on approximation will be found on pp. 18 to 21.

Approximations for Squares and Square Roots. An extension of this method can be made to apply to cases of squares and square roots, cubes and cube roots. As regards squaring and cubing, these may be considered as cases of multiplication, so that nothing further need be added. To find, say, a square root approximately, we must remember that the square root of 100 or 10^2 is 10, the square root of 10^4 is 10^2, and so on; the approximation, therefore, must be so arranged that an *even* number of tens are omitted or added. Hence the rule for this approximation may be expressed—

> Reduce the number whose square root is to be found to some number between 1 and 100, multiplied or divided by some even power of ten; then the approximate square root of this number, combined with half the number of strokes in the point fraction, gives the approximate square root of the number.

In the case of cube roots, the number must be reduced to some number between 1 and 100 multiplied or divided by 3, 6 or 9 . . . tens; then the approximate cube root of this number must be combined with one-third of the strokes in the point fraction.

Example 4.—Find an approximation for $\sqrt{498{\cdot}4}$.

In place of 498·4 write 500, which can be written as 5×10^2

or as $\qquad 5 \qquad \underline{11}$

Then the approximate square root is $\quad 2{\cdot}2 \quad \underline{1}$

or $\quad \underline{22.}$

If the number had been 4984 the number would read—

$$50 \qquad \underline{11}$$

and the square root—

$$7 \qquad 1, \qquad\qquad i.e., \ \underline{70.}$$

Example 5.—Find approximately the cube root of ·000182.

If for 182 we write 200, then ·000182 is replaced by—

$$200 \qquad \overline{111\,111}$$

so that the cube root of ·000182 is that of 200 divided by 10^2, since two strokes (viz. $\frac{1}{3}$ of 6) appear in the denominator of the point fraction.

Thus, the cube root is—

$$5{\cdot}8 \qquad \overline{11}$$

$$\underline{\text{or } {\cdot}058.}$$

Example 6.—Evaluate approximately $\sqrt{\dfrac{21{\cdot}43 \times {\cdot}097}{{\cdot}154 \times 2409}}$

Disregarding the square root sign for the moment, the approximation gives—

$$\frac{2 \times 1}{1{\cdot}5 \times 2} \qquad\qquad \frac{\underline{1}}{\overline{1}} \ \frac{\underline{1}}{\overline{111}}$$

$$i.e., \quad {\cdot}67 \qquad\qquad \overline{11}$$

For the application of the method of this paragraph this result would be written $\quad 67 \quad \overline{1111}$

of which the square root is $\quad 8{\cdot}2 \quad \overline{11}$

or the approximate square root is $\underline{{\cdot}082.}$

Exercises 1.—On Approximations.

Determine the approximate answers for Exercises 1 to 20.

1. $49{\cdot}57 \times {\cdot}0243$ **2.** ${\cdot}00517 \times {\cdot}1724$ **3.** $\dfrac{3597}{23{\cdot}4}$

4. $8{\cdot}965 \times 72{\cdot}49 \times {\cdot}094$ **5.** ${\cdot}1167 \times {\cdot}0004 \times 98{\cdot}1 \times 2710$

6. $\dfrac{4{\cdot}176 \times 25400}{87235}$ **7.** $\dfrac{151 \times {\cdot}00905}{{\cdot}847 \times 7500}$ **8.** $\sqrt{810{\cdot}5}$

9. $\dfrac{11540 \times {\cdot}3276 \times 3{\cdot}142 \times {\cdot}0078}{78560 \times {\cdot}0022 \times 53{\cdot}7}$ **10.** $\sqrt{{\cdot}00277}$ **11.** $\sqrt{9543{\cdot}8}$

12. $\sqrt{{\cdot}0277}$ **13.** $\sqrt{35{\cdot}2 \times {\cdot}195}$ **14.** $\sqrt{\dfrac{11 \times 4{\cdot}017}{9340 \times {\cdot}00015}}$

15. $\sqrt[3]{10570}$ **16.** $\sqrt[3]{{\cdot}185}$ **17.** ${\cdot}253 \times \sqrt[3]{{\cdot}00192}$

18. $\sqrt{{\cdot}907} \times \sqrt[3]{487{\cdot}2}$ **19.** $\sqrt[5]{\dfrac{55 \times 6{\cdot}43 \times {\cdot}0091}{{\cdot}000107 \times 985}}$

20. $\sqrt[4]{\dfrac{{\cdot}1109 \times {\cdot}9532}{{\cdot}00346 \times {\cdot}0209}}$

Indices.—The approximation being made, the actual figures can be determined either by logarithms or by the slide rule.

Napier, working in Scotland, and Briggs in England, during the period 1614–17 evolved a system which made possible the evaluation of expressions previously left severely alone. Without the aid of their system much of the experimental work of modern times would lose its application, in that the conclusions to be drawn could not be put into the most beneficial forms; and failing logarithms, arithmetic, with its cumbersome and exacting rules, would dull our faculties and prevent any advance.

The great virtue of the system of logarithms is its simplicity: rules with which we have long been acquainted are put into a more practical form and a new name given to them. Many are familiar with the simpler rules of indices, such as $a^3 \times a^4 = a^{3+4} = a^7$; $a^8 \div a^2 = a^{8-2} = a^6$; $(a^3)^4 = a^{3\times4} = a^{12}$, etc.

Following along these lines we can find meanings for $a^{\frac{1}{2}}$, a^0, and a^{-3}, *i. e.*, we can establish rules that will apply to all cases of positive, negative, fractional or integral indices. Thus, to find a meaning for a fractional power, consider the simplest case, viz. that in which the index is $\frac{1}{2}$.

When multiplying $a^3 \times a^4$ we add the indices; this can be done whatever the indices may be, hence—

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$$

i. e., $a^{\frac{1}{2}}$ is that quantity which multiplied by itself is equal to a, or, in other words, **$a^{\frac{1}{2}}$ is the square root of a.**

In like manner, since $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$, $a^{\frac{1}{3}}$ may be written as $\sqrt[3]{a}$. For example, $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$.

To carry this argument a step further we may consider a numerical example, *e. g.*, $64^{\frac{2}{3}}$, and from the meaning of this, derive a meaning for $a^{\frac{p}{q}}$

Thus $64^{\frac{2}{3}}$ might be written as $64^{\frac{1}{3}} \times 64^{\frac{1}{3}}$, which again may be put into the form $\sqrt[3]{64} \times \sqrt[3]{64}$, *i. e.*, $(\sqrt[3]{64})^2$ or $\sqrt[3]{64^2}$.

Hence the actual numerical value $= \sqrt[3]{64 \times 64} = 16$.

We see that the denominator of the exponent indicates the root, and the numerator the power; thus $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

To find a meaning for a^0—

$$a^m \times a^0 = a^{m+0} = a^m.$$

Dividing through by a^m, $a^0 = 1$, *i. e.*, any number or letter raised to the zero power equals 1: *e. g.*, $465^0 = 1$; $\cdot 2384^0 = 1$; $4x^0 = 4 \times 1 = 4$.

Assuming this result for a^0, we can show how to deal with negative powers, for—

$$a^m \times a^{-m} = a^{m-m} = a^0 = 1.$$

Hence, dividing through by a^m,

$$a^{-m} = \frac{1}{a^m}$$

Accordingly, in changing a factor (such as a^m) from the top to the bottom of a fraction or *vice versa*, we must change the sign before its index.

Thus—

$$\frac{b^2}{b^{-7}} = b^2 \times b^7 = b^{2+7} = b^9.$$

Example 7.—Simplify $(a^{-5}b^2c^3)^2 \times \sqrt{a^3b^{-4}c^6}$.

The expression	$= a^{-10}b^4c^6 \times a^{\frac{3}{2}}b^{-\frac{4}{2}}c^{\frac{6}{2}}$ . .	Removing brackets.
	$= a^{-10+\frac{3}{2}}\, b^{4-2}\, c^{6+3}$ . .	Collecting like letters.
	$= a^{-\frac{17}{2}}b^2c^9 = \dfrac{b^2c^9}{\sqrt{a^{17}}}$ or $\dfrac{b^2c^9}{a^{\frac{17}{2}}}$	

Example 8.—Simplify $\dfrac{(64x^{-3})^{\frac{1}{6}}}{2(5x^2)^3}$

The expression—

$$= \frac{64^{\frac{1}{6}}x^{-\frac{3}{6}}}{2 \times 5^3x^6} = \frac{\sqrt[6]{64}}{2 \times 125 \times x^{6+\frac{1}{2}}} = \frac{2}{250x^{\frac{13}{2}}} = \frac{1}{125x^{\frac{13}{2}}}$$

Exercises 2.—On Indices.

1. Express with positive indices—

$$b^{-8};\ 4b^{-7};\ (5a^2)^{-1};\ \sqrt[5]{9c^2};\ \sqrt[7]{x^2y^{-4}}.$$

2. Find the numerical values of—

$$32^{\frac{1}{5}};\ 64^{-\frac{5}{6}};\ \left(\frac{4}{9}\right)^{-\frac{3}{2}};\ 6\times512^{\frac{2}{3}};\ \{17\times\sqrt[8]{625^4}\}+\{15^2\times(\tfrac{9}{16})^{-\frac{3}{2}}\}$$

3. Simplify $(3a^2bc^{-3})^4\times(\frac{1}{720}a^9b^{-5}c)^{-\frac{1}{3}}$.

4. Simplify $\sqrt[3]{343x^{-4}y^2z^{\frac{1}{3}}}\div81x^{-1}y^{11}z^{\frac{2}{3}}$

5. Simplify $11a^2b^0cd^8\times(a^{-4}b^6c^3)^{-\frac{1}{2}}\div50\{b^3c^5d^9\}^{-\frac{1}{3}}$

6. Find the value of $\frac{v^{2n}nv^{n-1}}{4v^{3-n}}$ in terms of v, when $n=-1{\cdot}37$.

7. Find the value of $-vnCv^{-n-1}$ in terms of p when $n=1{\cdot}41$ and $pv^n=C$.

8. Simplify $\left\{a^{\frac{1}{n}}\sqrt{1-a^{\frac{n-1}{n}}}\right\}^2$, a formula referring to the flow of a gas through an orifice, a being the ratio of the outlet pressure to the pressure in the vessel.

9. Simplify $8(e^{x^2})^3\times\frac{1}{8}(e^5)^{x^2}\div(e^{2x})^{4x}$, and find its value when $e=2{\cdot}718$.

10. Simplify the expression—

$$\frac{4(a^{-5}b^3c^2)^{\frac{1}{5}}\times\sqrt[5]{32a^{-9}b^8c^3}}{(5c^{-\frac{1}{5}}d^2)^2\times(125d^9a^4c^5)^{-\frac{1}{3}}}$$

11. The work done in the adiabatic expansion of a gas from volume v_1 to volume v_2 may be written $W=\frac{C}{1-n}(v_2^{1-n}-v_1^{1-n})$. If $p_2v_2^n=p_1v_1^n=C$, by substituting for C, find a simpler expression for W.

Logarithms.—It is necessary to deal with indices at this stage, because logarithms and indices are intimately connected.

For example, $100=10^2$, and the logarithm of 100 to the base $10=2$ (written $\log_{10}100=2$). Here are two different ways of stating the same fact, for 2 is the index of the power to which the base 10 has to be raised to equal the number 100, but it is also called the logarithm of 100 to the base 10, *i. e.*, the index viewed from a slightly different standpoint is termed the logarithm. Hence the rules of logs (as they are called) must be the same as those connecting indices.

In general: **The logarithm of a number to a certain base is the index of the power to which the base must be raised to equal the number.**

It is not necessary to understand the theory of logs to be able to use them for ordinary calculations, but the knowledge of the principles involved is of very great assistance.

Consider the three statements—

$$64 = 2^6;\ 64 = 4^3;\ 64 = 8^2.$$

These could be written in the alternative forms—

$$\log_2 64 = 6;\ \log_4 64 = 3;\ \log_8 64 = 2;$$

where the numbers 2, 4 and 8 are called *bases*.

It will be noticed that the same number has different logs in the three cases, *i. e.*, if we alter the "base" or "datum" from which measurements or calculations are made, we alter the log; in consequence, as many tables of logs can be constructed as there are numbers. This shows the need for a standard base, and accordingly logs are calculated either to the base of 10 (such being called *Common* or *Briggian Logarithms*) or to the base of *e*, a letter written to represent a series of vast importance, the approximate value of which is 2·718. Logarithms calculated to the base of *e* are called *Natural*, *Napierian* or *Hyperbolic Logarithms*. At present we shall confine our attention to the Common logs; in the later parts of the work we shall find the importance and usefulness of the natural logs.

From the foregoing definition of a logarithm the logs of simple powers of ten can be readily written down; thus, $\log_{10}1000 = 3$, since $1000 = 10^3$, $\log_{10}1000000 = 6$, etc.; $\log_{10}1000 = 3$ is usually written in the shorter form $\log 1000 = 3$, the base 10 being understood when the small base figure is omitted.

For a number, such as 526·3, lying between 100 and 1000, *i. e.*, between 10^2 and 10^3, the log must lie between 2 and 3, and must therefore be 2 + some fraction. To determine this fraction recourse must be made to a table of logs.

To read logs from the tables.—The tables appearing at the end of this book are known as four-figure tables, and are quite full enough for ordinary calculations, but for particularly accurate work, as, for example, in Surveying, five- and even seven-figure tables are used. One soon becomes familiar with the method of using these tables, the few difficulties arising being dealt with in the following pages.

To return to the number 526·3: the fractional or decimal part of its logarithm is to be found after the following manner:—Look down the first column of the table headed "logarithms" (Table II at the end of the book) till 52 is reached, then along this line until under the column headed "6" at the top the figure 7210 is found; this is the decimal part of the log of 526, so that the 3 is at present unaccounted for. At the end of the line in which 7210 occurs are what are known as "difference" columns. Under that headed

"3" and in the same line as the 7210, the fourth figure of our number, the figure 2, occurs; this, added to 7210, making 7212, completes the decimal portion of the log of 526·3. The figure from the tables is thus 7212, and since this is to be the fractional portion the decimal point is placed immediately before the first figure. The log of 526·3 is therefore 2·7212, or, in other words, 526·3 = 10 raised to the power 2·7212; similarly the log of 52630 must be 4·7212, because 52630 is the same proportion of a power of 10 above 10,000 as 526·3 is above 100, and also it lies between 10^4 and 10^5, so that its logarithm must be 4 + some fraction.

The log thus consists of two distinct parts, the decimal part, which is always obtained from the tables and is called the *mantissa*, and the integral or whole-number part, settled by the position of the decimal point in the number, and called the *characteristic* or distinguisher. The logs of 526·3 and 52630 are alike as regards the decimal part, but must be distinguished from one another by the addition of the relative characteristic.

When the number was 526·3, *i. e.*, having 3 figures before the decimal point, the characteristic was 2, *i. e.*, 3 − 1; when the number was 52630, *i. e.*, having 5 figures before the decimal point, the characteristic was 4, *i. e.*, 5 − 1. This method could be applied for numbers down to 1, *i. e.*, 10^0, but for numbers of less value we are dealing with negative powers, and accordingly we must investigate afresh.

So far, then, we can formulate the rule: "**When the number is greater than 1, the characteristic of its log is positive and is one less than the number of figures before the decimal point.**"

E.g., if the number is 2507640, the characteristic of its log is 6, because there are seven figures in the number before the decimal point.

Referring to the figures 5263 already mentioned, place the decimal point immediately before the first figure, giving ·5263. The number now lies between ·1 and 1. Now—

$$\cdot 1 = \frac{1}{10} = 10^{-1} \text{ and } 1 = 10^0$$

so that the log of ·5263 lies between − 1 and 0, being greater than − 1 and less than 0, and therefore is −1 + a fraction. The mantissa is as before, viz. 7212, hence log ·5263 = − 1 + ·7212, or, as it is usually written, $\bar{1}$·7212, the minus sign being placed over the 1 to signify the fact that it applies only to the 1 and not to the ·7212, which latter is a positive quantity and must be kept as such.

$\bar{1}$·7212 actually means, then, − 1 + ·7212, or, in fact, − ·2788.

The figures taken from the tables are always positive, and accordingly the form $\bar{1}\cdot7212$ is adhered to throughout.

From similar reasoning, log ·005263 = $\bar{3}\cdot7212$ . . . . (1)
and log ·00005263 = $\bar{5}\cdot7212$ . . . . (2)

We can conclude, then, that: When dealing with the log of a decimal fraction the mantissa is found from the tables in just the same way as for a number larger than 1, or, in other words, no regard is paid, when using the tables to find the mantissa, to the position of the decimal point in the number whose log is required. The characteristic of the log, however, is negative, and one more than the number of zeros before the first significant figure.

In (1) there are 2 noughts before the first significant figure; therefore the characteristic is $\bar{3}$. In (2) the characteristic is $\bar{5}$, because there are 4 noughts before the first significant figure.

For emphasis, the rules for the determination of the characteristic of the log of any number are repeated—

If the number is greater than 1, the characteristic is positive and one less than the number of figures before the decimal point: if the number is less than 1, the characteristic is negative and one more than the number of noughts before the first significant figure.

It will be observed that in the earlier part of the table of logarithms at the end of the book there are, for each number in the first column, two lines in the "difference" column. This arrangement (the copyright of Messrs. Macmillan & Co., Ltd.) gives greater accuracy as regards the fourth figure of the log, since the differences in this portion of the table are large. The log is looked out as explained in the previous case, care being taken to read the "difference" figure in the same line as the third significant figure of the number whose log is being determined.

Thus the log of 1437 is 3·1553 + a difference of 21 = 3·1574, while the log of 1487 is 3·1703 + a difference of 20 = 3·1723.

We are now in a position to write down the value of the log of any number, and a few examples are given—

Number.	Log.
40700	4·6102
·2359	$\bar{1}\cdot3728$
70·08	1·8456
·0009	$\bar{4}\cdot9542$
500000	5·6990

(The mantissa for the log of 9, 90, 900, and 9000 is ·9542.)

Values of log 1 and log 0.—If a be any number, then, as proved earlier, $a^0 = 1$; or in the log form, $\log_a 1 = 0$.

Thus the log of 1 to any base = 0.

The log of 0 to any base is minus infinity; or if a be any number,

$$\log_a 0 = -\infty.$$

For, by writing this statement in the alternative form

$$a^x = 0$$

where x is the required logarithm, we see that x must be an infinitely small quantity; in fact, the smallest quantity conceivable.

Antilogarithms. Suppose the question is presented to us in the reverse way: "Find the number whose logarithm is 2·9053." The table of antilogarithms (Table III) will be found more convenient for this, although the log tables can be used in the reverse way. Just as the mantissa alone was found from the log tables when finding the logarithm, so this alone is used to determine the actual arrangement of the figures in the number. In the case under consideration the mantissa is ·9053, hence look down the first column until ·90 is reached, then along this line until in the column headed "5" 8035 is read off: to this must be added 6, the number found in the "difference" column headed "3," so that the actual figuring of the number is 8035 + 6 = 8041.

The characteristic 2 must now be considered so as to fix the position of the decimal point. Referring to our rule, we see that the characteristic is one less than the number of figures before the decimal point (since 2 is positive), hence, conversely, the number of figures before the decimal point must be one more than the characteristic; in this case there must be 3 figures before the decimal point, *i. e.*, the required number is 804·1.

If we had been asked to find the antilog of 2·0905, the line through ·09 would have been followed and not that through ·90, and the antilog is found to be 123·1. Many errors occur if this distinction is not appreciated; and the actual mantissa must be dealt with in its entirety, *no noughts being disregarded* wherever they may occur.

Examples—

Log.	No.
8·1164	130700000 or $1·307 \times 10^8$
·2549	1·799
1·0062	10·14
$\bar{3}$·8609	·007259

Failing a table of logs, the log scale on the slide rule can be used in the following manner. Reverse and invert the slide so that the S scale is now adjacent to the D scale: place the ends of the D and S scales level: then using the D scale as that of numbers, the corresponding logarithms are read off on the L scale—it being remembered that although the scale is inverted the numbers increase towards the right. The mantissa alone is found in this way, whilst the characteristic is settled according to the ordinary rules.

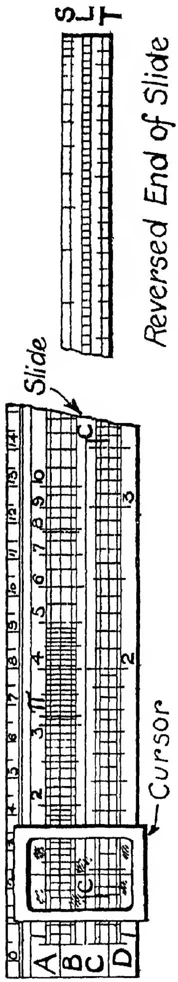

Fig. 1.—Slide Rule.

Fig. 1 shows the scales of an ordinary 10″ Slide Rule lettered as they will be referred to throughout this book. On the front are the scales A, B, C and D, the B and C scales being on the "slide." If the slide is taken out and reversed the S, L and T scales will be noticed (see right-hand end of figure). Any special markings referred to throughout the text are also indicated, and it is to this sketch that the reader should refer, no other sketch of the slide rule being inserted. The slide rule is referred to from time to time, wherever its use is required, and a word or two is then said about the method of usage, but no special chapter is devoted to its use. For a full explanation of the method of using the slide rule reference should be made to *Arithmetic for Engineers.**

Applications of Logarithms.—It will be granted that—

$$2+4=6$$

or—

$$\log 100 + \log 10{,}000 = \log 1{,}000{,}000 \text{ from definition.}$$

$$\text{But } 1{,}000{,}000 = 100 \times 10{,}000$$

$$\therefore \log (100 \times 10{,}000) = \log 100 + \log 10{,}000.$$

Simple powers of ten have been taken in this example, for convenience, but the rule demonstrated is perfectly general, holding for all numbers.

In general, **$\log (A \times B) = \log A + \log B$, where A and B are any numbers.** Thus, **the log of a product = the sum of the logs of the factors.**

This and the succeeding rules hold for bases other than 10; in fact, they are general in all respects.

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* *Arithmetic for Engineers*, by Charles B. Clapham, B.Sc. Chapman and Hall, Ltd., *7s. 6d.* net.

In like manner it can be shown that—

$$\log\left(\frac{\mathbf{A}}{\mathbf{B}}\right) = \log \mathbf{A} - \log \mathbf{B}$$

i. e., the log of a quotient = the difference of the logs of numerator and denominator.

Again, $3 \times 2 = 6$

$$\therefore 3 \times \log 100 = \log 1{,}000{,}000$$
$$= \log (100)^3$$

or in general, $\log (\mathbf{A})^n = n \log \mathbf{A}$, *this holding whatever value be given to* n.

E. g., (*a*) $\sqrt[4]{42 \cdot 76} = (42 \cdot 76)^{\frac{1}{4}}$

$$\therefore \log \sqrt[4]{42 \cdot 76} = \tfrac{1}{4} \log 42 \cdot 76.$$

(*b*) $\log (\cdot 0517)^{-4 \cdot 2} = -4 \cdot 2 \times \log \cdot 0517.$

Stated in words this rule becomes : **The log of a number raised to a power is equal to the log of that number multiplied by that power.**

Summarising, we see that multiplication and division can be performed by suitable addition and subtraction, whilst the troublesome process of finding a power or root resolves itself into a simple multiplication or division. (The application of logarithms to more difficult calculations is taken up in Chap. V.)

In any numerical example care should be taken to set the work out in a reasonable fashion; especially in questions involving the use of logs.

Example 9.—Find the value of $48 \cdot 21 \times 7 \cdot 429$.

Actual Working— *Approximation*— $50 \times 7 = 350$.

Let $x = 48 \cdot 21 \times 7 \cdot 429$

then $\log x = \log 48 \cdot 21 + \log 7 \cdot 429 = 1 \cdot 6831 + \cdot 8709$

$= 2 \cdot 5540$

$= \log 358 \cdot 1$ from the antilog tables.

$\therefore \underline{x = 358 \cdot 1.}$

Example 10.—If $C = \frac{V}{R}$, a formula relating to electric currents, find the value of C, a current, when the voltage V is $2 \cdot 41$ and the resistance R is $28 \cdot 7$.

Substituting the values of V and R—

$$C = \frac{2 \cdot 41}{28 \cdot 7}$$

$\therefore \log C = \log 2 \cdot 41 - \log 28 \cdot 7$

$= \cdot 3820 - 1 \cdot 4579$

$= \bar{2} \cdot 9241$, since 2 subtracted from $0 = \bar{2}$

$= \log \cdot 08397$

$\therefore \underline{C = \cdot 08397.}$

Approximation.

$\frac{2 \cdot 4}{3}$ $\bar{1}$

i. e., $\cdot 8 \div 10$ or $\cdot 08$

Example 11.—If F, the centrifugal force on a rotating body, $= \frac{Wv^2}{gr}$, find its value when $W = 28$, $v = 4\cdot75$, $g = 32\cdot2$, $r = 1\cdot875$.

Substituting the numerical values in place of the letters—

$$F = \frac{28 \times (4\cdot75)^2}{32\cdot2 \times 1\cdot875}$$

Taking logs throughout—

$$\log F = (\log 28 + 2 \log 4\cdot75) - (\log 32\cdot2 + \log 1\cdot875)$$

$$\begin{array}{ll} = 1\cdot4472 & - \; 1\cdot5079 \\ \underline{1\cdot3534} & \quad \underline{\cdot2730} \\ 2\cdot8006 & \quad 1\cdot7809 \end{array}$$

$$\begin{aligned} &= 2\cdot8006 - 1\cdot7809 \\ &= 1\cdot0197 \\ &= \log 10\cdot47 \\ \therefore\ &\underline{F = 10\cdot47.} \end{aligned}$$

Approximation.

$$\frac{3 \times 5 \times 5}{3 \times 2} \qquad \frac{1}{1}$$

i. e., 12·5.

Explanation.

$$\log 4\cdot75 = \cdot6767$$
$$\therefore 2 \times \log 4\cdot75 = 1\cdot3534.$$

Example 12.—If $f = \frac{gP}{W}$, an equation giving the acceleration produced by a force P acting on a weight W, find f when $g = 32\cdot2$, $P = 5\cdot934$, and $m = 487$.

Substituting the numerical values—

$$f = \frac{32\cdot2 \times 5\cdot934}{487}$$

Taking logs—

$$\begin{aligned} \log f &= (\log 32\cdot2 + \log 5\cdot934) - \log 4\cdot87 \\ &= (1\cdot5079 + \cdot7734) - 2\cdot6875 \\ &= 2\cdot2813 - 2\cdot6875 \\ &= \bar{1}\cdot5938 = \log \cdot3924 \\ \therefore\ &\underline{f = \cdot3924.} \end{aligned}$$

Approximation.

$$\frac{3 \times 6}{5} \qquad \frac{1}{11}$$

i. e., $3\cdot6 \div 10$ or ·36.

Example 13.—Find the value of $\frac{\cdot05229}{\cdot001872}$

Let $x = \frac{\cdot05229}{\cdot001872}$

$$\begin{aligned} \text{then } \log x &= \log \cdot05229 - \log \cdot001872 \\ &= \bar{2}\cdot7184 - \bar{3}\cdot2723 \\ &= 1\cdot4461 \\ &= \log 27\cdot94 \\ \therefore\ &\underline{x = 27\cdot94.} \end{aligned}$$

Approximation.

$$\frac{5}{2} \qquad \frac{111}{11}$$

i. e., $2\cdot5 \times 10$ or 25.

Note.—In the subtraction the minus 3 becomes plus 3 (changing the bottom sign and adding algebraically); and this, combined with minus 2, gives plus 1.

Example 14.—Find the value of the expression $s = \dfrac{\cdot 01154}{47\cdot 61 \times \cdot 0000753}$

Taking logs—

$$\log s = \log \cdot 01154 - (\log 47\cdot 61 + \log \cdot 0000753)$$
$$= \bar{2}\cdot 0622 - (1\cdot 6777 + \bar{5}\cdot 8768)$$
$$= \bar{2}\cdot 0622 - \bar{3}\cdot 5545$$
$$= \cdot 5077$$
$$= \log 3\cdot 219$$
$$\therefore \underline{s = 3\cdot 219.}$$

Approximation.

$$\frac{1\cdot 2}{4\cdot 8 \times 7\cdot 5} \qquad \frac{11111}{111}$$

i.e., $\cdot 033 \times 100$
or $3\cdot 3$.

Note.—In this subtraction the 1 borrowed for the 5 from 0 should be repaid by subtracting it from the 2, making it 3 : this, combined with + 3 (the sign being changed for subtraction), gives 0 as a result. Alternatively, the $\bar{3}$ must be increased by 1 to repay the borrowing, so that it becomes 2 ; and 2 subtracted from 2 gives 0.

Example 15.—The formula $V = \frac{4}{3} \times 3\cdot 142 \times r^3$ gives the volume of a sphere of radius r. Find the volume when the radius r is $\cdot 56$.

Substituting the numerical values—

$$V = \frac{4}{3} \times 3\cdot 142 \times (\cdot 56)^3.$$

Approximation.

$$\frac{4 \times 3 \times 6 \times 6 \times 6}{3} \qquad 111$$

i. e., $864 \div 1000$
or $\cdot 864$.

Taking logs—

$$\log V = (\log 4 + \log 3\cdot 142 + 3 \log \cdot 56) - \log 3$$
$$= (\cdot 6021 + \cdot 4972 + \bar{1}\cdot 2446) - \cdot 4771$$
$$= \cdot 3439 - \cdot 4771$$
$$= \bar{1}\cdot 8668$$
$$= \log \cdot 7358$$
$$\therefore \underline{V = \cdot 7358.}$$

Explanation.

$\log \cdot 56 = \bar{1}\cdot 7482$
$3 \times \log \cdot 56 = \bar{1}\cdot 2446$

i. e., there is + 2 to carry from the multiplication of the mantissa and this, together with $\bar{3}$ which is obtained when $\bar{1}$ is multiplied by 3, gives $\bar{1}$.

Example 16.—Find the fifth root of $\cdot 009185$.

$$\text{Let } x = \sqrt[5]{\cdot 009185} = (\cdot 009185)^{\frac{1}{5}}$$
$$\text{then } \log x = \tfrac{1}{5} \log \cdot 009185$$
$$= \tfrac{1}{5} \times \bar{3}\cdot 9630 \, *$$
$$= \tfrac{1}{5} \times \{\bar{5} + 2\cdot 9630\}$$
$$= \bar{1}\cdot 5926 = \log \cdot 3913$$
$$\therefore \underline{x = \cdot 3913.}$$

* We must not divide 5 into $\bar{3}\cdot 9630$ because the 3 is minus, whilst the $\cdot 9630$ is plus; but the addition of $\bar{2}$ to the whole number and of + 2 to the mantissa will permit the division of each part separately, while not affecting the value of the quantity as a whole.

Example 17.—Evaluate—

$$\frac{(\cdot 2164)^3 \times \sqrt{745\cdot 4}}{(\cdot 001762)^{\frac{1}{6}} \times (49\cdot 18)^{\frac{2}{5}}}$$

Let the whole fraction $= x$.

Then $\log x = \{3 \log \cdot 2164 + \frac{1}{2} \log 745\cdot 4\} - \{\frac{1}{6} \log \cdot 001762 + \frac{2}{5} \log 49\cdot 18\}$.

$$= (\bar{2}\cdot 0059 + 1\cdot 4362) - (\bar{1}\cdot 5410 + \cdot 6767)$$

$$= \bar{1}\cdot 4421 - \cdot 2177$$

$$= \bar{1}\cdot 2244$$

$$= \log \cdot 1677$$

$$\therefore \underline{x = \cdot 1677.}$$

Explanation.

$$\begin{aligned}
\log \cdot 2164 &= \bar{1}\cdot 3353 \\
3 \times \log \cdot 2164 &= \bar{2}\cdot 0059 \\
\log 745\cdot 4 &= 2\cdot 8724 \\
\tfrac{1}{2} \times \log 745\cdot 4 &= 1\cdot 4362 \\
\log \cdot 001762 &= \bar{3}\cdot 2460 \\
\tfrac{1}{6} \times \log \cdot 001762 &= \bar{1}\cdot 5410 \\
\log 49\cdot 18 &= 1\cdot 6918 \\
2 \times \log 49\cdot 18 &= 3\cdot 3836 \\
\therefore \tfrac{2}{5} \times \log 49\cdot 18 &= \cdot 6767.
\end{aligned}$$

Example 18.—If $x = \sqrt[8]{\dfrac{29\cdot 17 \times \cdot 1245}{9004 \times \cdot 0856}}$ find the value of x.

Taking logs—

$$\begin{aligned}
\log x &= \tfrac{1}{8}\{(\log 29\cdot 17 + \log \cdot 1245) - (\log 9004 + \log \cdot 0856)\} \\
&= \tfrac{1}{8}\{(1\cdot 4649 + \bar{1}\cdot 0951) - (3\cdot 9544 + \bar{2}\cdot 9325)\} \\
&= \tfrac{1}{8}(\cdot 5600 - 2\cdot 8869) = \tfrac{1}{8}(\bar{3}\cdot 6731) = \tfrac{1}{8}(\bar{8} + 5\cdot 6731). \\
&= \bar{1}\cdot 7091 = \log \cdot 5118
\end{aligned}$$

$$\therefore \underline{x = \cdot 5118.}$$

The following examples are worked by the slide rule.

Example 19.—Find the buckling stress P for a column of length l; from the formula $P = \dfrac{48000}{1 + 4c\left(\dfrac{l}{k}\right)^2}$ when $k^2 = \cdot 575$, $c = \dfrac{1}{30000}$ and $l = 180$.

Substituting these numerical values—

$$P = \frac{48000}{1 + \left(4 \times \dfrac{1}{30000} \times \dfrac{180^2}{\cdot 575}\right)}.$$

The second term of the denominator must be worked apart from the rest—

Thus, to evaluate $\dfrac{4 \times 180 \times 180}{30000 \times \cdot 575}$, proceed as follows—

Actual figuring, found from the slide rule, is 751, so that, in accordance with the approximation the value of this term is 7·51.

Approximation.

$$\frac{4 \times 2 \times 2}{3 \times 6}$$

i. e., ·88 × 10 or 8·8.

$$\therefore P = \frac{48000}{1 + 7\cdot 51} = \frac{48000}{8\cdot 51} = \underline{5630 \text{ lbs. per sq. in.}}$$

Example 20.—Find the value of E, Young's modulus for steel, from the formula $E = \frac{Wl}{8D}\left\{\frac{l^2}{6I} + \frac{5}{A}\right\}$ which expresses the result of a bending test on a girder.

Given that $A = {\cdot}924 \quad l = 60 \quad W = 5000$
$D = {\cdot}07 \quad I = 11{\cdot}15.$

Substituting values—

$$E = \frac{5000 \times 60}{8 \times {\cdot}07}\left\{\frac{60 \times 60}{6 \times 11{\cdot}15} + \frac{5}{{\cdot}924}\right\}$$
$$= 536000\ \{53{\cdot}8 + 5{\cdot}41\}$$
$$= \underline{31{\cdot}7 \times 10^6}.$$

Exercises 8.—On the Use of Logarithms and Evaluation of Formulæ.

Evaluate, using logarithms or the slide rule, Exs. 1 to 32; using approximations wherever possible.

1. $85{\cdot}23 \times 6{\cdot}917$
2. $876{\cdot}4 \times {\cdot}1194 \div 2{\cdot}356$
3. $75{\cdot}42 \times {\cdot}0002835$
4. $\frac{454{\cdot}2}{7{\cdot}965}$
5. ${\cdot}005376 \div {\cdot}1009$
6. $\frac{9543}{{\cdot}08176}$
7. $\frac{2{\cdot}896 \times 347{\cdot}2}{81{\cdot}48}$
8. $\frac{12{\cdot}08 \times {\cdot}02112}{{\cdot}01209}$
9. $\frac{{\cdot}0005}{{\cdot}007503}$
10. $\frac{4843 \times 29{\cdot}85}{75132}$
11. $\frac{1154 \div {\cdot}07648}{{\cdot}000914 \div 30{\cdot}42}$
12. $\frac{{\cdot}9867 \times {\cdot}4693}{{\cdot}0863 \times {\cdot}1842}$
13. $\frac{36{\cdot}87 \div 2{\cdot}57}{{\cdot}085 \div 13{\cdot}77 \times {\cdot}05}$
14. $\frac{24{\cdot}23 \times {\cdot}7529 \times {\cdot}00814}{3000 \times {\cdot}0115 \times 45{\cdot}27}$
15. $\frac{{\cdot}572 \div {\cdot}0086}{{\cdot}4539 \times {\cdot}0037 \div {\cdot}059}$
16. $\sqrt[4]{94{\cdot}03}$
17. $({\cdot}0517)^2$
18. $\sqrt[9]{{\cdot}1055}$
19. $\sqrt[5]{({\cdot}001769)^3}$
20. $\sqrt[5]{({\cdot}1182)^3}$
21. $(18{\cdot}24)^2 \times \sqrt[3]{{\cdot}2104}$
22. $\sqrt{\frac{94{\cdot}78 \times {\cdot}1109}{755 \times {\cdot}005}}$
23. $({\cdot}0254)^3 \div \sqrt[4]{{\cdot}7534}$
24. $\frac{({\cdot}0648)^2 \times \sqrt{2{\cdot}753}}{({\cdot}275)^3}$
25. $\frac{\sqrt[3]{94{\cdot}72} \times 8530}{({\cdot}2347)^{\frac{1}{2}} \times 5 \times 10^4}$
26. $\frac{(91{\cdot}56)^2 \times (3{\cdot}184)^{\frac{1}{3}}}{({\cdot}0459)^{\frac{1}{2}} \times ({\cdot}2743)^3}$
27. $\frac{(4{\cdot}72)^2 \times \sqrt[3]{20{\cdot}43}}{(2{\cdot}3)^2 \times \sqrt{8{\cdot}62}}$
28. $\sqrt[6]{\frac{{\cdot}00864 \times {\cdot}0372}{{\cdot}3807}}$
29. $\sqrt[7]{\frac{101{\cdot}4 \times ({\cdot}2801)^3}{({\cdot}00854)^4 \times 7094}}$
30. $\sqrt{\frac{42{\cdot}3 \times 1{\cdot}05}{{\cdot}0915 \times 7{\cdot}481}} \times \left(\frac{{\cdot}05006}{2411}\right)^2$
31. $\frac{\sqrt{{\cdot}0463} \times ({\cdot}086)^2}{\sqrt[3]{{\cdot}675} \times ({\cdot}091)^2}$
32. $\frac{(27{\cdot}63)^2 \times \log_{10} 3{\cdot}476}{\sqrt[8]{({\cdot}4349)^5} \times 5{\cdot}007 \times \frac{1}{7}}$

33. The formula $V = \pi r^2 l$ gives the volume of a cylinder. If $r = {\cdot}5$, $\pi = 3{\cdot}142$, $l = 12{\cdot}76$ find V.

34. Given that $L = \frac{8B - A}{3}$. Find L when $\frac{A}{2} = 11{\cdot}7$, $5B = 175{\cdot}5$.

35. If $R = \frac{l^2}{6a} + \frac{a}{2}$ find R when $l = 5{\cdot}1$ and $a = 0{\cdot}87$.

36. The velocity ratio of a differential pulley block is found from the formula—

$VR = \frac{2d_1}{d_2 - d_3}$ (where d_1, d_2 and d_3 are the diameters of the pulleys).

Find VR when $d_1 = 14{\cdot}57$, $d_2 = 5{\cdot}72$, $d_3 = 4{\cdot}83$.

37. If $v = u + ft$ and $s = ut + \frac{1}{2}ft^2$, find values of v and s when $u = 350$, $f = -27$, and $t = 4{\cdot}8$.

38. Find a velocity, v, from—

$$v = \sqrt{\frac{2gdh}{{\cdot}03l}}$$

when $g = 32{\cdot}2$, $d = 0{\cdot}84$, $h = 30$, $l = 5000$.

39. If $p = \left({\cdot}7854\frac{d^2}{t} \times \frac{f_s}{f_t}\right) + d$, find its value when $f_s = 5$, $t = 0{\cdot}75$ $f_t = 6$, $d = 1{\cdot}04$; p is the pitch of rivets, of diameter d, joining plates of thickness t.

40. Find the weight of a roof principal from Merriman's formula—

$$W = \frac{3}{4}al\left(1 + \frac{l}{10}\right) \text{ when } a = 10 \text{ and } l = 80.$$

41. To compare the cost of lighting by gas and electricity the following rule is often used, $a = \frac{\frac{b}{e} - c}{d}$

where a = price of 1 Board of Trade (B.O.T.) unit in pence; b = price per 1000 cu. ft. of gas in pence; d = watts per candle power (C.P.), e = candles per cu. ft. of gas per hour; c = cost in pence of lamp renewals per 1000 candle hours.

Find the equivalent cost per electric unit when lamps take 2·5 watts per C.P., $e = 2$, $c = 1\frac{1}{2}$ and gas is 2*s*. 2*d*. per 1000 cu. ft.

42. The length l of a trolley wire for a span L when the sag is d is given by the formula $l = L + \frac{8d^2}{3L}$. Find l, when $L = 500$, $d = 12$.

43. The input of an electric motor, in H.P., is measured by the product of the amperes and the volts divided by 746. What is the input in the case where 8·72 amps. are supplied at a pressure of 112·5 volts? If the efficiency of the motor at this load is 45 %, what is its output? (Output = efficiency × input.)

44. 2·4 lbs. of iron are heated from 60° F. to 1200° F. The specific heat of iron being ·13, find the number of British Thermal Units (B.Th.U.) required for this, given B.Th.U. = weight × rise in temp. × specific heat.

45. The following rules for the rating of motor-cars have been stated at various times.

(*a*) By Messrs. Rolls Royce, Ltd.—

$$H.P. = {\cdot}25(d - \tfrac{1}{2})^2 N\sqrt{S}$$

where d = diameter of cylinder in inches, N = no. of cylinders
S = stroke in inches.

(*b*) By the Royal Automobile Club—

$$\text{H.P.} = \cdot 197 d(d - 1)(r + 2)\text{N}$$

where N and d have the same meanings as before, and r = ratio of stroke to diameter. Find the rating of a 4-cylinder engine, whose cylinders are of 90 mms. diameter, and stroke 120 mms.; by the use of each of the rules.

46. 130 grms. of copper (W) at 95° C. (T) are mixed with 160 grms. (w) of water at 10° C. (t), the final temperature (t_1) being 16° C. Calculate the specific heat (s) of copper from

$$s = \frac{w(t_1 - t)}{\text{W}(\text{T} - t_1)}.$$

47. The volume v of a gas at a temperature of 0° C., or 273° C. absolute, and at a pressure corresponding to 760 mms. of mercury is 17·83 cu. ins. Find its volume at temperature t° C. and pressure H where $t = 83{\cdot}7$ and H = 797 from the formula—

$$\text{V} = v\frac{273 + t}{273} \cdot \frac{760}{\text{H}}.$$

48. If $\text{P} = \dfrac{nbt^2f}{3\text{L}}$ find its value when $b = 2\frac{1}{4}$, $t = \cdot 1$, $n = 8$, L = 12, $f = 80000$.

L = length of a railway spring on each side of the buckle, n = number of leaves, t = thickness of leaves, f = working stress, P = load applied and b = width of leaves.

49. The increase in length of a steel girder due to rise of temperature can be found from the formula, new length = old length $(1 + at)$ when t = rise in temperature, and a = coefficient of linear expansion. Find the increase in length of a girder of 80 ft. span due to change of temperature of 150° F. when $a = \cdot 000006$.

50. If $c = \frac{1}{10}\sqrt{(11p + 4d)(p + 4d)}$, find its value when $p = 3''$, $d = 1\frac{1}{8}''$. The meaning of c will be understood by reference to the riveted joint shown in Fig. 2.

51. Find the thickness (t_1) of a butt strap from the B.O.T. rule—

$$t_1 = \frac{5}{8}\left(\frac{p - d}{p - 2d}\right)t$$

when $p = 4\frac{3}{8}''$, $t = \frac{5}{8}''$, $d = \frac{13}{16}''$.

52. Find the thickness (t) of a pipe for pressure p lbs. per sq. in., when internal dia. = d, from—

$$t = \frac{p + 100}{7200}d + \cdot 333\left(1 - \frac{d}{100}\right)$$

$d = 2$, $p = 450$.

53. Taking $\quad p = \dfrac{f}{2}\left\{1 + \sqrt{1 + \dfrac{4s^2}{f^2}}\right\}$

which gives the principal (or maximum) stress p due to a normal stress f and a shearing stress s: determine p when $f = 3800$, $s = 2000$.

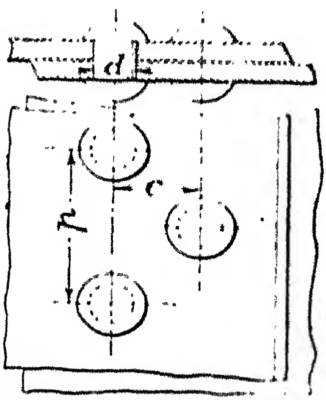

Fig. 2. Riveted Joint.

54. If $\text{P} = \dfrac{\text{F}}{1 + \dfrac{1}{1500} \cdot \dfrac{d^2}{\text{T}^2}}$ (Gordon's formula for the buckling load on struts), find P when F = 28, $d = 15$, $\text{T} = \frac{7}{16}$.

55. If $p = \frac{f}{1 + c\left(\frac{l}{k}\right)^2}$ (Rankine's formula for the buckling load on struts), find p when $f = 48000$, $l = 14\frac{1}{2} \times 12$, $c = \frac{1}{30000}$, $k^2 = 30{\cdot}7$.

56. The deflection d of a helical spring can be obtained from—

$$d = \frac{64wnr^3}{CD^4}.$$

Find the deflection for the case in which $w = 48$, $D = \frac{1}{4}$
$n = 12{\cdot}73$, $r = 1{\cdot}5$.
$C = 12 \times 10^6$.

57. If the deflection d of a beam of radius a and length l, due to a load of W is measured, Young's Modulus for the material of which the beam is composed, can be found from $E = \frac{4Wl^3}{3\pi da^4}$. If in a certain case the deflection was $4{\cdot}2$; and W, l, a and π had the values $14{\cdot}8$, $17{\cdot}56$, $\cdot39$ and $3{\cdot}142$ respectively, find the value of E.

58. For oval furnaces, if—

Δ = difference between the half axes before straining.
δ = ,, ,, ,, after ,,
p = pressure in lbs. per sq. in.
E = Young's Modulus.
D = diameter of furnace.

then $$\delta = \frac{\Delta \times 32EI}{32EI - pD^3}.$$

Find the value of δ when $\Delta = \cdot5$, $D = 40$, $p = 100$, $E = 30 \times 10^6$ and $I = \cdot0104$.

59. The modulus of rigidity C of a wire of length l and diameter d may be found by attaching weights of m_1 and m_2 respectively at the end of the wire and noting the times, t_1 and t_2 respectively, taken for a complete swing. The formula used in the calculation is—

$$C = \frac{128\pi l(m_1 - m_2)a^2}{gd^4(t_1^2 - t_2^2)}$$

Find C when $m_1 = 9{\cdot}8$, $m_2 = 1{\cdot}5$, $t_1 = 2{\cdot}1$, $t_2 = 1{\cdot}6$, $g = 32$, $d = \cdot126$, $l = 4{\cdot}83$, $a = \cdot97$ and $\pi = 3{\cdot}142$.

60. The weight W in tons of a flywheel is given by—

$$W = \frac{43257Hrn}{R^3N^3}$$

Find the weight when $R = \frac{40}{12}$, $r = \cdot2$, $n = 30$, $N = 120$, $H = 70$.

61. The approximate diameter of wire (in inches) to carry a given current C with $\theta°$ rise in temperature can be obtained from—

$$D = \sqrt[3]{\frac{4J\rho C^2}{\pi^2 m\theta}}$$

Find D when $J = \cdot0935$, $\theta = 35$, $\rho = \frac{1{\cdot}6}{10^6}$, $C = 55$, $m = \cdot0025$ and $\pi = 3{\cdot}142$.

62. Find the dimensions for the flanged cast-iron pipe shown in Fig. 3 (in each case to the nearest $\frac{1}{16}$th of an inch), when $P = 85$, $D = 7{\cdot}5$, $t = \frac{PD}{4000} + \cdot3$, $T = 1{\cdot}4t + \cdot15$, $d = \cdot83t + \cdot3$, $B = 2\frac{1}{4}d$.
There are n bolts and $n = \cdot6D + 2$.

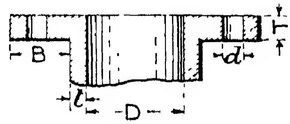

Fig. 3.

Investigation for Units.—A train covers a distance of 150 miles in 5 hours; what is its average speed?

Obviously it is $\frac{150 \text{ miles}}{5 \text{ hours}}$, *i. e.*, $\frac{30 \text{ miles}}{1 \text{ hour}}$ or 30 miles per hour.

It could with equal truth be expressed by $\frac{150 \times 5280 \text{ ft.}}{5 \times 60 \times 60 \text{ secs.}}$, *i. e.*, $\frac{44 \text{ ft.}}{1 \text{ sec.}}$, or 44 ft. per second. The figures in the results differ because they are measured in terms of different quantities, and it is essential that **the units in which results are expressed should be clearly stated.**

Here we have another form of investigation to be performed before the actual numerical working is attempted. To find the units in which the result is to be expressed, these units, with their proper powers attached, are put down in the form of a fraction, all figures and constants being disregarded, and are treated for cancelling purposes as though they were pure algebraic symbols.

Suppose a force of 100 lbs. weight is exerted through a distance of 15 ft., then the work done by this force is 100×15 or 1500 units: these units will be "foot lbs." since the result is obtained by multiplication of lbs. by feet. This statement might be written in the form, Work $=$ lbs. $\times$ feet $=$ foot lbs. If now we are told that the time taken over the movement was 12 minutes we can determine the average rate at which the work was done. The work done in 1 minute is evidently obtained by dividing the total work done in 12 minutes by the number of minutes: thus, rate of working $= \frac{1500}{12} = 125$. This figure gives the number of foot lbs. of work done in one minute, and the result would be expressed as, average rate of working $= 125$ foot lbs. *per* minute. It will be seen from this and from the previous illustration that the word *per* implies division. To obtain a velocity in miles per hour, the distance covered, in miles, must be divided by the number of hours taken, or

$$\text{velocity (miles per hour)} = \frac{\text{number of miles}}{\text{number of hours}} \text{ or, more shortly, } \frac{\text{miles}}{\text{hours}}.$$

$$\begin{aligned} \text{An acceleration} &= \text{rate of change of speed} \\ &= \text{feet per second added every second (say)} \\ &= \frac{\text{feet}}{\text{secs.} \times \text{secs.}} \text{ or } \frac{\text{feet}}{(\text{secs.})^2} \end{aligned}$$

Hence, wherever an acceleration occurs it must be written as $\frac{\text{distance}}{(\text{time})^2}$ in the investigation for units.

The "g" so frequently met with in engineering formulæ is an acceleration, being 32·2 ft. per sec. per sec. or $32{\cdot}2 \dfrac{\text{ft.}}{(\text{secs.})^2}$, and therefore must be treated as such wherever it occurs.

Example 21.—The steam pressure, as recorded by a gauge, is 65 lbs. per sq. in.; the area of the piston on which the steam is acting is 87 sq. ins. What is the total pressure on the piston?

Total pressure = area × intensity of pressure

$$\text{and is} \quad \cancel{(\text{ins.})^2} \times \frac{\text{lbs}}{\cancel{(\text{ins.})^2}} \quad \textit{i. e.}, \text{ is in lbs.}$$

The true pressure is 65 + 14·7 lbs. per sq. in., because the gauge records the excess over the atmospheric pressure—

$$\begin{aligned} \therefore \text{ total pressure} &= 79{\cdot}7 \times 87 \text{ lbs.} \\ &= \underline{6930 \text{ lbs.}} \end{aligned}$$

Example 22.—Find the force necessary to accelerate a mass of 10 tons by 12 ft. per sec. in a minute. The formula connecting these quantities is $P = \dfrac{Wf}{g}$ where W = weight, f = acceleration and g has its usual meaning.

Dealing merely with the units given, and forming our investigation for units—

$$P = \overbrace{\text{Tons}}^{W} \times \overbrace{\frac{(\text{secs.})^2}{\text{feet}}}^{\frac{1}{g}} \times \overbrace{\frac{\text{feet}}{\text{secs.} \times \text{mins.}}}^{f}$$

It will be seen that no cancelling can be done until the minutes are brought to seconds: then we have—

$$P = \text{Tons} \times \frac{\cancel{(\text{secs})^2}}{\cancel{\text{feet}}} \times \frac{\cancel{\text{feet}}}{\cancel{(\text{secs})^2}} = \text{Tons}$$

To find the force, therefore, the minutes must be multiplied by 60; *i. e.*, the denominator must be multiplied by 60.

$$\text{Hence} \qquad P = 10 \times \frac{1}{32{\cdot}2} \times \frac{12}{60} = \underline{{\cdot}0621 \text{ ton or } 139{\cdot}2 \text{ lbs.}}$$

Example 23.—The modulus of rigidity C of a wire can be found by noting the time of a complete swing of the pendulum shown in Fig. 4 and then calculating from the formula, $C = \dfrac{128\pi l I}{g d^4 t^2}$, where l is the length of the wire, d is its diameter, I is the moment of inertia

of the brass rod about the axis of suspension and t is the time of one swing.

If l and d are measured in inches, t in seconds, and I in lbs. × (feet)^2 [I being of the nature of mass × (distance)^2], in what units will C be expressed?

Investigating for units—

$$C = \underbrace{\text{constant}}_{128\pi} \times \underbrace{\text{ins.}}_{l} \times \underbrace{\text{lbs.} \times \text{feet}^2}_{I} \times \underbrace{\frac{\text{secs.}^2}{\text{feet}}}_{\frac{1}{g}} \times \underbrace{\frac{1}{\text{ins.}^4}}_{\frac{1}{d^4}} \times \underbrace{\frac{1}{\text{sec.}^2}}_{\frac{1}{t^2}}$$

$$= \frac{\text{constant} \times \text{lbs.} \times \text{feet}}{\text{ins.}^3}$$

If the numerator is multiplied by 12, then—

$$C = \frac{\text{constant} \times \text{lbs.} \times \text{ins.}}{\text{ins}^3.} = \frac{\text{lbs.}}{\text{ins.}^2}$$

or the result would be expressed in lbs. per sq. in. provided that the numerator was multiplied by 12.

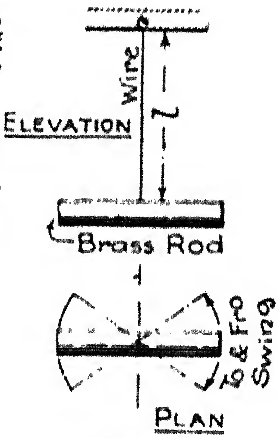

Fig. 4.

Example 24.—The head lost in a pipe due to friction is given by the formula $h = \cdot 03 \cdot \frac{l}{d} \cdot \frac{v^2}{2g}$. Find its value if the pipe is 3″ dia., 56 yards long, and the velocity of flow is 28 yards per min.

The meanings of the various letters will be better understood by reference to Fig. 5.

Dealing only with the units given, and disregarding the constants—

$$\text{Head lost} = \underbrace{\text{yards}}_{l} \times \underbrace{\frac{1}{\text{ins.}}}_{\frac{1}{d}} \times \underbrace{\frac{\text{yards}^2}{\text{mins.}^2}}_{v^2} \times \underbrace{\frac{\text{secs.}^2}{\text{feet}}}_{\frac{1}{g}}$$

This is not in a form convenient for cancelling; accordingly, bring all distances to feet and all times to seconds.

Then the head lost $= \text{feet} \times \frac{1}{\text{feet}} \times \frac{\text{feet}^2}{\text{secs.}^2} \times \frac{\text{secs.}^2}{\text{feet}} = \text{feet.}$

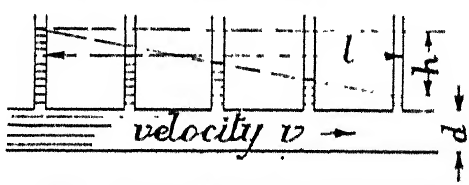

Fig. 5.—Flow of water through a pipe.

Substituting the numerical values in place of the symbols—

$$\text{Head lost} = h = \frac{\cdot 03 \times 56 \times 3 \times 12 \times 28 \times 3 \times 28 \times 3}{3 \times 60 \times 60 \times 64 \cdot 4}$$

$$= \underline{\cdot 612 \text{ foot.}}$$

Example 25.—Find the maximum deflection of a beam 24 ft. long, simply supported at its ends and loaded with 7 tons at the centre. The moment of inertia I of the section is 87·2 ins.^{4} units, and Young's Modulus E for the material is 30 × 10^6 lbs. per sq. in.

The maximum deflection $= \dfrac{Wl^3}{48IE}$

The investigation for units, as given, reads :—

$$\text{Deflection} = \overbrace{\text{tons}}^{W} \times \overbrace{\text{feet}^3}^{l^3} \times \overbrace{\frac{1}{\text{ins.}^4}}^{\frac{1}{I}} \times \overbrace{\frac{\text{ins.}^2}{\text{lbs.}}}^{\frac{1}{E}}$$

No cancelling can be attempted until the tons are brought to lbs. and the feet to inches or *vice versa*; assuming the former, then—

$$\text{Deflection} = \text{lbs.} \times \text{ins.}^3 \times \frac{1}{\text{ins.}^4} \times \frac{\text{ins.}^2}{\text{lbs.}} = \text{ins.}$$

So that, since 7 tons = 7 × 2240 lbs. and 24 ft. = 288 ins.

$$\text{Deflection} = \frac{7 \times 2240 \times 288^3}{48 \times 87{\cdot}2 \times 30 \times 10^6} \text{ ins.}$$

Calculation—

$$\begin{aligned}
\log d &= (\log 7 + \log 2240 + 3 \log 288) \\
&\quad - (\log 48 + \log 87{\cdot}2 + \log 30{,}000{,}000) \\
&= ({\cdot}8451 + 3{\cdot}3502 + 7{\cdot}3782) \\
&\quad - (1{\cdot}6812 + 1{\cdot}9405 + 7{\cdot}4771) \\
&= 11{\cdot}5735 - 11{\cdot}0988 \\
&= {\cdot}4747 = \log 2{\cdot}984
\end{aligned}$$

∴ deflection = 2·984 ins.

Approximation—

$$\frac{7 \times 2 \times 3 \times 3 \times 3}{5 \times 9 \times 3}$$

or 2·8.

Explanation—

log 288 = 2·4594

3 × log 288 = 7·3782.

Exercises 4.—On the Finding of Units.

1. In what units will f be expressed if $f = \dfrac{\delta E}{D}$ and δ is in inches, E in lbs. per sq. in. and D in ins.?

2. If a H.P. = 33000 foot lbs. of work per minute, find the H.P. necessary to raise 300 cwts. of water through a vertical height of 16½ yards in half an hour.

3. If $H = \dfrac{f^2}{2\rho E}$: find H in yards when $f = 18$ tons per sq. in.; E = 13000 per sq. in.; $\rho = 480$ lbs. per cu. ft.

4. Determine the stress f in a boiler plate in tons per sq. in. from—

$f = \dfrac{pd}{2t}$ when $t = {\cdot}63$ in., $d = 8$ feet, $p = 160$ lbs. per sq. in.

t is the thickness of plate, p is the pressure inside the boiler, and d is the diameter of the boiler.

5. The jump H of the wheels of a gun is given by—

$$H = \frac{AWk^2P}{Mh^2R}.$$

Find the jump in inches when A = 40 ins., k = 10 ft., h = 1 yd., P = 47·5 cwts., R = 1·15 tons, M = 1·2 tons, W = 9 cwts.

6. The tension in a belt due to centrifugal action can be calculated from $T = \frac{wv^2}{g}$. If w = wt. per foot run of belt in lbs., v = veloc. in ft. per sec., and g has its usual value, in what units will T be expressed?

7. If, in the previous example, w = ·43 lb. per foot length of belt per sq. in. of surface, find a simple relation between the stress (in lbs. per sq. in.) and the velocity (ft./sec.).

8. The I.H.P. of an engine is determined from the formula—

$$\text{I.H.P.} = \frac{2\text{PLAN}}{33000}$$

where P = mean effective pressure in lbs. per sq. in., L = stroke in feet, A = area of piston in sq. ins., and N = revolutions per minute. If l is the stroke in ins. and A = ·7854D² show that this equation may be written $\text{I.H.P.} = \frac{4\text{PN}l\text{D}^2}{1{,}000{,}000}$ approximately.

9. Given that $f = \frac{wv^2}{g}$, where w = weight in lbs. per cu. in., v = veloc. in feet per sec.

(a formula relating to tensile stress in revolving bodies).

Arrange the formula so that f is given in lbs. per sq. in.

10. Investigate for units answer in the following formula for the Horse Power transmitted by a shaft.

$$\text{H.P.} = \frac{p\text{R}^3\text{N}\pi^2}{33000}$$ where R is inches, N is Revolutions per minute, π is a constant, and p is in lbs. per sq. in.

If these are not found to be H.P. units, viz. foot lbs. per minute, state what correction should be made.

11. The formula $Q = a_1a_2\sqrt{\frac{2g(p_1 - p_2)}{\rho(a_1^2 - a_2^2)}}$ gives the quantity of water passing through a Venturi Meter.

In what units will Q be expressed if a_1 and a_2 are in sq. ft.; p_1 and p_2 in lbs. per sq. ft.; g in feet per sec. per sec.; ρ in lbs. per cu. ft.?

12. Given that 1 lb. = 454 grms., 1″ = 2·54 cms.

1 erg. = work done when 1 dyne acts through 1 cm.

1 grm. weight = 981 dynes.

and 1 watt = 10^7 ergs per sec.,

find the number of watts per H.P.

13. The extension x of a rubber shock absorber for an aeroplane chassis is given by—

$$x = \frac{6\cdot04\ \text{WD}}{nd^2\ \text{E}}$$

where W = wt. of machine (lbs.). D = dia. of coil (ins.).
n = number of coils. d = dia. of rubber cord (ins.).
E = Young's modulus (lbs. per sq. in.).

Find x when W = 1500, D = 5, n = 50, d = $\frac{5}{8}$ and E = 300, stating the units in which the answer is expressed.

CHAPTER II

EQUATIONS

Simple Equations.—A *simple equation* consists of a statement connecting an unknown quantity with others that are known; and the process of "Solving the equation" is that of finding the particular value of the unknown that satisfies the statement. To many, this chapter, on the methods of solving equations and of transposing formulæ, must be as important and useful as any in the book, for it is impossible to proceed very far without a working knowledge of the ready manipulation of formulæ. The methods of procedure always followed is the isolation of the unknown, involving the transposition of the known quantities, which may be either letters or numbers, from one side of the equation to the other. The transposition may be of either (*a*) terms or (*b*) factors; and the rule for each change will now be developed.

To deal first with the *transposition of terms*:—

When turning the spindle shown in Fig. 6 it was necessary to calculate the length of the "plain turned" portion, or the length marked l in the diagram. The conditions here are that the required length, together with the radius $\cdot 375''$, must add to $1\cdot 5''$. A statement of conditions may thus be made, in the form—

$$l + \cdot 375 = 1\cdot 5.$$

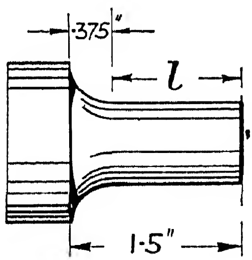

Fig. 6.

The truth of this statement will be unaltered if the same quantity, viz. $\cdot 375$, is subtracted from each side, so that—

$$l + \cdot 375 - \cdot 375 = 1\cdot 5 - \cdot 375$$

or $$l = 1\cdot 5 - \cdot 375 = \underline{1\cdot 125}.$$

Thus, in changing the $\cdot 375$ from one side of the equation to the other, the sign before it has been changed; $+ \cdot 375$ on the one side becoming $- \cdot 375$ when transferred to the other side.

Again, suppose the excess of the pressure within a cylinder over that of the atmosphere (taken as $14\cdot 7$ lbs. per sq. in.) is $86\cdot 2$

lbs. per sq. in., and we require to determine the absolute pressure in the cylinder.

Let p represent the absolute pressure, *i. e.*, the excess over zero pressure. Then—

$$p - 14{\cdot}7 = 86{\cdot}2.$$

To each side add $14{\cdot}7$; then—

$$p = 86{\cdot}2 + 14{\cdot}7 = \underline{100{\cdot}9} \text{ lbs. per sq. in.}$$

Thus, $-14{\cdot}7$ on the left-hand side becomes $+14{\cdot}7$ when transferred to the right-hand side of the equation.

Accordingly, we may say that :— **When transferring a TERM from one side of an equation to the other, the sign before the term must be changed, plus becoming minus, and *vice versa*.**

To deal with the *transposition of factors* :—

Suppose we are told that 3 tons of pig iron are bought for £7 10*s.* : we should say at once that the price per ton was $\frac{1}{3}$ of £7 10*s.*, or £2 10*s.*

We might, however, use this case to illustrate one of the most vital rules in connection with transpositions, by expressing the statement in the form of an equation and then solving the equation.

The unknown in this case is the price per ton, which may be called p shillings. Our equation then becomes —

$$3 \times p = 150 \quad \text{. . . . .} \quad (1)$$

Divide both sides by 3, which is legitimate, since the equation is not changed if *exactly the same operation is performed on either side.*

$$\therefore p = \frac{150}{3} = 50 \quad \text{. . . . .} \quad (2)$$

or <u>the cost is 50*s.* per ton.</u>

Again, had we been told that $\frac{1}{2}$ a ton could be bought for 25*s.* we could express this in the form—

$$\tfrac{1}{2} p = 25 \quad \text{. . . . .} \quad (3)$$

If we multiply both sides by 2 we find that

$$p = 25 \times 2 = 50 \quad \text{. . . . .} \quad (4)$$

which, of course, agrees with the above.

It will be seen that, to isolate p and so find its absolute value, we transfer the multiplier in equation (1) or the divisor in equation (3) to the other side, when its effect is exactly reversed : thus the multiplier 3 in equation (1) becomes a divisor when transferred to the other side of the equation, as in (2); and the dividing 2 in equation (3) becomes the multiplying 2 in equation (4).

The motion of a swinging pendulum furnishes an illustration of the transposition of a factor which is preceded by a minus sign. The acceleration of the pendulum *towards* the centre of the movement increases proportionately with the displacement *away from* the centre. Taking a numerical case, suppose that we wish to find the displacement s when the acceleration f is 4·6 units and the relation between f and s is $f = -25s$.

Substituting the numerical value for f

$$4{\cdot}6 = -25s.$$

To isolate s we must divide both sides of the equation by -25, and then—

$$\frac{4{\cdot}6}{-25} = s$$

or $$\underline{s = -{\cdot}184 \text{ unit.}}$$

The rule for the transposition of factors can now be stated, viz. **To change a FACTOR (i. e., a multiplier or a divisor) from one side of an equation to the other, change also its position regarding the fractional dividing line, viz., let a denominator become a numerator and conversely; and let the sign of the factor be kept unchanged.**

We have thus established the elementary rules of term and factor changing in simple equations. The following examples, as illustrations of these fundamental laws, should be most carefully studied, every step being thoroughly grasped before proceeding to another.

Example 1.—Solve for x, in the equation, $\frac{5x}{4} = \frac{7}{1{\cdot}8}$

Transferring the 5 and 4 so that x is by itself, the 5 must change from the top to the bottom and the 4 from the bottom to the top, since 5 and 4 are factors.

Then—
$$x = \frac{7}{1{\cdot}8} \times \frac{4}{5} = \underline{3{\cdot}11.}$$

Example 2.—Solve for a, in the equation, $4a + 17 = 2{\cdot}5a - 9$

Transposing, to get the unknowns together on one side—

$$4a - 2{\cdot}5a = -9 - 17.$$

Here the change is that of terms, hence the change of signs.

Grouping, or collecting the terms—

$$1{\cdot}5a = -26$$

$$\therefore \quad a = \frac{-26}{1{\cdot}5} = \underline{-17{\cdot}33.}$$

Example 3.—The weight of steam required per hour for an engine was a constant 60 lbs., together with a variable 25 lbs. for each H.P. developed. If, in a certain case, 210 lbs. of steam were supplied in an hour, what was the H.P. developed?

Let h represent the unknown H.P.

Then $25h$ represents the amount of steam for this H.P., apart from the constant, and the equation including the whole of the statement of conditions is—

$$25h + 60 = 210.$$

Transferring the term $+60$ to the other side, where it becomes -60,

$$25h = 210 - 60 = 150.$$

Dividing throughout by 25 $\quad h = \frac{150}{25} = 6$

or, the H.P. developed was 6.

Example 4.—To convert degrees Fahrenheit to degrees Centigrade use is made of the following relation—

$$F - 32 = \frac{9}{5}C.$$

Find the number of degrees C., corresponding to 457° F.

Substituting for F its numerical value—

$$457 - 32 = \frac{9}{5}C$$

$$\therefore \quad 425 = \frac{9}{5}C$$

Transposing factors 5 and 9, $\quad \frac{425 \times 5}{9} = C$

$$\therefore \quad 236 \cdot 1 = C$$

i.e., 236° C. correspond to 457° F.

It might happen that in an engine or boiler trial only thermometers reading in Centigrade degrees were available, whereas for purposes of calculation it might be necessary to have the temperatures expressed in degrees Fahrenheit. This would mean that a number of equations would have to be solved; but the work involved could be shortened by a suitable transposition of the formula given above.

$$F - 32 = \frac{9}{5}C \quad \dots\dots\dots\dots \quad (1)$$

$$\therefore \quad F = \frac{9}{5}C + 32 \quad \dots\dots\dots\dots \quad (2)$$

Equation (2) is far more suitable for our purpose than equation (1), although the change is so slight.

Example 5.—Convert 80°, 15°, 120°, and 48° C. to degrees F.

When C = 80, $F = \left(\frac{9}{5} \times 80\right) + 32 = 176°$, and so on.

Or, we might tabulate, for the four readings given, thus :—

C	$\frac{9}{5}C+32$	F
80	144 + 32	176
15	27 + 32	59
120	216 + 32	248
48	86·4 + 32	118·4

Example 6.—Ohm's law states that the drop in electrical pressure E when a current C flows through a resistance R, is given by the formula E = CR. Transpose this for R and C.

To find C, $E = CR$

Transposing the factor R, $\therefore\ C = \frac{E}{R}$.

In like manner $R = \frac{E}{C}$.

Brackets occurring in equations must be removed before applying the rules of transposition, and the same remark applies to fractions, which may always be regarded as brackets written in a different form.

Example 7.—Solve for *w* in the equation—

$$(3w-4{\cdot}1)-2(7-{\cdot}1w) = 15({\cdot}3w+{\cdot}62).$$

Removing brackets—

$$3w-4{\cdot}1-14+{\cdot}2w = 4{\cdot}5w+9{\cdot}3.$$

Dissociating knowns and unknowns—

$$3w+{\cdot}2w-4{\cdot}5w = 9{\cdot}3+4{\cdot}1+14$$

$$\therefore\quad -1{\cdot}3w = 27{\cdot}4$$

$$\therefore\quad w = \frac{27{\cdot}4}{-1{\cdot}3} = \underline{-21{\cdot}1}.$$

Note that the sign of 1·3 is kept unchanged.

Example 8.—When finding the latent heat L of steam, the following equation was used—

$$w(T-t) = qL+t_1-T.$$

Transpose this for L, *i. e.*, find an expression for L in terms of the other letters, which must be regarded as representing known quantities.

Here L is the unknown, since the values of all the other letters are supposed to be known.

Clearing of brackets, $wT - wt = qL + t_1 - T$

Transposing terms, $wT - wt - t_1 + T = qL$

Transposing the factor q, $\dfrac{wT - wt - t_1 + T}{q} = L$

Example 9.—Solve the equation—

$$\frac{4x}{5} - \frac{7x}{2} + \frac{8\cdot1}{4} = \frac{1\cdot9x}{5} + \frac{7\cdot21}{3}.$$

The L.C.M. of 5, 2, 4 and 3 is 60, and multiplication throughout by this figure will remove the denominators.

$$(4x \times 12) - (7x \times 30) + (8\cdot1 \times 15) = (1\cdot9x \times 12) + (7\cdot21 \times 20)$$

$$48x - 210x + 121\cdot5 = 22\cdot8x + 144\cdot2$$

$$48x - 210x - 22\cdot8x = 144\cdot2 - 121\cdot5$$

or $$-184\cdot8x = 22\cdot7$$

$$\therefore \quad x = \frac{22\cdot7}{-184\cdot8} = \underline{-\cdot123}.$$

Example 10.—The electro-motive force E of a cell was found on open circuit, and also the drop in potential V when a resistance of R was placed in the circuit. The internal resistance of the cell may be calculated from the equation $(E - V) = \frac{V}{R} \times R_i$ where R_i is the internal resistance. Find the internal resistance for the case for which $E = 1\cdot34$, $V = \cdot8965$ and $R = 5$.

It being required to find R_i we transpose the $\frac{V}{R}$ and treat the bracketed letter as one quantity for the time being; then

$$\frac{R}{V}(E - V) = R_i$$

which completes the transposition.

Substituting the numerical values—

$$R_i = \frac{5}{\cdot8965}(1\cdot34 - \cdot8965) = \frac{5 \times \cdot4435}{\cdot8965}$$

$$= \underline{2\cdot47 \text{ ohms.}}$$

Example 11.—Solve the equation—

$$\frac{3y-5}{4} - \frac{7y+9}{16} + \frac{8y+19}{8} + 8\frac{5}{8} = 0.$$

Before proceeding to find the L.C.M. it will be found the safest plan to place brackets round the numerators of the fractions. This

emphasises the fact that the whole of each numerator is to be treated as one quantity. Thus—

$$\frac{(3y-5)}{4}-\frac{(7y+9)}{16}+\frac{(8y+19)}{8}+\frac{69}{8}=0.$$

Failing this step, mistakes are almost certain to arise, especially with signs, *e. g.*, the minus before the second fraction applies equally to the 9 and to the $7y$. This fact would probably be overlooked if the bracket were not inserted.

Multiplying throughout by 16, the L.C.M. of 4, 16 and 8

$$4(3y-5)-(7y+9)+2(8y+19)+(2\times 69)=0$$

i.e., $$12y-20-7y-9+16y+38+138=0$$

$\therefore$ $$12y-7y+16y=20+9-38-138$$

$$21y=-147$$

$$y=\frac{-147}{21}$$

$$\underline{\underline{=-7.}}$$

Example 12.—If p is the intensity of pressure over an annular plate of outside diameter D and inside diameter d, then the total pressure on the plate is given by—

$$P=\cdot 7854p(D^2-d^2).$$

Assuming that p, P and D are known, transpose this equation into a form convenient for the calculation of the value of d.

Treating the $\cdot 7854p$ as one quantity, and transposing it—

$$D^2-d^2=\frac{P}{\cdot 7854p}$$

Transferring D^2 to the right-hand side—

$$-d^2=\frac{P}{\cdot 7854p}-D^2$$

Changing signs throughout—

$$d^2=D^2-\frac{P}{\cdot 7854p}$$

Taking the square root of both sides—

$$\underline{d=\sqrt{D^2-\frac{P}{\cdot 7854p}}}$$

Example 13.—If $t=2\pi\sqrt{\frac{l}{g}}$, giving the time in seconds of 1 swing (periodic, or to and fro) of a simple pendulum of length l feet; find an expression for l.

It will be easiest in this case to square both sides (*i. e.*, to remove the square root sign which is merely one form of bracket).

Then— $$t^2=4\pi^2\frac{l}{g}$$

or, transposing the factors, 4, π^2 and g, $$\underline{\frac{gt^2}{4\pi^2}=l.}$$

Example 14.—Transpose for q, the dryness fraction of steam found by the Barrus test for superheated steam, in the equation—

$$\cdot 48(T_A - T_B - n) = (1 - q)L + \cdot 48(T_S - T).$$

T_A, T_B, T_S and T are temperatures, L is the latent heat of the steam, and n = loss of temperature of the superheated steam when the supply of moist steam is cut off.

Treating $\cdot 48(T_S - T)$ as a term, it may be transferred to the other side with change of sign before it—

$$\cdot 48(T_A - T_B - n) - \cdot 48(T_S - T) = (1 - q)L$$

or, since ·48 multiplies each bracket, we can take it outside one large bracket—

$$\cdot 48\{T_A - T_B - n - T_S + T\} = (1 - q)L.$$

Dividing both sides by L—

$$(1 - q) = \frac{\cdot 48}{L}\{T_A - T_B - n - T_S + T\}$$

$$\therefore \qquad q = 1 - \frac{\cdot 48}{L}\{T_A - T_B - n - T_S + T\}.$$

Example 15.—The equation $\frac{f^2 AL}{2E} = W(H + e)$ refers to the stress produced in a bar by a weight W falling through a height H on to the bar. Transpose this equation for f and also for e.

To find f:—

Transposing factors, $f^2 = W \times \frac{2E}{AL}(H + e)$

Extracting the square root of both sides of the equation.

$$f = \sqrt{\frac{2EW(H + e)}{AL}}.$$

To find e—

$$\frac{f^2 AL}{2E} = W(H + e)$$

$$H + e = \frac{f^2 AL}{2EW}$$

$$\therefore \qquad e = \frac{f^2 AL}{2EW} - H.$$

Example 16.—One hundred electric glow lamps, each of 150 ohms resistance and each requiring ·75 ampere, are connected in parallel. How many cells, each of ·0052 ohm resistance and giving 2·08 volts, will be required to light these lamps? (Cells to be in series.)

Total resistance = Internal resistance + external resistance.

External resistance $= \frac{150}{100} = 1 \cdot 5$ ohms (because lamps in parallel offer less resistance, *i. e.*, an easier path is made for the current).

Suppose x cells are required—

$$\begin{aligned}
\text{Total E.M.F.} &= x \times 2{\cdot}08 \\
\text{Total internal resistance} &= x \times {\cdot}0052 \\
\therefore \quad \text{Total resistance} &= {\cdot}0052x + 1{\cdot}5 \\
\text{Current} &= \frac{\text{E.M.F.}}{\text{Resistance}} \\
\text{and } 100 \times {\cdot}75 &= \frac{2{\cdot}08x}{{\cdot}0052x + 1{\cdot}5}.
\end{aligned}$$

Multiplying across, *i. e.*, multiplying throughout by the common denominator $\cdot 0052x + 1\cdot 5$.

$$\begin{aligned}
75({\cdot}0052x + 1{\cdot}5) &= 2{\cdot}08x \\
{\cdot}39x + 112{\cdot}5 &= 2{\cdot}08x \\
112{\cdot}5 &= 2{\cdot}08x - {\cdot}39x = 1{\cdot}69x \\
\therefore x &= 66{\cdot}6
\end{aligned}$$

Or 67 cells would suffice.

Exercises 5.—On Simple Equations and Transpositions.

Solve the equations in Exs. 1 to 6

1. $5x + 7(x - 2) = 3 - 4(x + 6)$

2. $\frac{1}{8}a + \frac{2}{7}a - 3a = 5 - \frac{2a}{5}$

3. $\frac{4{\cdot}2p}{7{\cdot}45} = \frac{9{\cdot}58}{4{\cdot}69}$

4. $\frac{y - 6}{5} + \frac{4y - 3}{2} = 17 - \frac{6y - 17}{9}$

5. $\frac{15x}{4{\cdot}7} - \frac{3{\cdot}15}{1{\cdot}08} = \frac{37{\cdot}5}{2{\cdot}95} + \frac{8{\cdot}4x}{9{\cdot}11}$

6. $8{\cdot}2x - 4{\cdot}75(3 - 2x) + 2{\cdot}14(5x + 7) = 17 - (1 - {\cdot}8x) + 5{\cdot}43$

7. Transpose for c in the equation $\frac{4ab}{7} = \frac{2d}{5ac}$

8. If $H = ws(T - t)$, find an expression for T.

9. If $P = CTAE$, find E when $A = 19{\cdot}25$, $C = {\cdot}000006$, $T = 442$, $P = 1{,}532{,}000$.

10. Transpose for L, the latent heat of steam, in the equation $w_1(t_1 - T + L) = w(T - t)$, and hence find its value when $w_1 = \frac{3}{32}$, $t_1 = 212$, $w = 1\frac{1}{4}$, $T = 145$, and $t = 70$.

11. A formula occurring in connection with Tacheometric Surveying is $D = \frac{fS}{\delta} + f + d$. Determine the value of δ to satisfy this when $D = 3600$, $f = 12$, $d = 6$ and $S = 36$.

12. Using the equation in Exercise 11, find the value of f to satisfy it when $D = 310{\cdot}7$, $S = 4{\cdot}63$, $\delta = {\cdot}015$, and $d = {\cdot}5$.

13. If $w = \frac{Wl^2}{cds - l^2}$, find s when $w = 8{\cdot}15$, $l = 50$, $W = 83{\cdot}5$, $d = 4$, and $c = 1400$. w is the weight of a girder in tons to carry an external load

W tons, d is the effective depth of the girder in feet, s is the shearing stress in tons per sq. in., and c is a coefficient depending on the type of girder.

14. If $\frac{2}{E}\left(1+\frac{1}{m}\right)=\frac{1}{C}$, find E in terms of C for the case when $m=4$. In other words, find the relation between Young's modulus and the Rigidity modulus when "Poisson's ratio" is 4.

15. Find the value of R_t from $E-V=\frac{V}{R}\times R_t$ when $E=130\cdot4$, $V=97\cdot9$, $R=5$. The letters have the same meanings as in *Example* 10, page 36.

16. Given that $A=\frac{(R-a)(D-d)}{D}$, transpose for R and hence find its value when $A=35$, $D=6\cdot5$, $d=4\cdot7$, $a=\cdot25$.

17. The equation $\frac{1\cdot83\times50\times50}{6}=(18t+2\times6\frac{1}{2}\times\frac{1}{2})\times5\times4\frac{1}{2}$ occurred when finding the thickness of the flange of the section of a girder for an overhead railway. Find the value of t to satisfy this.

18. Transpose for q in the equation $W(h_2-h_1)=w(qL+h-h_2)$. [q is the dryness fraction of a sample of steam.]

19. How many electric cells, each having an internal resistance of 1·8 ohms, and each giving 2 volts, must be connected up in series so that a current of ·086 amperes may be passed through an external resistance of 12·2 ohms?

20. If $D=\frac{SC}{n}+K$, find n when $D=500$, $S=12$, $C=950$, and $K=1\cdot5$.

21. The tractive pull P that a two-cylinder locomotive can exert is given by

$$P=\frac{\cdot8pd^2L}{D}$$

where p = steam pressure in lbs. per sq. in., d = diameter of cylinders in ins., L = stroke in ins., and D = diameter of driving wheels in inches.

Find the diameter of the cylinders of the engine for which the pull is 10,000 lbs., the steam pressure 200 lbs. per sq. in., the stroke 2′-3″, and driving wheels are 4′-6″ in diameter.

22. To determine the diameter of a crank the following rule is used

$$\frac{Pl}{8}=\frac{\pi}{32}fd^3 \qquad \{\pi=3\cdot142\}.$$

Put this equation in a form convenient for the calculation of the value of d.

23. Lloyd's rule for the strength of girders supporting the top of the combustion chamber of a boiler is $P=\frac{ch^2t}{(W-p)DL}$, where P = working pressure in lbs. per sq. in.; t = thickness of girder at the centre; L = width between tube plates; p = pitch of stays; h = depth of girder at the centre; and D = distance from centre to centre of the girders.

Find the value of p when $c=825$, $W=27$, $L=2\frac{1}{2}$, $D=7\frac{1}{2}$, $t=1\frac{1}{2}$, $h=6\frac{5}{8}$, and $P=160$.

24. Find the thickness of metal t (ins.) for Morrison's furnace tube from each of the given formulæ—

(*a*) Board of Trade rule

$$P = \frac{14000t}{D}$$

(*b*) Lloyd's rule

$$P = \frac{1259(16t-2)}{D}$$

where P = pressure in lbs. per sq. in., and D = diameter (ins.) outside corrugations. Given that P = 160 and D = 43″.

25. (*a*) Transpose the given equation for Δ

$$W(h+\Delta) = \frac{Ea\delta\Delta}{2}$$

where δ = proof strain of iron, a = area of section of bar of length l on to which a weight W is dropped from a height h inches; Δ being the extension produced.

(*b*) Find the value of l, which equals $\frac{\Delta}{\delta}$, when $E = 30 \times 10^6$; $a = 1{\cdot}2$, $\delta = {\cdot}001$, $h = 132$, and $W = 40$.

26. If $t = \sqrt{\frac{WL}{A+2L}}$, find the value of L when W = 7000, A = 8000, and $t = 1{\cdot}62$.

27. Find the pitch p of the rivets in a single-riveted lap joint from ${\cdot}7854d^2f_s = (p-d)tf_t$ where $d = t + \frac{7}{16}$, $t = \frac{3}{4}$, $f_s = 23$, and $f_t = 28$.

28. Calculate the value of p to satisfy the equation—

$B = C\sqrt{p}A$ when $C = {\cdot}02$, $A = 200$, $B = 2{\cdot}53$.

29. The stress f in the material of a cylinder for a steam-engine may be found from $t = \frac{pd}{2f} + \frac{1}{2}$ where p = steam pressure = 80 lbs. per sq in., d = diameter = 14″, and t = thickness of metal = $\frac{7}{8}$″. Find f for this case.

30. Determine the value of p to satisfy the equation—

$(p-d)tf_t = 1{\cdot}571d^2f_s$, relating to riveted joints,

when $f_s = 23$, $f_t = 28$, $d = 1\frac{1}{8}$, and $t = \frac{3}{4}$.

31. The diameter of shaft to transmit a torque T when the stress allowable is f is found from $T = \frac{\pi}{16}fd^3$. Find the diameter of shaft to transmit a torque of 22,000 lbs. ft, if the maximum permissible stress in the material is 5000 lbs. per sq in ($\pi = 3{\cdot}142$)

32. The formula $d = \sqrt{\frac{2M}{xbc(1-\frac{1}{3}x)}}$ occurs in reinforced concrete design. Find M (a bending moment) when $b = 9$, $c = 600$, $x = {\cdot}36$, $d = 15{\cdot}3$.

33. $D = d\sqrt{\frac{f+p}{f-p}}$ is Lamé's formula for thick cylinders of outside diameter D and inside diameter d Calculate the value of p when $D = 9{\cdot}5''$, $d = 6''$, and $f = 6$ tons per sq. in.

34. An important formula in structural work is $\frac{M}{I} = \frac{E}{R}$ where M is the bending moment applied to a beam, I is the moment of inertia of the section of the beam, E is Young's modulus for the beam, and

R is the radius of curvature of the bent beam. If M = 5600 lbs. ft., I = ·7854 in.^{4} units, E = 28 × 10^6 lbs. per sq. in.; find the value of R, stating clearly the units in which it is expressed.

35. Compare the deflection d_m of a beam due to bending moment with that d_s due to shear, for the following cases—

(*a*) length = 10 × depth, *i. e.*, $l = 10d$.

(*b*) length = 3 × depth.

You are given that—

$$d_m = \frac{Wl^3}{48EAk^2}, \quad d_s = \frac{1\cdot5Wl}{4AC}, \quad k^2 = \frac{d^2}{12}, \quad \text{and } E = 2\cdot5C.$$

36. If $\dfrac{mc}{l} = \dfrac{k}{1-k}$ and $c = \dfrac{2lr}{k}$, find an expression for r in terms of m and k: hence find its value when $k = \cdot36$, $m = 15$.

37. If $E = 3K\left(1 - \dfrac{2}{m}\right)$ and $E = 2C\left(1 + \dfrac{1}{m}\right)$, find the relation between K, the bulk modulus, and C, the rigidity modulus.

Find also an expression for E, Young's modulus, in terms of K and C only.

38. Find an expression for x from the equation—

$$\frac{16Wx}{\pi d^3} = \frac{16W(r-x)}{\pi d^3} + \frac{2W}{\pi d^2}$$

39. Find the internal pressure p for a thick cylinder from Lamé's formula—

$$\frac{D}{d} = \sqrt{\frac{f+p}{f-p}}$$

where D = 12·74″, d = 9″, f = 2100 lbs./□″. State the units in which p is expressed.

40. Given that $W = \dfrac{n}{n-1}\,pv\,\dfrac{T_1 - T_2}{T_1}$, a formula occurring in Thermodynamics, and also that W = 33000, T_2 is [illegible] T_1, T_1 = 2100, v = 12·4, and p = 2160, find the value of n.

41. A takes 2 hours longer than B to travel 60 miles; but if he trebles his pace he takes 2 hours less than B. Find their rates of walking.

42. If $H = \dfrac{4flv^2}{2gd}$ and $f = 0\cdot01\left(1 + \dfrac{1}{12d}\right)$, find v when H = 22·1, $d = \frac{1}{2}$, l = 380, and g = 32.

(H is the head lost when water flows through a length l of pipe of diameter d, and f is the coefficient of resistance.)

43. If M = moment of a magnet, H = strength of the earth's field, p = time of a complete oscillation of the magnet, and I = moment of inertia of the magnet, then $\dfrac{M}{H} = \dfrac{d^3T}{2}$ (expressing the result of a deflection experiment, d being the distance between the centre of the magnet and that of the needle, and T being a measure of the deflection) and also $MH = \dfrac{4\pi^2 I}{p^2}$ (expressing the result of an oscillation experiment). Find the values of M and H when d = 20, I = 169, p = 13·3, π = 3·142, and T = ·325.

44. In finding the swing radius k (ins.) of a connecting rod, the following measurements were made:—

t = time of a complete oscillation = 2·03 secs.

ρ = distance of centre of gravity from the centre of suspension = 31·43″.

If h = distance of centre of percussion from centre of suspension

$$t = 2\pi\sqrt{\frac{h}{g}}; \quad \text{and also } k^2 = \rho h.$$

Find h in inches $\{\pi = 3{\cdot}142,\ g = 32{\cdot}2 \text{ f.p. sec.}^2\}$.

45. The maximum stress in a connecting rod can be found from the equation $f = 1{\cdot}05\frac{D^2p}{d^2} + {\cdot}00429\frac{v^2l^2}{rd}$

If $f = 4700$, D = diameter of cylinder = 14, d = diameter of rod = 2·5, p = steam pressure at mid stroke = 65, v = velocity of crank, r = crank radius = 8, and l = length of connecting rod = 60; find the value of v.

46. It is required to find the diameter D of one pipe of length L, equivalent to pipes of length l_1 and l_2 and diameters d_1 and d_2 respectively, from—

$$\frac{L}{D^5} = \frac{l_1}{d_1^{\,5}} + \frac{l_2}{d_2^{\,5}}$$

Put this equation in a form suitable for this calculation.

47. If $\frac{r-y}{r} = \left(\frac{\frac{l}{2}-x}{\frac{l}{2}}\right)^2$ find an expression for y.

48. Transpose the equation $\frac{c-y}{h-y} = \frac{3c^2}{2h^2+2ch-c^2}$, occurring in structural design, to give an expression for y.

Simultaneous Equations.—So long as only one of the quantities with which we are dealing is unknown, one equation, or one statement of equality, is sufficient to determine its value.

Cases often present themselves in which two, and in rarer cases three or even more, quantities are unknown; then the equations formed from the conditions are termed *simultaneous equations.* Taking the more common case of two unknowns, one equation would not determine absolutely the value of either, but would simply connect the two, *i. e.*, would give the value of one in terms of the other. For two unknowns we must have two sets of conditions or two equations. This rule holds throughout, that for complete solution **there must be as many equations as there are unknowns.**

The treatment of such equations will be best understood by the aid of worked examples.

Example 17.—What two numbers add up to 5·4 and differ by 2·6?

For shortness, take x and y to represent the numbers, substituting these to form an equation to satisfy the first condition—

$$x+y = 5{\cdot}4 \quad \ldots\ldots\ldots\ldots \quad (1)$$

Here, by taking various values of y we could calculate corresponding values of x, and there would be no limit to the number of "solutions." The first statement in the question is, however, qualified by the second, from which we form equation (2), viz.—

$$x - y = 2{\cdot}6 \quad \ldots \ldots \ldots \ldots \quad (2)$$

If equations (1) and (2) are added—

$$2x = 8{\cdot}0$$
$$\therefore \quad x = 4;$$

or, in other words, y has been eliminated, *i. e.*, the number of unknowns has been reduced by one. Our plan must therefore be to "eliminate," by some means, one unknown at a time until all become "knowns." This method will be followed in all cases.

Reverting to our example, x is found, but y is still unknown.

To find y, substitute the value found for x in either equation (1) or equation (2).

In (1) $\quad 4 + y = 5{\cdot}4$

and $\quad y = 5{\cdot}4 - 4 = 1{\cdot}4$

$$\therefore \quad \left.\begin{aligned} x &= 4{\cdot}0 \\ y &= 1{\cdot}4 \end{aligned}\right\}$$

and we have completely solved our problem.

Example 18.—Determine values of a and b to satisfy the equations

$$4a + 3b = 43 \quad \ldots \ldots \ldots \ldots \quad (1)$$
$$3a - 2b = 11 \quad \ldots \ldots \ldots \ldots \quad (2)$$

If equations (1) and (2), as they stand, were either added or subtracted, both a and b would remain, so that we should be no nearer a solution. To eliminate a, say, we must make the coefficients of a the same in both lines.

E. g., if equation (1) be multiplied by 3

and equation (2) be multiplied by 4, each line would contain $12a$, so that the *subtraction* of the equations would cause a to vanish.

Thus—
$$12a + 9b = 129$$
$$12a - 8b = 44$$

Subtracting—
$$17b = 85$$

whence
$$b = 5.$$

Substituting this value for b in equation (2)—

$$3a - 10 = 11$$
$$3a = 21$$
$$\therefore \quad a = 7$$

Grouping the results—
$$\left.\begin{aligned} a &= 7 \\ b &= 5 \end{aligned}\right\}$$

Note.—If it were desired to eliminate b, equation (1) would have to be multiplied by 2 and equation (2) by 3, and the resulting equations *added*, since there would then be $+6b$ in the top line and $-6b$ below, which on addition would cancel one another.

Example 19.—The effort E, to raise a weight W, by means of a screw jack, is given by the general formula, $E = aW + b$. If $E = 2{\cdot}5$ when $W = 5$; and if $E = 5{\cdot}5$ when $W = 20$, find the values of a and b, and thence the particular equation connecting E and W.

Substituting the numerical values for E and W—

$$2{\cdot}5 = 5a + b \quad \ldots\ldots\ldots\ldots (1)$$
$$5{\cdot}5 = 20a + b \quad \ldots\ldots\ldots\ldots (2)$$

In this case it is easier to subtract straight away; thus eliminating b.

Thus— $$-3 = -15a$$

or $$a = \frac{-3}{-15} = {\cdot}2$$

Substituting in equation (1), $2{\cdot}5 = 1 + b$

$\therefore$ $b = 1{\cdot}5$

so that $\underline{E = {\cdot}2W + 1{\cdot}5.}$

Example 20.—Keeping the length of an electric arc constant and varying the resistance of the circuit, the values of the volts V and amperes A were taken. These are connected by the general equation—

$$V = m + \frac{n}{A}$$

Find the value of m and n for the following case—

$$\left.\begin{array}{l} V = 54{\cdot}5 \text{ when } A = 4 \\ V = 48{\cdot}8 \text{ when } A = 10 \end{array}\right\}$$

Substituting the numerical values, in the general equation—

$$54{\cdot}5 = m + \frac{n}{4}$$

$$48{\cdot}8 = m + \frac{n}{10}$$

Changing the fractions into decimals to simplify the calculation—

$$54{\cdot}5 = m + {\cdot}25n \quad \ldots\ldots\ldots\ldots (1)$$
$$48{\cdot}8 = m + {\cdot}1n \quad \ldots\ldots\ldots\ldots (2)$$

Subtracting— $$5{\cdot}7 = {\cdot}15n$$

$$\therefore \quad n = \frac{5{\cdot}7}{{\cdot}15} = 38$$

Substituting this value in equation (2)—

$$48{\cdot}8 = m + 3{\cdot}8$$

$$\therefore \quad m = 45$$

or $$\underline{V = 45 + \frac{38}{A}}$$

Example 21.—Karmarsch's rule states that the total strength P of a wire in lbs. is given by $P = ad + bd^2$, where d is the diameter in inches. For copper (unannealed)—

$$\left.\begin{array}{ll} P = 421 & \text{when } d = {\cdot}1 \\ P = 55212 & \text{when } d = 1{\cdot}2 \end{array}\right\}$$

Find the actual law connecting P and d.

By substitution of the numerical values

$$55212 = (a \times 1{\cdot}2) + (b \times 1{\cdot}44) \quad \ldots \quad (1)$$
$$421 = (a \times {\cdot}1) + (b \times {\cdot}01) \quad \ldots \quad (2)$$

To eliminate a multiply equation (2) by 12 and subtract.

Thus—
$$55212 = 1{\cdot}2a + 1{\cdot}44b$$
$$5052 = 1{\cdot}2a + {\cdot}12b.$$

Subtracting—
$$50160 = 1{\cdot}32b$$

$$\therefore \quad b = \frac{50160}{1{\cdot}32} = 38000.$$

Substituting in equation (2)—

$$421 = {\cdot}1a + 380$$
$${\cdot}1a = 41$$
$$\therefore \quad a = 410.$$
$$\therefore \quad \underline{P = 410d + 38000d^2}$$

i. e., for a diameter of ·5″, the total strength is

$$(410 \times {\cdot}5) + (38000 \times {\cdot}25) = \underline{9705 \text{ lbs.}}$$

Solution of Equations involving three unknowns. These may also be solved by the process of elimination, the method being similar to that employed when there are two unknowns only. Three equations are necessary and these may be taken together in pairs, the same quantity being eliminated from each pair, whence the question resolves itself into a problem having two equations and two unknowns.

Example 22.—Find the values of a, b and c to satisfy the equations—

$$4a - 5b + 7c = -14 \quad \ldots \quad (1)$$
$$9a + 2b + 3c = 47 \quad \ldots \quad (2)$$
$$a - b - 5c = 11 \quad \ldots \quad (3)$$

The unknowns must be eliminated one at a time. Suppose we decide to commence with the elimination of c. This may be done by taking equation (1) and equation (2) together, multiplying equation (1) by 3 and equation (2) by 7, and then subtracting; an equation containing a and b only being thus obtained. For complete solution one other equation must be found to combine with this; if equation (2) and equation (3) are taken together, equation (2) must be multiplied by 5 and equation (3) by 3 and the resulting equations then added.

Hence, considering equations (1) and (2), and multiplying according to our scheme—

$$12a - 15b + 21c = -42$$
$$63a + 14b + 21c = 329.$$

Subtracting—
$$-51a - 29b = -371 \quad \ldots \quad (4)$$

Combining equations (2) and (3), multiplying equation (2) by 5 and equation (3) by 3—

$$45a+10b+15c = 235$$
$$3a-\ 3b-15c = 33$$

Adding— $$48a+\ 7b \qquad = 268 \quad \ldots\ldots\ldots \quad (5)$$

Equations (4) and (5) may now be combined and either a or b eliminated.

To eliminate a, multiply equation (4) by 16 and equation (5) by 17 and add.

Then— $$-816a-464b = -5936$$
$$816a+119b = 4556$$

Adding— $$-345b = -1380$$
$$\therefore \quad b = 4$$

Substitute this value of b in equation (5) and the value for a is found—

$$i.\ e., \quad 48a+28 = 268$$
$$\text{or} \quad 48a = 240$$
$$\therefore \quad a = 5$$

For a write 5, and for b write 4, in equation (2).

Then— $$45+8+3c = 47$$
$$\text{or} \quad 3c = -6$$
$$\therefore \quad c = -2$$

Collecting the results— $$\left.\begin{aligned} a &= 5 \\ b &= 4 \\ c &= -2 \end{aligned}\right\}$$

Example 23.—A law is required, in the form $E = a+bT+cT^2$, for the calibration of a thermo-electric couple. The corresponding values of E and T are—

T (C.°)	100	600	1000
E (micro-volts)	450	3900	5600

In other words, we wish to find the values of the three unknowns, a, b, and c.

The three equations formed from the given values are—

$$5600 = a+1000b+1000000c \quad \ldots\ldots\ldots \quad (1)$$
$$3900 = a+\ 600b+\ 360000c \quad \ldots\ldots\ldots \quad (2)$$
$$450 = a+\ 100b+\ \ 10000c \quad \ldots\ldots\ldots \quad (3)$$

Grouping equations (1) and (2) and subtracting, a is eliminated; and similarly for equations (2) and (3).

Thus—

$$5600 = a + 1000b + 1000000c$$
$$3900 = a + 600b + 360000c$$
$$\therefore \quad 1700 = 400b + 640000c \quad \ldots\ldots\ldots\ldots \quad (4)$$

Also—

$$3900 = a + 600b + 360000c$$
$$450 = a + 100b + 10000c$$
$$\therefore \quad 3450 = 500b + 350000c \quad \ldots\ldots\ldots\ldots \quad (5)$$

To eliminate b, multiply equation (4) by 5 and equation (5) by 4, and subtract.

Then—

$$8500 = 2000b + 3200000c$$
$$13800 = 2000b + 1400000c$$
$$-5300 = 1800000c$$
$$\therefore \quad c = \frac{-5300}{1800000} = -\cdot 002944$$

Substituting in equation (4)—

$$1700 = 400b - 1884$$
$$\text{or} \quad 400b = 3584$$
$$\therefore \quad b = 8\cdot 96.$$

Substituting for b and c in equation (5)—

$$450 = a + 896 - 29$$
$$\therefore \quad a = -417.$$

Hence the law of calibration is—

$$\underline{\mathrm{E} = -417 + 8\cdot 96\mathrm{T} - \cdot 002944\mathrm{T}^2.}$$

Exercises 6. On Solution of Simultaneous Equations.

Solve the equations in Exercises 1 to 9.

1. $7x + 3y = 10$
$35x - 6y = 1$

2. $2a - 9b = 32$
$3a + 10b = 1$

3. $5m - 6n = -6\cdot 6$
$11n - 25 = 2m$

4. $48x - 27y = 48$
$y - 51x = 51$

5. $y + 1\cdot 37 = 4x$
$9x - 17y = 49\cdot 87$

6. $\frac{1}{2}x - \frac{1}{3}y = \frac{1}{12}x - \frac{1}{10}y$
$19x + 2y = 208$

7. $\dfrac{4x - 3y}{2} = x + \frac{1}{2}$
$19x - \frac{1}{3}y = 31y + 1$

8. $2a + 3b - 5c = 4\cdot 5$
$3c - 7a + 15b = 62\cdot 7$
$9b - 10a = 50\cdot 3$

9. $2p - 5s + 4t = -33$
$4s + 10t - 8p = -84$
$3p - 12t + 2s = 89$

10. If $\mathrm{E} = a + bt + ct^2$, and also

for values we have			
E	4·6	9	4·5
t	2	10	5

find the values of a, b, and c.

11. If $\mathrm{P} = ad + bd^2$ and $\left.\begin{array}{l}\mathrm{P} = 17830 \text{ when } d = \cdot 5 \\ \mathrm{P} = 2992 \text{ when } d = \cdot 2\end{array}\right\}$
find the values of a and b.

(P and d have the same meanings as in Example 21, page 45.)

12. You are given the following corresponding values of the effort E necessary to raise a load W on a machine. Find the connection between E and W in the form $E = aW + b$, given that $E = 7$ when $W = 20$; and $E = 14{\cdot}2$ when $W = 80$.

13. Corresponding values of the volts and amperes (obtained in a test on an electric arc) are—

$$V = 48{\cdot}75 \text{ when } A = 4; \text{ and } V = 75{\cdot}75 \text{ when } A = {\cdot}8.$$

Find the law connecting V and A in the form $V = m + \frac{n}{A}$.

14. The I.H.P. (I) of an engine was found to be $3{\cdot}19$ when the B.H.P. (B) was 2, and $6{\cdot}05$ when the B.H.P. was 5. Find the I.H.P. when the B.H.P. is $3{\cdot}7$. $\{I = aB + b.\}$

15. The law connecting the extension of a specimen with the gauge length may be expressed in the form, $e = a + bL$, where $L =$ length and $e =$ extension on that length.

The extension on 6″ was found to be $2{\cdot}062''$, and that on 8″ was $2{\cdot}444''$. Find the values of the constants a and b.

16. The electrical resistance R_t of a conductor at temperature $t°$ may be found from $R_t = R_0(1 + \alpha t)$ where $R_0 =$ resistance at 0°, and $\alpha =$ temperature coefficient.

If the resistance at 20° is $5{\cdot}38$ ohms and at 90° is $7{\cdot}71$ ohms, find the resistance at 0° and also the temperature coefficient.

17. Find a simple law connecting the latent heat L with the temperature t when you are given that—

L	975	800
t	200	450

Find also the latent heat at 212°.

18. Unwin's law for the connection between the length, the area, and the extension of a specimen is—

$$\text{Percentage elongation } e = \frac{c\sqrt{\text{area}}}{\text{length}} + b.$$

If the area a is $\cdot 75$ and $e = 30{\cdot}11$ when the length $l = 5''$, and if $25{\cdot}6$ when $l = 8''$, find the law for this case (Mild steel specimen).

19. Repeat as for No. 18, when $a = 2{\cdot}12$, and $l = 3''$ when $e = 59{\cdot}2$ and 10″ when $e = 24{\cdot}5$ (Rolled brass specimen).

20. The difference in potential E between the hot and cold junction of a thermal couple for a difference of temperatures T is given by—

$$E = a + bT + cT^2.$$

Find the law connecting E and T for the values—

T	50	100	300
E	202·2	570·1	2058

21. f is the tenacity (in tons per sq. in) of copper at $t°$ F. f and t are connected by an equation of the form $f = a - b(t - 60)^2$. Find this equation, given that $f = 14{\cdot}8$ at 60° F. and $f = 13{\cdot}2$ at 400° F.

22. Repeat as for No. 21, the values of f and t (for cast phosphor-bronze) being—

f	16·06	13·1
t	100	400

23. Given that $W = a + \frac{b}{p+4}$. Find the law connecting W and p if $W = 21 \cdot 11$ when $p = 80$; and also $W = 16 \cdot 56$ when $p = 120$. W is the weight of water used by a steam engine per H.P. hour, and p is the absolute pressure.

24. If w = steam per H.P. hour and I = H.P., then

$$w = a + \frac{b}{I}$$

If 12000 lbs. of steam were used per hour when the H.P. was 1000 and 3554 lbs. when the H.P. was 180, find the law connecting w and I.

25. 500 cu. ins. of cast iron together with 240 cu. ins. of copper weigh 206·8 lbs., whilst 13 cu. ins. of copper weigh as much as 16 cu. ins. of cast iron. Find the number of cubic inches per ton of each of these metals.

26. Measurements to find the constants of a telescope with stadia wires resulted in the following. At 1 chain distance from the instrument the difference between the readings on the staff for the top and bottom wires was ·65 ft.; and at 2 chains the difference was 1·311 ft. Find the constants, C and K from $CS + K = D$ where S = difference of staff readings and D = distance. (1 chain = 22 yds.)

27. Three wires A, B, and C are successively looped together and the resistance of each loop measured. The resistance of A and B is found to be 260 ohms, of A and C is 280 ohms, and of B and C is 400 ohms. Determine the individual resistances of A, B, and C.

28. The following equations occurred when finding the fixing couples of a built-in girder—

$$10m_1 + 10m_2 = 762 \cdot 5$$

$$\frac{200}{3}m_1 + \frac{400}{3}m_2 = 7186.$$

Solve these equations for m_1 and m_2.

29. The "dead weight" tonnage of a ship is 700 tons, whilst the cubic capacity of its hold is 42000 cu. ft. To ensure the most profitable voyage, a mixed cargo of heavy and lighter goods must be carried, and the complete capacity of the hold must be utilised. Prove the truth of the following rule: "To obtain the weight of the lighter cargo, multiply the specific volume (*i. e.*, the number of cu. ft. per ton) of the heavy cargo by the dead weight tonnage. Subtract this result from the total cubic capacity and divide the difference by the difference between the specific volumes of the heavy and light goods."

If, in a certain case, the densities of the heavy and light goods are 35 cu. ft. per ton (saltpetre), and 80 cu. ft. per ton (ginger in bags) respectively, determine the weight of saltpetre carried and also the weight of the ginger.

Methods of Factorisation.—Reference has already been made to the word "factor" as denoting a number or symbol that multiplies or divides some other numbers or symbols in an expression. Thus $3\times5=15$, and 3 and 5 are called factors of 15, *i.e.*, when multiplied together their product is 15.

Again— $$26a^3=2\times13\times a\times a\times a.$$

Here the quantity has been broken up into 5 factors. The process of breaking up a number or expression into the simple quantities, which, when multiplied together, reproduce the original, is known as factorisation. Little is said about this in works on Arithmetic, but the process is used none the less for that.

To illustrate by a numerical example—

Find the L.C.M. of 18, 24, 15, and 28.

These numbers could be factorised and written as follows—

$$2\times3\times3,\quad 2\times2\times2\times3,\quad 3\times5,\quad 2\times2\times7.$$

The L.C.M. must contain each of these; it must, therefore, contain the first, any factor in the second not already included, and so on for the four.

$$i.\,e.,\ \text{L.C.M.}=\underbrace{2\times3\times3}_{\text{1st}}\times\underbrace{2\times2}_{\text{2nd}}\times\underbrace{5}_{\text{3rd}}\times\underbrace{7}_{\text{4th}}=2520.$$

The necessity for the presence of the two 2's in the second group should be realised. There must be as many 2's as factors in the result as there are 2's in the number having the greatest quantity of 2's in its factors: *i. e.*, there must here be three 2's as factors in the result.

It is, however, in Algebra that this process finds its widest application. Rather difficult equations can often be put into simpler forms from which the solution can be readily obtained, and by its use much arithmetical labour can be saved. Generally speaking, the factorised form of an expression demonstrates its nature and properties rather more clearly than does its original form. For practical purposes the following methods of factorisation will be found sufficient.

Rule I.—**Often every term of an expression contains a common factor: this factor can be taken out beforehand and put outside a bracket. The multiplication is then done once instead of many times.**

$$35+60-55 \text{ is, we know } =40$$

But— $$35+60-55=(5\times7)+(5\times12)-(5\times11)$$

and the factor 5 is common to each term. If this factor is taken outside a bracket, the arrangement then becomes $5(7+12-11)$,

or $5 \times 8 = 40$, which agrees with the previous result. The final arrangement is to be preferred, because the numbers with which we have to deal are much simpler. Hence for this numerical case we see that the common factor must be taken outside a bracket, whilst the terms inside are the quotients of this factor derived from the original terms.

Numbers have been taken for clearness of demonstration, but the method holds equally well for symbols of all kinds.

Example 24.—Factorise the expression, $7a^4b^2 - 28a^3bc^2 + 42a^5b^3c^4$.

In this expression, 7 is common to each term, a^3 is the highest power of a common to each term, b the highest power of b, whilst no c occurs in the first term, and c is, therefore, not a factor common to all terms.

Then, the factor to be taken outside a bracket $= 7a^3b$.

Hence the expression $= \underline{7a^3b(ab - 4c^2 + 6a^2b^2c^4)}$, or we have broken it up into two factors.

Example 25.—Find the volume of a hollow cylindrical column, 12 ft. long, 1 ft. external radius, and 9 ins. internal radius, from the formula—

$$\text{Volume of a cylinder} = \pi r^2 l \quad (\pi = 3{\cdot}142)$$

In this case the net volume will be the difference between the volumes of the outside and inside cylinders :

$\therefore \text{V} = (\pi \times 1^2 \times 12) - (\pi \times (\tfrac{3}{4})^2 \times 12)$ . . . . working in feet

$= 12\pi\{1^2 - (\tfrac{3}{4})^2\}$ because 12π is a factor common to both terms

$= 12\pi\{1 - \tfrac{9}{16}\} = 12\pi \times \tfrac{7}{16}$

$= \underline{16{\cdot}48 \text{ cu. ft.}}$

Rule 2.— **The expression may be of a form similar to one whose factors are known, and the factors may be written down from inspection.**

If $(A+B)$ be multiplied by $(A-B)$ the resulting product is A^2-B^2.

Conversely, then, the factors of $A^2 - B^2$ are $(A-B)$ and $(A+B)$,

or $\quad \mathbf{A^2 - B^2 = (A - B)(A + B)}$,

i. e., **to factorise the difference of two squares, multiply the sum of the quantities by their difference.** This rule is of wide application.

Example 26.—Write down the value of $9154^2 - 9151^2$.

Squaring each and subtracting the results is far longer than making use of the rule just given—

Thus— $\quad 9154^2 - 9151^2 = (9154 + 9151)(9154 - 9151)$

$= 18305 \times 3 = \underline{54915}.$

Example 27.—Find the factors of $81a^8 - 16b^4$.

$$\begin{aligned} 81a^8 - 16b^4 &= (9a^4)^2 - (4b^2)^2, \text{ which is the difference of two squares,} \\ \text{and therefore } &= (9a^4 - 4b^2)(9a^4 + 4b^2) \\ &= [(3a^2)^2 - (2b)^2][9a^4 + 4b^2] \\ &= \underline{(3a^2 - 2b)(3a^2 + 2b)(9a^4 + 4b^2).} \end{aligned}$$

In this example the rule is applied twice.

Two other standard forms are here added, although their use is by no means so frequent as that of the above.

$$\begin{aligned} A^3 - B^3 &= (A - B)(A^2 + AB + B^2) \\ A^3 + B^3 &= (A + B)(A^2 - AB + B^2). \end{aligned}$$

Example 28.—Find the factors of $27a^6b^3 + 125a^3c^9$.

Let E denote the expression, then—

$$\begin{aligned} E &= a^3(27a^3b^3 + 125c^9) \quad \text{by } \textit{Rule } 1. \\ &= a^3[(3ab)^3 + (5c^3)^3] \\ &= \underline{a^3(3ab + 5c^3)(9a^2b^2 - 15abc^3 + 25c^6).} \end{aligned}$$

Rule 3.- **In many cases of trinomial, *i. e.*, three-term expressions, the factors must be found by trial, at any rate to a very large extent.**

There are certain rules applying to the signs, which can best be followed by first considering the following products:—

$$\begin{aligned} (x+5)(x+6) &= (x \times x) + (x \times 6) + (5 \times x) + (5 \times 6) \\ &= x^2 + 11x + 30 \quad \ldots\ldots\ldots\ldots \quad (1) \\ (x-5)(x-6) &= x^2 - 11x + 30 \quad \ldots\ldots\ldots\ldots \quad (2) \\ (x+5)(x-6) &= x^2 - x - 30 \quad \ldots\ldots\ldots\ldots \quad (3) \\ (x-5)(x+6) &= x^2 + x - 30 \quad \ldots\ldots\ldots\ldots \quad (4) \end{aligned}$$

In (1) and (2) there are *like signs* in the brackets and *a plus sign precedes the third term* in the expansion, which must be written in the order of ascending or descending powers of x or its equivalent.

In (3) and (4) there are *unlike signs* in the brackets and a *minus sign* comes before the 30. Hence the first rule of signs may be stated :—So arrange the signs that the one before the first term is plus, an adjustment of signs throughout being made if necessary. Look to the sign before the third term of the expression; if this is a *plus* then we conclude that the signs in the brackets will be *like*, and if this sign is a *minus* then the signs in the brackets will be *unlike*. If they are to be like, they must be either both plus or both minus, and the sign before the second term in the given expression indicates which of these is accepted. Thus, a plus sign before the second term indicates that the signs in the brackets are both plus.

If, however, the signs in the brackets are to be unlike, one product must be the greater and the sign before the second term indicates whether it is the product obtained by using the plus or the minus sign.

E.g., in (3) we have -30 as the third term; accordingly the signs in the brackets will be unlike: also the second term is $-x$ so that the minus product is to be the greater; hence the minus sign in the brackets must be before the 6.

The actual numbers in the brackets must be found by trial. They must in each of the four instances multiply together to give 30; also, in (1) and (2) they must add together to give 11, and in (3) and (4) their difference must be 1.

Example 29.—Find the factors of $x^2+17x-110$.

In the given expression the third term is -110, so that there must be unlike signs in the brackets. Also, the $+$ product must be the greater, since $+17x$ is the second term.

Since the signs in the brackets are to be *unlike*, two numbers must be found which when multiplied together give 110, and which *differ* by 17.

These numbers are 5 and 22; and the signs placed before these must be so chosen that $+17x$ results when the brackets are removed. Thus the plus sign must be placed before the 22, and hence

$$\underline{x^2+17x-110=(x+22)(x-5).}$$

Example 30.—Factorise the expression $-2x^2-28x-90$.

Applying *Rule* 1—

$$\text{The expression}=-2(x^2+14x+45).$$

(Note the adjustment of signs, to ensure $+$ before the first term.)

Dealing with the part of the expression in brackets: plus signs throughout denote $+$ in brackets; hence two numbers are required that multiplied give 45, and added give 14; these being 9 and 5.

$$\therefore \quad \underline{\text{The factors}=-2(x+9)(x+5).}$$

Example 31.—Find the factors of $6m^2+11m-35$.

This expression could be reduced to the form of the previous examples by dividing by 6, but the fractions so obtained would render the further working rather involved. It is better, therefore, to proceed as follows:

There will be unlike signs in the brackets, since the sign before the third term is minus, and the factors of 6 have to be combined with those of 35 to give a difference of products of $+11$. The varying of the factors at either end may result in many arrangements being tried

before the correct one is found. After a little practice, however, the student disregards absurd arrangements and so reduces his work.

The correct arrangement in this case is $(3m-5)(2m+7)$.

The first terms when multiplied together give $6m^2$, the last ones give -35, the extreme terms give $+21m$, and the middle terms $-10m$, *i. e.*, the last two combine to give $+11m$.

The arrangement is more clearly shown if written down as—

$$\begin{array}{ccc}(3 & - & 5)\\(2 & + & 7)\end{array}$$

The end terms are easily settled, but for the middle term the multiplication must be performed as indicated by the arrows, and the results must be added or subtracted as the case may demand. When the correct arrangement of the figures has been found, the letters must be inserted. Hence, the expression has for its factors $\underline{(3m-5)(2m+7)}$.

Example 32.—Factorise the expression $72a^2+18ab-77b^2$.

In the first place disregard the letters; dealing only with the numbers.

The factors of 72 are to be combined with those of 77 to give a difference of 18.

72 has many factors, but $77 = 7\times 11$ or 77×1.

The trial arrangements would be of this nature—

$\begin{array}{cc}8 & 11\\9 & 7\end{array}$ For the middle term, the difference $= 43$.

$\begin{array}{cc}8 & 7\\9 & 11\end{array}$ " " " " $= 25$.

$\begin{array}{cc}24 & 11\\3 & 7\end{array}$ " " " " $= 135$.

$\begin{array}{cc}6 & 7\\12 & 11\end{array}$ " " " " $= 18$

The last is the arrangement desired. To allocate the signs:—the net result of the products is to be $+18$: 7×12 gives the greater product, hence the $+$ must be placed before the 7.

$\therefore$ The expression $= \underline{(6a+7b)(12a-11b)}$.

The Remainder and Factor Theorems.—Suppose we have to deal with an expression such as—

$$x^2+bx^3+cx+dx^5.$$

If this expression be divided by $(x-a)$, the remainder will be—

$$a^2+ba^3+ca+da^5,$$

which could have been more simply obtained by substituting a for x in the original expression.

If $(x-a)$ is to be a factor of the original expression then the remainder after division by $(x-a)$ must be zero. Hence we obtain a rule enabling us to find factors of rather complicated expressions:—

Find the value of the main quantity (usually the x) which makes the suggested factor zero; substitute this value in place of the x in the expression, and if the result is zero one factor has been found.

E. g., if it be conjectured that $(x+3)$ is a factor of an expression, its value would be found when x had the value -3.

Example 33. Find the factors of $x^3+x^2-14x-24$.

Let us try if $(x-4)$ is a factor; we will substitute, therefore, $+4$ for x in the expression, which becomes

$$(4)^3+(4)^2-14(4)-24=64+16-56-24=0$$

$\therefore\ (x-4)$ is a factor.

Another likely factor would be $(x+3)$, for 3×4 is part of 24, and there must be a plus sign to combine with the minus in $(x-4)$ to give -24.

Substitute -3 for x, and the expression becomes—

$$(-3)^3+(-3)^2-14(-3)-24=-27+9+42-24=0$$

$\therefore\ (x+3)$ is a factor.

The other factor may be found to be $(x+2)$

$$\therefore\ x^3+x^2-14x-24=\underline{(x+2)(x+3)(x-4)}.$$

Multiplication and Division of Algebraic Fractions.

The simplification of algebraic fractions furnishes useful examples on the application of the rules of indices and of factorisation.

When a number of fractions are to be multiplied together, cancelling can be performed as in the case of arithmetic fractions, always provided that the complete factors are cancelled and not portions thereof.

E. g., $\dfrac{2x+3}{4x+3}$ is in its lowest terms; we cannot cancel $2x$ into $4x$ or strike out the 3's, because $(2x+3)$ must be treated as one quantity, as also must $(4x+3)$.

Example 34.—Simplify $\dfrac{48a^3bc^2}{7a^4b^2c^5}\times\dfrac{35c^8b^{\frac{1}{2}}}{18a^3c^4}\div\dfrac{2a^{\frac{3}{5}}b^{\frac{1}{2}}c^{\frac{4}{5}}}{3a^2c^6}$

$$\text{The fraction}=\frac{48a^3bc^2}{7a^4b^2c^5}\times\frac{35c^8b^{\frac{1}{2}}}{18a^3c^4}\times\frac{3a^2c^6}{2a^{\frac{3}{5}}b^{\frac{1}{2}}c^{\frac{4}{5}}}$$

$$=20a^{3+2-4-3-\frac{3}{5}}\,b^{1+\frac{1}{2}-2-\frac{1}{2}}\,c^{2+8+6-5-4-\frac{4}{5}}$$

$$=20a^{-\frac{13}{5}}b^{-1}c^{\frac{31}{5}}=\underline{\frac{20c^{\frac{31}{5}}}{a^{\frac{13}{5}}b}}.$$

Example 35.—Simplify $\dfrac{x^2+8x+15}{2x^2+3x-35} \times \dfrac{12x^2-147}{20x^2+28x-96}$.

No cancelling must be made until numerators and denominators are expressed in terms of their factors. Thus the fraction—

$$= \frac{\overset{①}{(x+3)}\overset{⑤}{(x+5)}}{\underset{③}{(2x-7)}\underset{⑥}{(x+5)}} \times \frac{3\overset{④}{(2x-7)}(2x+7)}{4\underset{②}{(x+3)}(5x-8)}$$

and in this fraction— ① cancels with ②
③ cancels with ④
⑤ cancels with ⑥

giving the answer $\dfrac{3(2x+7)}{4(5x-8)}$ in which no further cancelling can occur.

Example 36.—Simplify $\dfrac{4x^2+x-14}{6xy-14y} \times \dfrac{4x^2}{x^2-4} \times \dfrac{x-2}{4x-7} \div \dfrac{2x^2+4x}{3x^2-x-14}$

The numerators and denominators are first factorised giving the fraction in the form—

$$\frac{(4x-7)(x+2)}{2y(3x-7)} \times \frac{4x^2}{(x-2)(x+2)} \times \frac{(x-2)}{(4x-7)} \times \frac{(3x-7)(x+2)}{2x(x+2)}$$

which by cancelling reduces to $\dfrac{x}{y}$

Example 37.—Simplify the fraction $\dfrac{2x^3-41x-x^2+70}{3x^2+11x-20}$

The factors for the denominator are the more easily found; they are $(x+5)$ and $(3x-4)$. The first of these is a possible factor of the numerator also; applying the remainder theorem, the value of the numerator when $x=-5$ is $2(-125)-41(-5)-(-5)^2+70$, *i.e.*, 0, hence $(x+5)$ is a factor. In like manner it would be found that $(x-2)$ was also a factor; and by division of the numerator by the product of these, viz by $x^2+3x-10$, the remaining factor is found to be $(2x-7)$.

Hence the fraction $= \dfrac{(x-2)(x+5)(2x-7)}{(x+5)(3x-4)} = \dfrac{(x-2)(2x-7)}{(3x-4)}$

Addition and Subtraction of Algebraic Fractions.—The same rules are adopted as for arithmetical fractions.

The L.C.M. of the denominators (L.C.D.) must first be found by factorising the separate denominators according to the plan detailed on page 51.

Example 38.—Simplify the fraction—

$$\frac{5a}{4a-7}+\frac{10a}{12a-21}-\frac{51}{20a-35}$$

This becomes (after factorisation of the denominators) —

$$\frac{5a}{(4a-7)}+\frac{10a}{3(4a-7)}-\frac{51}{5(4a-7)}$$

and the L.C.D. $= (4a-7)\times 3\times 5 = 15(4a-7)$

whence the expression $= \dfrac{75a+50a-153}{15(4a-7)} = \underline{\dfrac{125a-153}{15(4a-7)}}$

Example 39.—Simplify $\dfrac{x}{x^2+5x+6}+\dfrac{15}{x^2+9x+14}-\dfrac{12}{x^2+10x+21}$

This becomes (after factorisation of the denominators)

$$\frac{x}{(x+3)(x+2)}+\frac{15}{(x+7)(x+2)}-\frac{12}{(x+7)(x+3)}$$

and the L.C.D. is $(x+3)(x+2)(x+7)$.

Dealing with the first term only and multiplying both numerator and denominator by this L.C.D.—

$$\frac{x}{(x+3)(x+2)}=\frac{x}{(x+3)(x+2)}\times\frac{(x+3)(x+2)(x+7)}{(x+3)(x+2)(x+7)}$$

which after cancelling reduces to $\dfrac{x(x+7)}{(x+3)(x+2)(x+7)}$

In like manner the second and third terms reduce to—

$$\frac{15(x+3)}{(x+3)(x+2)(x+7)} \text{ and } \frac{12(x+2)}{(x+3)(x+2)(x+7)} \text{ respectively.}$$

Hence the fraction $= \dfrac{x(x+7)+15(x+3)-12(x+2)}{(x+3)(x+2)(x+7)}$

$$=\frac{x^2+7x+15x+45-12x-24}{(x+3)(x+2)(x+7)}$$

$$=\frac{x^2+10x+21}{(x+3)(x+2)(x+7)}$$

$$=\frac{(x+3)(x+7)}{(x+3)(x+2)(x+7)}=\underline{\frac{1}{x+2}}$$

Example 40.—Show that if $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{a+b}{b}=\dfrac{c+d}{d}$ and $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

From $\dfrac{a}{b}=\dfrac{c}{d}$, by adding 1 to each side—

$$\frac{a}{b}+1=\frac{c}{d}+1$$

Taking the L.C.D. of each side—

$$\frac{a+b}{b}=\frac{c+d}{d} \quad \ldots\ldots\ldots\ldots \quad (1)$$

In like manner by subtracting 1 from each side of the original equation—

$$\frac{a-b}{b} = \frac{c-d}{d} \quad \ldots\ldots\ldots\ldots \quad (2)$$

Hence, dividing (1) by (2)—

$$\frac{a+b}{a-b} = \frac{c+d}{c-d} \quad \ldots\ldots\ldots\ldots \quad (3)$$

These results are of importance.

Exercises 7.—On Factors, and on Multiplication and Addition of Algebraic Fractions.

Factorise the expressions in Examples 1 to 20.

1. $x^2+18x-88$ **2.** $x^2-19x+88$

3. $x^2-26x+105$ **4.** $8a^3-125b^6$ **5.** $24x^2-x-44$

6. $(2a+b)^2-(3a-4b)^2$ **7.** $a^2+4ab-45b^2$

8. $12x^2-73xy+105y^2$ **9.** $88-3x^2-13x$

10. $20m^2n+20n^3-58mn^2$ **11.** $\frac{wx^2l^2}{4}-\frac{wxl^3}{24}+\frac{5wx^3l}{384}-\frac{wx^4}{16}$

12. $\frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3$, giving the volume of a hollow sphere of outside radius R, and internal radius r.

13. $94x^2+39x-963$.

14. $\frac{wlx^3}{12EI}-\frac{wx^4}{24EI}-\frac{wl^3x}{24EI}$, an expression occurring in connection with the deflection of beams.

15. $54a^4b-300a^2bc^2-42a^3bc$ **16.** $4a^2-16c^2-12ab+9b^2$

17. $64c^6+\frac{8}{27}b^3a^9$

18. $V-v$ (giving the volume of the frustum of a cone; R and r being the radii of the ends of the frustum and h its thickness) where $V=\frac{\pi R^2}{3}(h+k)$, $v=\frac{\pi r^2k}{3}$ and $k=\frac{rh}{R-r}$

19. $2x^3+7x^2-44x+35$. [*Hint*.—Try $(x+7)$ as a factor.]

20. $6p^3+23p^2+6p-35$. [$(p-1)$ is one factor.]

21. Find, by the methods of this chapter, the value of $(199\times 46)+(398\times 69)-(199\times 92)$.

22. Find the value of $\pi R^2l-\pi r^2l$, which gives the volume of a hollow cylinder, when $\pi=3{\cdot}142$, $R=12{\cdot}72$, $r=9{\cdot}58$, $l=64{\cdot}3$.

23. Find the L.C.M. of x^2-x-6, $3x^2-21x+36$, and $4x^2-8x-32$.

24. Simplify $\frac{10a^{\frac{1}{2}}b^2c^{\frac{1}{4}}}{17a^3b^3c^4}\times\frac{51a^{\frac{1}{4}}b^0c^{\frac{1}{2}}}{4c^{\frac{1}{4}}}\div\frac{3a^{-7}c^3b^{\frac{1}{3}}}{15^{-1}}$

25. Simplify $\frac{8x^2-24x-80}{20x^2+15x-110}\times\frac{8x^2+58x+99}{4x^2-2x-90}$

26. Simplify $\frac{2x}{x-4}+\frac{5x}{2x+4}-\frac{9}{x^2-2x-8}$

27. Simplify $\frac{3a^3+3b^3}{4a^4+4b^4+4a^2b^2}\div\frac{9a^2-5ab-14b^2}{7a^3-7b^3}$

28. Simplify

$$\frac{5x}{18x^2 - 100 + 30x} - \frac{8x^2 + 7x}{24x^2 - 4x - 280} + \frac{14 - 8x}{30x^2 + 175 - 155x}$$

29. Solve the equation $\frac{4 + 2x}{5x - 3} + \frac{4x}{10x} = \frac{7}{3}$

30. Solve the equation $\frac{3}{x + 3} - \frac{5}{2x - 7} = \frac{9}{10x^2 - 105 - 5x}$

[*Hint.* Multiply through by the L.C.D.]

31. A unit pole is attracted by a magnetic pole of strength m with a force $\frac{m}{(d - l)^2}$ and repelled by a force of $\frac{m}{(d + l)^2}$. What is the resultant attractive force? Find the value of this force if l is very small compared with d.

32. Find the factors of (*a*) $3x^3 + 6x^2 - 180x$; (*b*) $24 + 37x - 72x^2$; (*c*) $(3x + 7y)^2 - (2x - 3y)^2$.

33. Find the factors of $(x^2 + 7x + 6)(x^2 + 7x + 12) - 280$.

34. M, a bending moment, is given by

$$M = \frac{Pa(2s + 3)}{8(s + 2)} - \frac{Pa(18s^2 + 35s + 9)}{24(6s^2 + 2 + 13s)}$$

Find a more simple expression for M.

35. The expression $p_1v_1 - \frac{1}{n - 1}(p_2v_2 - p_1v_1) - p_2v_2$ relates to the work done in the expansion of a gas. State this in a more simple form.

36. The depth of the centre of pressure of a rectangular plate, of width h, immersed vertically in a liquid, the top being a and the bottom b units below the level of the surface of the liquid, is given by $\dfrac{\frac{\rho h}{3}(b^3 - a^3)}{\frac{\rho h}{2}(b^2 - a^2)}$

Express this in a simpler form.

Quadratic Equations. Any equation in which the square, but no higher power, of the unknown, occurs, is termed a *quadratic* equation. The simplest type, or *pure* quadratic, is $d^2 = 25$; to solve which, take the square root of both sides. Then d = either $+5$ or -5, because $(+5)^2 = 25$ and also $(-5)^2 = 25$. This result would be written in the shorter form $d = \pm 5$.

It is essential that the two solutions should be stated, although in most practical cases the nature of the problem shows that the positive solution is the one required.

The solution of the pure quadratic is elementary; but in the case of an equation of the type $x^2 + 7x + 12 = 0$ (spoken of as an *adfected* quadratic, *i. e.*, one in which both the first and the second power of the unknown occur) new rules must be developed or stated. Three rules or methods of procedure are suggested for the solution of adfected quadratics, viz.—

Method 1.—Solution of a Quadratic by Factorisation.

Group all the terms to the left-hand side and factorise the expression so obtained. Next, let each of these factors in turn = 0: thus two solutions are determined.

For all quadratics there must be two solutions or "roots"; in some cases they may be equal, and in rare cases "imaginary."

Applying this method to the example under notice:—

Example 41.—Solve the equation $x^2 + 7x + 12 = 0$.

By factorisation of the left-hand side

$$(x+3)(x+4) = 0.$$

Then either $x + 3 = 0$, in which case $x = -3$,

or $x + 4 = 0$, in which case $x = -4$,

because, if one factor is zero, the product of the two factors must also be zero; *e.g.*, if $x = -3$, $(x+3)(x+4) = 0 \times 1 = 0$.

Hence $\underline{x = -3 \text{ or } -4.}$

Example 42.—Solve the equation $24a^2 + 17a = 20$.

Collecting terms, $24a^2 + 17a - 20 = 0$.

Factorising, $(8a - 5)(3a + 4) = 0$.

$\therefore$ either $8a - 5 = 0$, *i.e.*, $a = \frac{5}{8}$

or $3a + 4 = 0$, *i.e.*, $a = -\frac{4}{3}$

If no factors can be readily seen we may proceed to—

Method 2.—Solution of a Quadratic by completion of the Square.

All the terms containing the unknown must be grouped to one side of the equation and the knowns or constants to the other side.

The left-hand side, viz. that on which the unknown is placed, is next made into a perfect square by a suitable addition, the same amount being added also to the right-hand side, and then the square root of both sides is taken. The solution of the two simple equations thus obtained gives the "roots" of the original equation.

Before proceeding further with this method a little preliminary work is necessary, the principle of which must be grasped if the reason of the method of solution is to be understood.

$$(a+24)^2 = a^2+48a+576.$$

Suppose that the first two terms of the right-hand side are

given, and it is desired to add the necessary quantity to make it into a complete square.

$a^2+48a+576$ might be written as $(a)^2+[2\times(24)\times(a)]+\left(\frac{48}{2}\right)^2$ so that if a^2+48a is given, the term to be added is $\left(\frac{48}{2}\right)^2$, *i. e.*, is the square of half the coefficient of a.

Similarly, x^2+7x could be expressed as a perfect square if $\left(\frac{7}{2}\right)^2$ were added : it is then the square of $\left(x+\frac{7}{2}\right)$.

Returning to the method; a numerical example will best illustrate the processes.

Example 43.—Solve the equation $x^2+15x+9=0$.

Grouping terms $\qquad x^2+15x=-9.$

Adding the square of half the coefficient of x, viz. $\left(\frac{15}{2}\right)^2$, to each side,

$$x^2+15x+\left(\frac{15}{2}\right)^2=-9+\left(\frac{15}{2}\right)^2=-9+56\tfrac{1}{4}$$

$$\text{or}\qquad\left(x+\frac{15}{2}\right)^2=47\cdot25.$$

Extracting the square root of both sides—

$$x+\frac{15}{2}=\pm6\cdot88$$

$$\begin{aligned}\therefore\quad x&=-7\cdot5\pm6\cdot88\\&=-7\cdot5+6\cdot88\quad\text{or}-7\cdot5-6\cdot88\\&=\underline{-\cdot62\quad\text{or}\quad-14\cdot38.}\end{aligned}$$

The change from $x^2+15x+\left(\frac{15}{2}\right)^2$ to $\left(x+\frac{15}{2}\right)^2$ often presents difficulty: the reason for the omission of the $15x$ does not seem clear. It must be remembered that it is represented in the second form, for

$$\left(x+\frac{15}{2}\right)^2=(1^{st})^2+(2^{nd})^2+2\text{ (product)}\quad x^2+\left(\frac{15}{2}\right)^2+\left[2\times x\times\left(\frac{15}{2}\right)\right]$$
$$=x^2+15x+\left(\frac{15}{2}\right)^2$$

If the coefficient of x^2 is not unity it must be made so by division throughout by its coefficient.

Example 44.—Find a value of B (the breadth of a flange) to satisfy the equation $3\cdot64B^4-51\cdot8B^2-900=0$.

This equation, though not a quadratic, may be treated as a quadratic and solved first for B^2; *i. e.*, if for B^2 we write A the equation becomes

$$3\cdot64A^2-51\cdot8A-900=0.$$

Dividing through by 3·64 (the coefficient of A^2) and transferring the constant term to the right-hand side—

$$A^2 - 14{\cdot}24A = 247.$$

The coefficient of A is 14·24; half of this is 7·12, hence add $7{\cdot}12^2$, *i. e.*, 50·8 to each side—

$$\textit{i. e.,}\quad A^2 - 14{\cdot}24A + (7{\cdot}12)^2 = 247 + 50{\cdot}8 = 297{\cdot}8$$
$$\text{or}\quad (A - 7{\cdot}12)^2 = 297{\cdot}8.$$

Extracting the square root throughout—

$$A - 7{\cdot}12 = \pm 17{\cdot}26$$
$$\textit{i.e.,}\quad A = 7{\cdot}12 \pm 17{\cdot}26$$
$$\text{whence}\quad A = 24{\cdot}38 \text{ or } -10{\cdot}14.$$

Now $A = B^2$, so that $B^2 = 24{\cdot}38$ or $-10{\cdot}14$. Of these values the former only is taken, since we cannot extract the square root of a negative quantity.

Thus $B^2 = 24{\cdot}38$ or $B = \pm 4{\cdot}94$, but evidently the negative solution has no meaning in this case.

$$\therefore\quad \underline{B = 4{\cdot}94.}$$

Example 45.—If $4a^2 - 15ab + 2b^2 = 0$, find the values of a to satisfy this equation.

Dividing through by 4 and transferring the constant term to the right-hand side, $\quad a^2 - \frac{15}{4}ab = -\frac{b^2}{2}$

The coefficient of a is $\frac{15}{4}b$; half of this is $\frac{15}{8}b$: hence we must add $\left(\frac{15b}{8}\right)^2$ to each side.

Thus $\quad a^2 - \frac{15}{4}ab + \left(\frac{15b}{8}\right)^2 = -\frac{b^2}{2} + \frac{225b^2}{64} = \frac{193b^2}{64}$

Extracting the square root—

$$\left(a - \frac{15b}{8}\right) = \pm \frac{\sqrt{193}}{8}\,b$$
$$\therefore\quad a = \frac{15 \pm \sqrt{193}}{8}\,b$$
$$= \underline{3{\cdot}61b \quad \text{or } {\cdot}14b.}$$

Method 3.—Solution of a Quadratic by the use of a Formula.

It will be evident from the foregoing examples that all quadratics reduce to the general form

$$\mathbf{A}x^2 + \mathbf{B}x + \mathbf{C} = 0.$$

If *Method* 2 is applied to the solution of this, the result is a

formula giving the roots of any quadratic, provided that the particular values of A, B and C are substituted in it. Thus—

$$Ax^2 + Bx + C = 0$$

Dividing through by A and transposing the constant term—

$$x^2 + \frac{B}{A}x = -\frac{C}{A}$$

To each side add the square of half the coefficient of x, viz. $\left(\frac{B}{2A}\right)^2$

$$x^2 + \frac{B}{A}x + \left(\frac{B}{2A}\right)^2 = -\frac{C}{A} + \left(\frac{B}{2A}\right)^2$$

$$= -\frac{C}{A} + \frac{B^2}{4A^2} = \frac{B^2 - 4AC}{4A^2}$$

or $$\left(x + \frac{B}{2A}\right)^2 = \frac{B^2 - 4AC}{4A^2}$$

Extracting the square root of both sides—

$$x + \frac{B}{2A} = \pm\frac{\sqrt{B^2 - 4AC}}{2A}$$

whence $$\boldsymbol{x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}}$$

Thus the "roots" of the general quadratic $Ax^2 + Bx + C = 0$ are $\frac{-B + \sqrt{B^2 - 4AC}}{2A}$ and $\frac{-B - \sqrt{B^2 - 4AC}}{2A}$

Example 46. Solve the equation $5x^2 = 8x + 12$.

Collecting all the terms to one side, $5x^2 - 8x - 12 = 0$.
Then for this to be identical with the standard form—

$$A = 5, \quad B = -8, \quad C = -12$$

$$\therefore \quad x = \frac{+8 \pm \sqrt{64 + 240}}{10}$$

$$= \frac{8 \pm \sqrt{304}}{10} = \frac{8 \pm 17{\cdot}4}{10}$$

$$= \underline{2{\cdot}54 \text{ or } -{\cdot}94}.$$

Great care must be exercised to avoid errors of sign. To obtain the value of $-4AC$, first find $AC = 5 \times (-12)$ or -60;

$$4AC \text{ then} = -240$$

and $$-4AC = +240.$$

Example 47.—Solve for y, in $-\cdot 4y^2 - 1\cdot 5y - \cdot 32 = 0$.

It is always advisable to have the first term positive, so change all the signs before applying the formula.

Then— $$\cdot 4y^2 + 1\cdot 5y + \cdot 32 = 0.$$

Here— $$A = \cdot 4, \quad B = 1\cdot 5, \quad C = \cdot 32.$$

$$\therefore \quad x = \frac{-1\cdot 5 \pm \sqrt{2\cdot 25 - \cdot 512}}{\cdot 8}$$

$$= \frac{-1\cdot 5 \pm \sqrt{1\cdot 738}}{\cdot 8}$$

$$= \frac{-1\cdot 5 + 1\cdot 32}{\cdot 8} \text{ or } \frac{-1\cdot 5 - 1\cdot 32}{\cdot 8}$$

$$= \underline{-\cdot 23 \text{ or } -3\cdot 52.}$$

Example 48.—The stresses on the section of a beam due to the loading are a normal stress f_n and a shearing stress q. These produce an entirely normal stress f on a plane known as the plane of principal stress. Find an expression for f from the equation $f(f - f_n) = q^2$.

Removing the bracket and grouping the terms to one side—

$$f^2 - ff_n - q^2 = 0.$$

Here— $$A = 1, \quad B = -f_n, \quad C = -q^2$$

$$\therefore \quad f = \frac{+f_n \pm \sqrt{f_n^2 + 4q^2}}{2}$$

$$= \underline{\frac{f_n}{2} \pm \sqrt{\frac{f_n^2}{4} + q^2} \text{ or } \tfrac{1}{2}\{f_n \pm \sqrt{f_n^2 + 4q^2}\}}$$

The next example is instructive as showing the advantage of resolving large or small numbers into integers multiplied by powers of ten.

Example 49.—Solve the equation $Lx^2 + Rx + \frac{1}{K} = 0$ (an equation occurring in electrical work) when $L = \cdot 0015$, $R = 400$, $K = \cdot 45 \times 10^{-6}$.

Substituting the numerical values for L, R, and K—

$$\cdot 0015x^2 + 400x + \frac{1}{\cdot 45 \times 10^{-6}} = 0.$$

The last term may be written in the more convenient form $2\cdot 22 \times 10^6$

since $$\frac{1}{\cdot 45} = 2\cdot 22 \quad \text{and} \quad \frac{1}{10^{-6}} = 10^6.$$

Thus— $$\cdot 0015x^2 + 400x + (2\cdot 22 \times 10^6) = 0$$

Comparing with the standard form—

$$A = (1\cdot 5 \times 10^{-3}), \quad B = (4 \times 10^2), \quad C = (2\cdot 22 \times 10^6)$$

$$\text{Hence}\quad x = \frac{-(4\times 10^2) \pm \sqrt{(16\times 10^4) - (6\times 10^{-3}\times 2\cdot 22\times 10^6)}}{3\times 10^{-3}}$$

$$= \frac{-(4\times 10^2) \pm \sqrt{(16\times 10^4) - (1\cdot 33\times 10^4)}}{3\times 10^{-3}},$$

the second term under the radical sign being written in this form so that 10^4 is a factor common to both terms; and the square root of 10^4 is readily found.

$$[6\times 10^{-3}\times 2\cdot 22\times 10^6 = 13\cdot 32\times 10^3 = 1\cdot 332\times 10^4.]$$

The square root of 10^4 is 10^2: this may be placed outside the radical sign, and then—

$$x = \frac{-(4\times 10^2) \pm 10^2\sqrt{16 - 1\cdot 33}}{3\times 10^{-3}}$$

$$= \frac{10^2(-4 \pm \sqrt{14\cdot 67})}{3\times 10^{-3}}$$

$$= \frac{10^5}{3}(-4 \pm 3\cdot 83)$$

$$= (10^5\times -2\cdot 61) \quad\text{or}\quad 10^5\times -\cdot 053$$

$$= \underline{-261000 \quad\text{or}\quad -5600.}$$

Example 50.—A formula given by Prony (in connection with the flow of water through channels) connecting the hydraulic gradient i with the velocity v and the hydraulic mean depth m was of the form $mi = av + bv^2$. Under certain conditions $a = \cdot 000044$, $b = \cdot 000094$.

Show that this is in close agreement with the formula given by Chezy, viz. $v = 103\sqrt{mi}$.

$$mi = av + bv^2$$

$$\text{or}\qquad bv^2 + av - mi = 0$$

$$\therefore\qquad v = \frac{-a \pm \sqrt{a^2 + 4mib}}{2b}$$

Inserting the numerical values for a and b—

$$4\,mib = 4\times \cdot 000094mi$$
$$= \cdot 000376mi.$$

Also, a^2 is very small, even in comparison with $\cdot 000376$, and can therefore be neglected.

$$\text{Hence}\qquad v = -\frac{\cdot 000044}{2\times \cdot 000094} \pm \frac{\sqrt{\cdot 000376mi}}{2\times \cdot 000094}$$

$$= -\cdot 234 \pm 104\cdot 3\sqrt{mi}.$$

Taking the $+$ sign and neglecting the first term, $v = 104\cdot 3\sqrt{mi}$, which agrees well with the $v = 103\sqrt{mi}$ given by Chezy.

Quadratics with "imaginary" Roots.—The question may have presented itself: What is done when (B^2-4AC) in the formula for the solution of the quadratic becomes negative? How can the square root of a negative quantity be extracted?

The square root of a negative quantity is known as an *imaginary* quantity, and all imaginaries are reduced to terms of the square root of -1, which is denoted by j. At present no meaning can be stated for this, but it is referred to again in a later chapter.

Thus— $j=\sqrt{-1}$, $j^2=-1$, $j^3=-\sqrt{-1}$, etc.

$$E.g., \sqrt{-30}=\sqrt{30\times -1}=\sqrt{30}\times\sqrt{-1}=\pm 5\cdot 47j.$$

Example 51.—Solve the equation $2x^2-3x+15=0$, employing *Method* 3.

$$x=\frac{+3\pm\sqrt{9-120}}{4}$$
$$=\frac{+3\pm\sqrt{111}\times\sqrt{-1}}{4}$$
$$=\frac{3\pm 10\cdot 55j}{4}=\cdot 75\pm 2\cdot 64j.$$

Expressions of the type $a\pm bj$, where a and b may have any values, occur in Electrical theory and in the theory of Vibrations, such being referred to in Chapter VI. they are also of importance when the stability of aeroplanes is considered.

Cubic Equations.—Cubic Equations, *i.e.*, equations containing the cube of the unknown as its highest power, may be solved graphically, in a manner to be demonstrated in a later chapter, or use may be made of what is known as *Cardan's solution.*

The three roots of a cubic equation may be either, *one real and two imaginary*, or, *three real.* Cardan's solution applies only to the former of these cases and gives the *real* root only.

If $x^3+ax+b=0$ be taken as the standard type of cubic equation, then the real solution is given by Cardan as—

$$x=\left\{-\frac{b}{2}+\sqrt{\frac{a^3}{27}+\frac{b^2}{4}}\right\}^{\frac{1}{3}}+\left\{-\frac{b}{2}-\sqrt{\frac{a^3}{27}+\frac{b^2}{4}}\right\}^{\frac{1}{3}}$$

The proof of this result is too difficult to be inserted here, but it is outlined in *A Treatise on Algebra*, by C. Smith (Macmillan and Co., Ltd., 7*s*. 6*d*.).

If $\frac{a^3}{27}+\frac{b^2}{4}$ be negative, the three roots are all real, but Cardan's solution cannot be applied.

Example 52. —Solve the equation $x^3 - 12x + 65 = 0$. (Imaginary roots are not required.)

Here $a = -12$, $b = 65$, in comparison with the standard form.

$$\therefore \quad x = \left\{-\frac{65}{2} + \sqrt{-\frac{1728}{27} + \frac{4225}{4}}\right\}^{\frac{1}{3}} + \left\{-\frac{65}{2} - \sqrt{-\frac{1728}{27} + \frac{4225}{4}}\right\}^{\frac{1}{3}}$$

$$= \{-32\cdot5 + 31\cdot5\}^{\frac{1}{3}} + \{-32\cdot5 - 31\cdot5\}^{\frac{1}{3}}$$

$$= (-1) + (-4) = \underline{-5.}$$

If the equation is not of the form, $x^3 + ax + b = 0$, it can be made so in the following manner.

Example 53.—Find a solution of the equation—

$$v^3 + 24v^2 + 144v - 1944 = 0, \ v \text{ being a velocity.}$$

For this to be reduced to the standard form, the term containing v^2 must be eliminated.

By writing $(V + a)$ for v and suitably choosing a, this can be done, for—

$$(V + a)^3 + 24(V + a)^2 + 144(V + a) - 1944 = 0$$

i.e., $$V^3 + 3V^2a + 3a^2V + a^3 + 24V^2 + 24a^2 + 48aV + 144V + 144a - 1944 = 0 \quad . \quad . \quad (1)$$

Equating the coefficients of V^2 to zero (since the term containing V^2 is to be made to vanish), $3a + 24 = 0$, *i.e.*, $a = -8$;

so that $$v = V - 8.$$

Equation (1) can now be written (-8 being substituted for a)

$$V^3 - 24V^2 + 192V - 512 + 24V^2 + 1536 - 384V + 144V - 1152 - 1944 = 0$$

or $$V^3 - 48V - 2072 = 0$$

Comparing with our standard formula—

$$a = -48, \ b = -2072.$$

Therefore, by Cardan—

$$V = \left\{+\frac{2072}{2} + \sqrt{-\frac{48^3}{27} + \frac{2072^2}{4}}\right\}^{\frac{1}{3}} + \left\{\frac{2072}{2} - \sqrt{-\frac{48^3}{27} + \frac{2072^2}{4}}\right\}^{\frac{1}{3}}$$

$$= (2070)^{\frac{1}{3}} + (2)^{\frac{1}{3}} = 12\cdot75 + 1\cdot26 = 14.$$

Hence— $$v = V - 8 = \underline{6.}$$

Equations of degree higher than the third (if not reducible to any of the forms already given) are best solved graphically. (Compare with Chapter IX.)

Exercises 8.—On Quadratic and Cubic Equations.

Solve the equations in Exercises 1 to 10.

1. $x^2 + 5x + 4 = 0$ **2.** $2x^2 - 7x + 15 = 4x$

3. $-3x^2 + 9x + 14 = 0$ **4.** $8p^2 - 7p = 4p^2 + 5p + 16$

5. $\cdot 001a^2 - \cdot 234a - \cdot 764 = \cdot 417a - \cdot 325a^2$ **6.** $9x^2 + 5x + 2 = 0$

7. $\dfrac{x+4}{2x-5} = \dfrac{29-5x}{3x-7}$ **8.** $\dfrac{2x}{x^2+5x+6} + \dfrac{17}{x+2} = \dfrac{5x}{x+3}$

9. $-5x^2 - 2x - 12 = 3x^2$ **10.** $1700 + \cdot 0126F = \cdot 000003F^2$

11. If $\dfrac{V^2}{2g} = \dfrac{2 \times 230 \times l - l^2}{2 \times 36}$, find the values of l when $\left.\begin{array}{l} V = 33\cdot 5 \\ g = 32\cdot 2 \end{array}\right\}$

12. If $r = \dfrac{a^2 + h^2}{2h}$, find h when $r = 15$, $a = 5\cdot 5$. This equation gives the radius of a circle when the height of arc h and length of chord $2a$ are known.

13. Solve for F the equation—

$$\frac{3F^2}{30 \times 10^6} - \frac{3F}{7200} = 100.$$

14. We are told that $W(h + \triangle) = \frac{1}{2}F\triangle$ (a formula relating to the strength of bodies under impact), and also $\triangle = \dfrac{F}{1500}$, $W = \cdot 45$, and $h = 2\cdot 4$. Find values of F to satisfy these conditions.

15. The equation $b^2 - ab - a^2 = \dfrac{\rho h^2}{w}$ relates to masonry dams, where b = width in feet of base of a dam a feet wide at the top, and h feet deep, w being the weight of 1 cu. ft. of masonry, and ρ being the weight of 1 cu. ft. of water. Find b for the case when $a = 5$, $h = 30$, $w = 144$, and $\rho = 62\cdot 4$.

16. Find expressions for f_s from—

$$\left(f_1 - \frac{r}{2}\right)^2 = f_s^2 - nrf_s.$$

17. If $t = \dfrac{abu + agv}{2u^2 + 3v^2}$ solve (a) for u and (b) for v.

18. To find n (the depth from the compression edge to the neutral axis of a reinforced concrete beam of breadth b) it was necessary to solve the equation $bn^2 + 2A_Tmn - 2mA_Td = 0$ Determine the value of n to satisfy the conditions when $m = 15$, $A_T = 1\cdot 56$, $b = 5$, and $d = 10$.

19. Solve for C the equation $75 \times 10^6C^2 - 10^{10}C + 12 \times 10^{10} = 0$.

20. Find the values of t when $L = \dfrac{(P-t)(T-t)}{P-T} \times \cdot 009C$, and $P = 120$, $C = 3375$, $T = 67$, $L = 1765$.

21. Find the ratio, $\dfrac{\text{length of arc of approach } a}{\text{pitch } p}$ (of teeth of involute wheels) from $\dfrac{\cdot 3p}{a} = \dfrac{3a}{np} + \dfrac{1}{4}$, where n = number of teeth in the follower wheel = 24.

22. The equation $mi = av + bv^2$ relates to the flow of water in channels. If $a = \cdot 000024$ and $b = \cdot 000014$, put this in the form $v = c\sqrt{mi}$, making any justifiable approximation. (Compare *Example* 50, p. 66.)

23. Solve the equation $2x^2 - 53 + 5x = \dfrac{42 \times 81}{154}$

24. Find values of l to satisfy the equation $W = \frac{3}{4}al\left(1 + \dfrac{l}{10}\right)$ (Merriman's formula for the weight of roof principals), given that $W = 5400$ and $a = 10$.

25. x is the distance of the point of contraflexure of a fixed beam of length l from one end. If x and l are connected by the equation $1 - \dfrac{6x}{l} + \dfrac{6x^2}{l^2} = 0$, find the positions of the points of contraflexure.

26. To find the position of a mechanism so that the angular velocities of two links should be the same it was necessary to solve the equation $f^3 - 19\cdot 5f^2 + 42\cdot 5f + 546 = 0$. Find values of f to satisfy this.

27. To find d, the depth of flow through a channel under certain conditions of slope, etc., it was necessary to solve the equation $d^3 - 1\cdot 305d - 1\cdot 305 = 0$. Find the value of d to satisfy this.

28. The values of the maximum and minimum stresses in the metal of a rivet due to a shearing stress q and a tensile stress f_n due to contraction in cooling are given by the roots of the equation—

$$f(f - f_n) = q^2.$$

If $q = 4\frac{1}{2}$ tons per sq. in. and $f_n = 3\frac{1}{2}$ tons per sq. in., find the two values of f.

29. The length L of a wire or cable hanging in the form of a parabola is given by—

$$L = S + \frac{8D^2}{3S}$$

where S = span and D = droop or sag.

Find the span if the sag is 3′-9″ and the length of cable is 100·4023 ft.

Simultaneous Quadratics.—Consider the two equations

$$\frac{2x}{y} = 1\cdot 8 \quad \ldots\ldots\ldots \quad (1)$$

and

$$5\cdot 6(5\cdot 6 - y) = x^2 \quad \ldots\ldots\ldots \quad (2)$$

Values of x and y are to be found to satisfy both equations at the same time (hence the term "simultaneous"): also the second equation is of the second degree as regards x, and is therefore a quadratic.

In most practical examples (the above being part of the investigation dealing with compound stresses) one equation is somewhat more complicated than the other, and therefore, for purposes of elimination, we substitute from the simpler form into the more difficult.

In this example equation (1) is the simpler, and from it, by transposition—

$$x = \frac{1\cdot8y}{2} = \cdot9y.$$

Substitute for x, wherever it occurs in equation (2), its value in terms of y. Then—

$$5\cdot6(5\cdot6 - y) = (\cdot9y)^2 = \cdot81y^2$$
$$31\cdot36 - 5\cdot6y = \cdot81y^2$$
or $$\cdot81y^2 + 5\cdot6y - 31\cdot36 = 0.$$

Hence $$y = \frac{-5\cdot6 \pm \sqrt{31\cdot36 + 4 \times \cdot81 \times 31\cdot36}}{1\cdot62}$$
$$= \frac{-5\cdot6 \pm 11\cdot53}{1\cdot62}$$
$$= \frac{-17\cdot13}{1\cdot62} \text{ or } \frac{5\cdot93}{1\cdot62} = -10\cdot56 \text{ or } 3\cdot67$$

To find x—

$x = \cdot9y$ and, substituting in turn the two values found for y,

$$x = \cdot9 \times -10\cdot56 \quad \text{or } \cdot9 \times 3\cdot67$$
$$= -9\cdot51 \text{ or } 3\cdot30$$

$$\left.\begin{array}{ll} \therefore & x = -9\cdot51 \text{ or } 3\cdot30 \\ \text{and} & \underline{y = -10\cdot56 \text{ or } 3\cdot67} \end{array}\right\}$$

Example 54.—In a workshop calculation for the thickness x of a packing strip or distance piece in a lathe the following equations occurred—

$$(9\cdot9)^2 = (1\cdot5 + y)^2 + (\cdot8 + x)^2 \quad \dots \dots \quad (1)$$
$$(9\cdot9)^2 = (1\cdot75 + y)^2 + x^2 \quad \dots \dots \quad (2)$$

Solve these equations for x and y. The packing strip was required for a check for a gauge, and great accuracy was necessary in the calculation.

By removal of the brackets the equations become—

$$98\cdot01 = 2\cdot25 + y^2 + 3y + \cdot64 + x^2 + 1\cdot6x \quad \dots \quad (3)$$
$$98\cdot01 = 3\cdot0625 + y^2 + 3\cdot5y + x^2 \quad \dots \dots \quad (4)$$

By subtracting equation (4) from equation (3) an equation is obtained giving y in terms of x, thus—

$$0 = -\cdot8125 - \cdot5y + \cdot64 + 1\cdot6x$$
or $$\cdot5y = 1\cdot6x - \cdot1725$$
whence $$y = 3\cdot2x - \cdot345\cdot$$

Substituting for y in equation (4)—

$$98\cdot01 = 3\cdot0625 + (3\cdot2x - \cdot345)^2 + 3\cdot5(3\cdot2x - \cdot345) + x^2$$
$$= 3\cdot0625 + 10\cdot24x^2 + \cdot1200 - 2\cdot208x + 11\cdot2x - 1\cdot2075 + x^2.$$

Collecting terms, $11 \cdot 24x^2 + 8 \cdot 992x - 96 \cdot 035 = 0$

whence $$x = \frac{-8 \cdot 992 \pm \sqrt{(8 \cdot 992)^2 + (4 \times 96 \cdot 035 \times 11 \cdot 24)}}{22 \cdot 48}$$

$$= \frac{-8 \cdot 992 \pm 66 \cdot 3219}{22 \cdot 48}$$

$$= \frac{57 \cdot 3299}{22 \cdot 48} \text{ or } \frac{-75 \cdot 3139}{22 \cdot 48}$$

$$= 2 \cdot 5503 \text{ or } -3 \cdot 3503.$$

i.e., the thickness of the strip was $2 \cdot 5503$ (inches); the negative solution being disregarded.

The two values of y would be obtained by substituting the two values found for x in the equation $y = 3 \cdot 2x - \cdot 345$.

Thus— $y = (3 \cdot 2 \times 2 \cdot 5503) - \cdot 345$ or $y = (3 \cdot 2 \times - 3 \cdot 3503) - \cdot 345$

$\therefore$ $y = 7 \cdot 816$ or $- 11 \cdot 066$.

The positive solutions alone have meaning in this example, so that

$$\underline{\underline{x = 2 \cdot 5503 \text{ and } y = 7 \cdot 816.}}$$

Example 55.—Solve the equations—

$$5x^2 + y^2 + 2x - 7y - 4 = 91 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$7x + 3y = 9 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From equation (2)— $3y = 9 - 7x$

or $$y = \frac{9 - 7x}{3}$$

Substituting in equation (1)—

$$5x^2 + \left(\frac{9 - 7x}{3}\right)^2 + 2x - 7\left(\frac{9 - 7x}{3}\right) = 95.$$

$$\therefore \quad 5x^2 + \frac{81 + 49x^2 - 126x}{9} + 2x - 7\left(\frac{9 - 7x}{3}\right) = 95.$$

Multiplying through by 9—

$$45x^2 + 81 + 49x^2 - 126x + 18x - 189 + 147x = 855.$$

Collecting terms—

$$94x^2 + 39x - 963 = 0$$

Factorising—

$$(94x + 321)(x - 3) = 0$$

$$i.e., \; x = -\frac{321}{94} \text{ or } 3.$$

Now, $$y = \frac{9 - 7x}{3}$$

Substituting the two values for x—

$$y = \frac{9 + \frac{2247}{64}}{3} \quad \text{or } y = \frac{9 - 21}{3}$$

$$= \frac{1031}{94} \qquad \text{or} \quad = -4$$

$$\therefore \left.\begin{aligned} x &= 3 \quad \text{or } -\frac{321}{94} \\ y &= -4 \quad \text{or } \frac{1031}{94} \end{aligned}\right\}$$

The method of substitution indicated in the previous three examples is to be recommended in preference to the "symmetrical" methods usually given, but which only apply to special cases.

Occasionally one meets with an equation or pair of equations to which this method is not applicable. Thus if the equations are *homogeneous* and of the second degree, *i. e.*, all the terms containing the unknowns are of the second degree in those unknowns, proceed as in the following example; the method being in reality an extension of the preceding.

Example 56.—Solve the equations—

$$x^2 + 3xy = 54 \quad \dots\dots\dots\dots \quad (1)$$
$$xy + 4y^2 = 115 \quad \dots\dots\dots\dots \quad (2)$$

Divide equation (1) by equation (2).

Then—
$$\frac{x^2 + 3xy}{xy + 4y^2} = \frac{54}{115}$$

Multiplying across—

$$115x^2 + 345xy = 54xy + 216y^2.$$

Collecting terms—

$$115x^2 + 291xy - 216y^2 = 0.$$

Factorising—

$$(23x + 72y)(5x - 3y) = 0$$

whence
$$x = -\frac{72}{23}y \quad \text{or } x = \frac{3}{5}y.$$

Substitute each of these values for x in turn in equation (2).

Thus, taking $x = -\frac{72}{23}y$

$$-\frac{72}{23}y^2 + 4y^2 = 115$$

$$\therefore \quad \frac{20y^2}{23} = 115$$

$$\therefore \qquad y^2 = \frac{115 \times 23}{20}$$

$$y = \pm \frac{23}{2}$$

and $$x = -\frac{72}{23}y = -\frac{72}{23} \times \left(\pm \frac{23}{2}\right) = \mp 36.$$

i.e., $$y = -36 \quad \text{when } x = +\frac{23}{2}$$

and $$y = +36 \quad \text{when } x = -\frac{23}{2}$$

Taking $x = \frac{3}{5}y$ $$\frac{3}{5}y^2 + 4y^2 = 115$$

$$\frac{23}{5}y^2 = 115$$

$$y^2 = \frac{115 \times 5}{23}$$

and $$y = \pm 5$$

and $$x = \pm 3$$

Grouping results— $$\therefore \quad \left.\begin{aligned} x &= \pm 3 \quad \text{or } \mp 36 \\ y &= \pm 5 \quad \text{or } \pm \frac{23}{2} \end{aligned}\right\}$$

Surds and Surd Equations.—One often meets with such quantities as $\sqrt{3}$, $\sqrt[3]{7}$ or $\sqrt[4]{a}$: such are known as *surds* or irrational quantities, since their exact values cannot be found.

The value of $\sqrt{3}$ can be found to as many places of decimals as one pleases, but for ordinary calculations two, or at the most three, figures after the decimal point are quite sufficient: thus $\sqrt{3} = 1{\cdot}73$ approximately, or $1{\cdot}732$ more nearly.

It is both easier and more accurate to multiply by a surd than to divide by it, and therefore, if at all possible, one must rid the denominator of the surds by suitable multiplication.

The process is known as *rationalising* the denominator.

Example 57.—Rationalise the denominator of $\frac{5}{\sqrt{3}}$

To do this, multiply both numerator and denominator by $\sqrt{3}$, since $\sqrt{3} \times \sqrt{3} = 3$.

Then— $\frac{5}{\sqrt{3}} = \frac{5 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{5\sqrt{3}}{3}$, or if the result is required in decimals it is $\underline{2{\cdot}89}$.

Example 58.—Find the value of $\frac{7}{4 - \sqrt{5}}$

Multiply numerator and denominator by $4 + \sqrt{5}$.

$$\text{Then the fraction} = \frac{7(4+\sqrt{5})}{(4-\sqrt{5})(4+\sqrt{5})} = \frac{7(4+\sqrt{5})}{4^2-(\sqrt{5})^2}$$
$$= \frac{7(4+2\cdot236)}{16-5} = \frac{7\times 6\cdot236}{11}$$
$$= \underline{3\cdot968}.$$

Surds occurring in equations must be eliminated as early as possible by squaring or cubing as the case may demand.

Example 59.—Solve the equation $\sqrt[3]{p-2} = 7$.

Cubing both sides—
$$p-2 = 343$$
$$\therefore \quad p = \underline{345}.$$

Example 60.—Find x from the equation—
$$4-\sqrt{2x+3}=5.$$

Transposing so that the surd is on one side by itself—
$$-\sqrt{2x+3}=1.$$
Squaring both sides—
$$2x+3=1$$
$$\text{whence} \quad x = \underline{-1}.$$

Note.—The solutions of all surd equations should be tested.

Reverting to the original equation and substituting -1 for x—
$$4-\sqrt{2x+3} = 4-\sqrt{-2+3} = 4-1 = 3$$
and not 5, so that -1 is not a solution of this equation, but it would be of the somewhat similar equation—
$$4+\sqrt{2x+3} = 5.$$

When squaring, either $(-\sqrt{2x+3})^2$ or $(+\sqrt{2x+3})^2 = 2x+3$, so that the solution obtained may be that of either the one equation or the other.

In this case, then, there is no solution to the equation as given.

Example 61.—Wöhler's law for repeated stresses can be expressed by—
$$f_2 = \frac{S}{2}+\sqrt{f_1^2-xSf_1}$$
where f_1 = original breaking stress, S = stress variation in terms of f_2, and f_2 = new breaking stress.

Find f_2 when $S = \cdot537f_2$, $f_1 = 52$, $x = 2$.

Substituting the numerical values—
$$f_2 = \frac{\cdot537f_2}{2}+\sqrt{52^2-(2\times\cdot537f_2\times 52)}$$
$$i.\,e., \quad f_2 = \cdot2685f_2+\sqrt{2704-55\cdot8f_2}$$

Arranging so that the surd is isolated—

$$f_2 - \cdot 2685 f_2 = \sqrt{2704 - 55 \cdot 8 f_2}$$

or

$$\cdot 7315 f_2 = \sqrt{2704 - 55 \cdot 8 f_2}$$

Squaring—

$$\cdot 535 f_2^2 = 2704 - 55 \cdot 8 f_2$$

or

$$\cdot 535 f_2^2 + 55 \cdot 8 f_2 - 2704 = 0.$$

Solving for f_2 by means of the formula—

$$f_2 = \frac{-55 \cdot 8 \pm \sqrt{3110 + 5790}}{1 \cdot 07}$$

$$= \frac{-55 \cdot 8 \pm 94 \cdot 5}{1 \cdot 07} = \frac{38 \cdot 7}{1 \cdot 07} \text{ or } -\frac{150 \cdot 3}{1 \cdot 07}$$

$$= 36 \cdot 2 \text{ or } -140 \cdot 5.$$

The negative solution has no meaning in the cases to which this formula is applied, hence the positive solution alone is taken.

$$\therefore \qquad \underline{f_2 = 36 \cdot 2.}$$

Example 62.—Find the value or values of x to satisfy the equation

$$\sqrt{4x - 7} + 3\sqrt{2x + 17} = 18.$$

It is best to separate the surds thus—

$$3\sqrt{2x + 17} = 18 - \sqrt{4x - 7}.$$

Square both sides and then—

$$9(2x + 17) = (18)^2 + (\sqrt{4x - 7})^2 - 2 \times 18 \times \sqrt{4x - 7}$$

i. e., $\quad 18x + 153 = 324 + 4x - 7 - 36\sqrt{4x - 7}$

or, isolating the surd—

$$36\sqrt{4x - 7} = 164 - 14x$$

or

$$18\sqrt{4x - 7} = 82 - 7x.$$

Squaring again—

$$324(4x - 7) = 6724 + 49x^2 - 1148x$$

whence—

$$49x^2 - 2444x + 8992 = 0.$$

Factorising—

$$(49x - 2248)(x - 4) = 0.$$

$$\therefore \qquad x = \frac{2248}{49} \text{ or } 4.$$

To test whether these values satisfy the original equation—

When $x = 4$, left-hand side $= \sqrt{16 - 7} + 3\sqrt{8 + 17}$

$= 3 + 15 = 18$, which is the value of the right-hand side— $\underline{x = 4}$ satisfies.

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* Always remember that when squaring a "two-term" expression three terms result, of the character: (1st squared) + (2nd squared) ± (twice product of 1st and 2nd).

i. e., $\quad (A + B)^2 = A^2 + B^2 + 2AB$

$\quad (A - B)^2 = A^2 + B^2 - 2AB.$

When $x = \frac{2248}{49}$, left-hand side $= \sqrt{\frac{8992}{49} - 7} + 3\sqrt{\frac{4496}{49} + 17}$

$$= \frac{93}{7} + \frac{219}{7}$$

$$= \frac{312}{7} = 44\tfrac{4}{7} \text{ which does not } = 18.$$

Hence $x = \frac{2248}{49}$ is not a solution of the given equation; it however satisfies the equation $3\sqrt{2x+17} - \sqrt{4x-7} = 18$.

Exercises 9.—On the Solution of Simultaneous Quadratic Equations and Surd Equations.

1. Find values of x and y to satisfy the equations—

$$x - 3y = 16$$
$$x^2 + 3y^2 - 2x + 4y = 50$$

2. Solve the equations— $a^2 = 8 + 4y^2$

$$2a + 2y = 7$$

3. Solve for p and q the equations—

$$3p^2 - pq - 7q^2 = 5$$
$$p^2 + 5pq + 5q^2 = -1$$

4. Solve the equations $5x^2 - 9x + 12xy - 2y^2 = 132$

$$7x + 8y = 54$$

5. Determine the values of a and b to satisfy the equations—

$$a^2 - 2ab + 3 = 0$$
$$2a + b = 4$$

6. Solve for m the equation $\sqrt{3m^2 - 4m} = 5$.

7. The following formula is used to calculate the length of hob required to cut a worm wheel, for throat radius r and depth of tooth d

$$f = 2\sqrt{d(2r - d)}$$

Find the depth of tooth when the hob is 3″ long and the throat radius is 2″.

8. Find a from the equation $\sqrt{8a+9} - 3 = 4$.

9. Solve for H the equation $25{\cdot}6\text{H} - 346\sqrt{\text{H}} = 10000$.

10. The formula $f_2 = \frac{\text{S}}{2} + \sqrt{f_1{}^2 - x\text{S}f_1}$ is that given by Unwin, and refers to variation of stress. Bauschinger's experiments in a certain case gave $\text{S} = {\cdot}41f_2$, $f_1 = 22{\cdot}8$, and $x = 1{\cdot}5$. Find the value of f_2.

11. Using the same formula as in the previous example (No. 10) find f_2 when $\text{S} = \frac{1}{5}f_2$, $f_1 = 30$, and $x = 2$.

12. Solve for a the equation $\sqrt{a+2} + \sqrt{a} = \frac{4}{\sqrt{a+2}}$

13. Find a value or values of x to satisfy the equation—

$$\sqrt{1 + 9x} = \sqrt{x+1} - \sqrt{4x+1}$$

14. The length x of a strut in a roof truss was required from the equation $\sqrt{x^2 - 36} + \sqrt{x^2 - 4} = 16$. Find this length.

15. Find the value of k to satisfy the equation referring to the discharge of water from a tank—

$$t_1 - t_2 = \frac{2A}{a}\sqrt{\frac{1+k}{2g}}(\sqrt{h_1} - \sqrt{h_2})$$

Given that $t_2 = \cdot 45 \times 60$, $t_1 = 2\cdot 6 \times 60$, $A = 15\cdot 6$, $a = \frac{12\cdot 57}{144}$, $g = 32\cdot 2$, $h_1 = 36$, and $h_2 = 25$.

16. The following equations occurred when calculating a slope of a form gauge—

$$m^2 = (\cdot 06 + n)n$$
$$\frac{m}{\cdot 03} = \frac{\cdot 0175 - n}{\cdot 035}$$

Find the values of m and n to satisfy these conditions.

17. If l = span of an arch $\qquad d$ = rise
h = height of roadway above the lowest point of the arch
c = " " " " " highest " " " "

then $c + d = h$ and also $c = \frac{1}{2}d + \frac{l^2}{8d}$

Find the values of d when $h = 23$ ft., and $l = 24$ ft.

18. The dimensions for cast iron pipes for waterworks are related by the equation

$$t = \cdot 000054\text{H}d + \cdot 15\sqrt{d}$$

where H = head of water in feet
t = thickness of metal in inches
d = internal dia. of pipe in inches

If H = 300 and $t = \cdot 5$ find d.

CHAPTER III

MENSURATION

Introduction.—Mensuration is that part of practical mathematics which deals with the measurement of lengths, areas, and volumes. A sound knowledge of it is necessary in all branches of practical work, for the draughtsman in his design, the works' manager in his preparation of estimates, and the surveyor in his plans, all make use of its rules.

Our first ideas of mensuration, apart from the tables of weights and measures, are usually connected with the areas of rectangles. How much floor space will be required for a planer 4 ft. wide and 12 ft. long? Here we have the simplest of the rules of mensuration, viz. the multiplication together of the two dimensions. Thus, in this case, the actual space covered is $4 \times 12 = 48$ sq. ft.

Area of Rectangle and Triangle.—If the rectangle is bisected diagonally, two equal triangles result, the area of each being one-half that of the original rectangle, or we might state it, $\frac{1}{2}$ (length $\times$ breadth), or as it is more generally expressed, $\frac{1}{2}$ base $\times$ height or $\frac{1}{2}$ height $\times$ base. (Note that the $\frac{1}{2}$ is used but once; thus we do not multiply $\frac{1}{2}$ base by $\frac{1}{2}$ height.) This rule for the area of the triangle will always hold, viz. that **the area of the triangle is one-half that of the corresponding rectangle**, *i. e.*, the rectangle on the same base and of the same height. Thus in Fig. 7, the triangles ABC, AB′C, and AB″C are all equal in area, this area being one-half of the rectangle ACB′D, *i. e.*, $\frac{1}{2}bh$. It is the most widely used of the rules for the area of the triangle, because if sufficient data are supplied to enable one to construct the triangle, one side can be considered as the base, and the height (*i. e.*, the perpendicular from the opposite angular point on to this side or this side produced) can be readily measured, whence one-half the product of these two is obtained.

A special case occurs when one of the angles is a right angle; then the rule for the area becomes: Area (to be denoted by Δ) equals one-half the product of the sides including the right angle.

One further point in connection with the right-angled triangle must be noted, viz. the relation between the sides.

The square on the hypotenuse (the side opposite the right angle, *i. e.*, the longest side) **is equal to the sum of the squares on the other sides.** (*Euclid*, I. 47.)

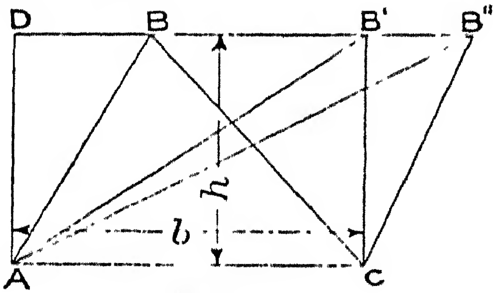


Fig. 7.

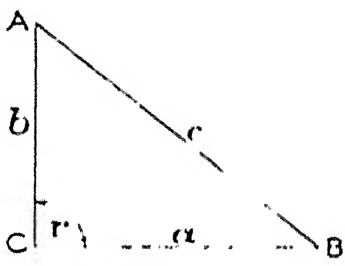


Fig. 8.

In Fig. 8, AB is the hypotenuse, because the right angle is at C, and—

$$(AB)^2 = (AC)^2 + (BC)^2$$

or

$$c^2 = b^2 + a^2.$$

A word about the lettering of triangles will not be out of place here. It is the convention to place the large letters A, B and C at the angular points of the triangle, to keep these letters to represent the angles, *e. g.*, the angle ABC is denoted by B, and to letter the sides opposite to the angles by the corresponding small letters. Thus the side BC is denoted by *a*, because it is the side opposite the angle A.

Rule for Area of Triangle when the three sides are given.—As previously indicated the rule $\frac{1}{2}bh$ can here be applied if the triangle is drawn to scale and a height measured. (The triangle can be constructed so long as any two sides are together greater than the third.) If, however, instruments are not handy proceed along the following lines:—

Add together the three sides a, b, and c, and call half their sum s;

$$i.e., \ s = \frac{a+b+c}{2}$$

Then the area is given by—

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

This rule will be referred to as the "s" *rule*, and the proof of it will be found in Chapter VI.

Logarithms or the slide rule can be employed directly when using this formula, since products and a root alone are concerned.

Example 1.—One end of a lock gate, 7 ft. broad, is 2 ft. further along the stream than the other when the gates are shut: find the width of the stream.

Let $2x$ = width of stream. (See Fig. 9.)

Then— $7^2 = x^2 + 2^2$

$x^2 = 7^2 - 2^2 = (7-2)(7+2) = 45$

or $x = 6{\cdot}71$

so that the width of the stream $= 2x =$ <u>13·42 ft.</u>

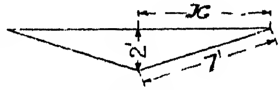


Fig. 9.

Example 2.—Find the area of the triangle ABC, Fig. 10.

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}\ \text{base} \times \text{height}\\ &= \tfrac{1}{2} \times 14{\cdot}6 \times 11{\cdot}4\\ &= \underline{83{\cdot}2\ \text{sq. ins.}}\end{aligned}$$

(Note that 11·4 is the perpendicular on to AB produced.)

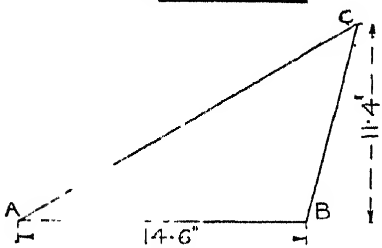


Fig. 10.

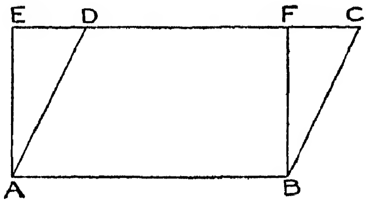


Fig. 13.

Example 3.—The pressure on a triangular plate immersed in a liquid is 4·5 lbs. per sq. ft. The sides of the plate measure 18·1″, 25·3″, and 17·4″ respectively: find the total pressure on the plate.

Let $a = 18{\cdot}1$, $b = 25{\cdot}3$, $c = 17{\cdot}4$.

Using these figures, the area will be in sq. ins.—

$$s = \frac{18{\cdot}1 + 25{\cdot}3 + 17{\cdot}4}{2} = \frac{60{\cdot}8}{2} = 30{\cdot}4$$

Then $\Delta = \sqrt{30{\cdot}4(30{\cdot}4 - 18{\cdot}1)(30{\cdot}4 - 25{\cdot}3)(30{\cdot}4 - 17{\cdot}4)}$

$= \sqrt{30{\cdot}4 \times 12{\cdot}3 \times 5{\cdot}1 \times 13}$

Taking logs throughout—

$$\log \Delta = \tfrac{1}{2}\{\log 30{\cdot}4 + \log 12{\cdot}3 + \log 5{\cdot}1 + \log 13\}$$

$$= \tfrac{1}{2}\left\{\begin{array}{r} 1{\cdot}4829\\ 1{\cdot}0899\\ {\cdot}7076\\ 1{\cdot}1139\\ \hline 4{\cdot}3943 \end{array}\right\} = 2{\cdot}1972$$

$= \log 157{\cdot}5$

$\therefore\ \Delta = 157{\cdot}5$ sq. ins.

Then total pressure $= \dfrac{157{\cdot}5}{144} \times 4{\cdot}5$ lbs. $\left[\text{feet}^2 \times \dfrac{\text{lbs.}}{\text{feet}^2} = \text{lbs.}\right]$

$=$ <u>4·92 lbs.</u>

A further rule for the area of a triangle will be found in Chapter VI.

A rule likely to prove of service is—

Area of an equilateral triangle = ·433 × (side)².

Thus if the sides of a triangle are each 8 units long its area is $\cdot 433 \times 8^2$, *i.e.*, 27·7 sq. units.

Exercises 10. On Triangles and Rectangles.

1. A boat sails due E. for 4 hours at 13·7 knots and then due N. for 7 hours at 10·4 knots. How far is it at the end of the 11 hours from its starting-point?

2. Find the diagonal pitch of 4 boiler stays placed at the corners of a square, the horizontal and vertical pitch being 16".

3. If a right-angled triangle be drawn with sides about the right angle to represent the electrical resistance (R), and reactance ($2\pi f L$), respectively, then the hypotenuse represents the impedance. Find the impedance when $f = 50$, $L = \cdot 159$, $R = 50$, and $\pi = 3\cdot 142$.

4. It is required to set out a right angle on the field, a chain or tape measure only being available. Indicate how this might be done, giving figures to illustrate your answer.

5. A floor is 29'-5" long and 11'-10" broad. What is the distance from one corner to that opposite?

6. At a certain point on a mountain railway track the level is 215 ft.; 500 yds. further along the track the level is 227 ft. Express the gradient as—

(*a*) 1 in x (x being measured along the track).
(*b*) 1 in x (x being measured along the horizontal).

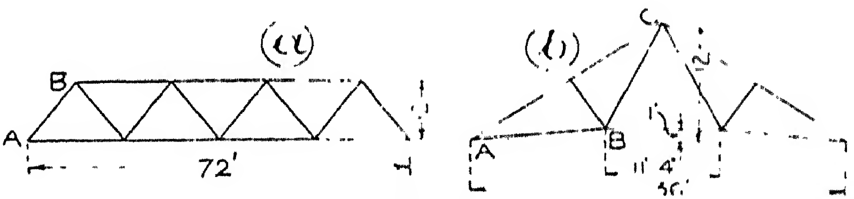


Fig. 11.

7. For the Warren Girder shown at (*a*), Fig. 11, find the length of the member AB.

8. A roof truss is shown at (*b*), Fig. 11. Find the lengths of the members AB, BC and AC.

9. A field is $24\frac{1}{2}$ chains long and 650 yds. wide. What is its area in acres? (Surveyors' Measure is given on p. 87.)

10. Find how many "pieces" of paper are required for the walls of a room 15 ft. long, 12'-6" wide and 8 ft. high, allowing 8 % of the space for window and fireplace (a "piece" of paper being 21" wide and 9 ft. long).

11. A courtyard 15 yds. by 12 yds. is to be paved with grey stones measuring 2 ft. × 2 ft. each, and a border is to be formed, 2 ft. wide, of red stones measuring 1 ft. × 1 ft. How many stones of each kind are required?

12. A room 15 ft. by 12 ft. is to be floored with boards $4\frac{1}{2}$" wide. How many foot run will be required?

13. Before fracture the width of a mild steel specimen was 2·014" and its thickness ·387". At fracture the corresponding dimensions were 1·524" and ·250". Find the percentage reduction in area.

14. A rectangular plot of land $\frac{1}{2}$ mile long and 400 ft. wide is to be cut up into building plots each having 40 ft. frontage and 200 ft. depth. How many such plots can be obtained?

15. The top of a tallboy is in the form of a cone; the diameter of the base is 4", and the vertical height is $1\frac{1}{4}$". Find the slant height.

16. A bar of iron is at the same time subjected to a direct pull of 5000 lbs., and a pull of 3500 lbs. at right angles to the first. Find the resultant force due to these.

17. At a certain speed the balls of a governor are 5" distant from the governor shaft; the length of the arms is 10". Find the "height" of the governor h and hence the number of revs. per sec. n from $h = \dfrac{\cdot 816}{n^2}$.

18. A load on a bearing causes a stress of 520 lbs./□". If the stress is reckoned on the "projected area" of the bearing, the diameter of which is 4" and the length $5\frac{1}{2}$", find the load applied.

19. The sides of a triangle are 17·4", 8·4" and 15·7" respectively. Find its area, by—

(*a*) Drawing to scale and use of $\frac{1}{2}$ base × height rule.
(*b*) Use of "*s*" rule. (See p. 80.)

20. Find the rent of a field in the form of a triangle having sides 720, 484 and 654 links respectively, at the rate of £2 10*s*. per acre. (See note to Ex. 9.)

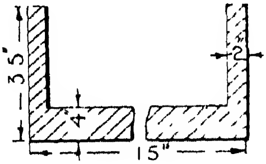


Fig. 11*a*.

21. Find the area of the joist section shown in Fig. 11*a*. (Thickness of flange is 0·2".)

22. Neglecting the radii at the corners, calculate the areas of the

sections in Fig. 12: viz. (*b*) channel section, (*c*) unequal angle, and (*d*) tee section.

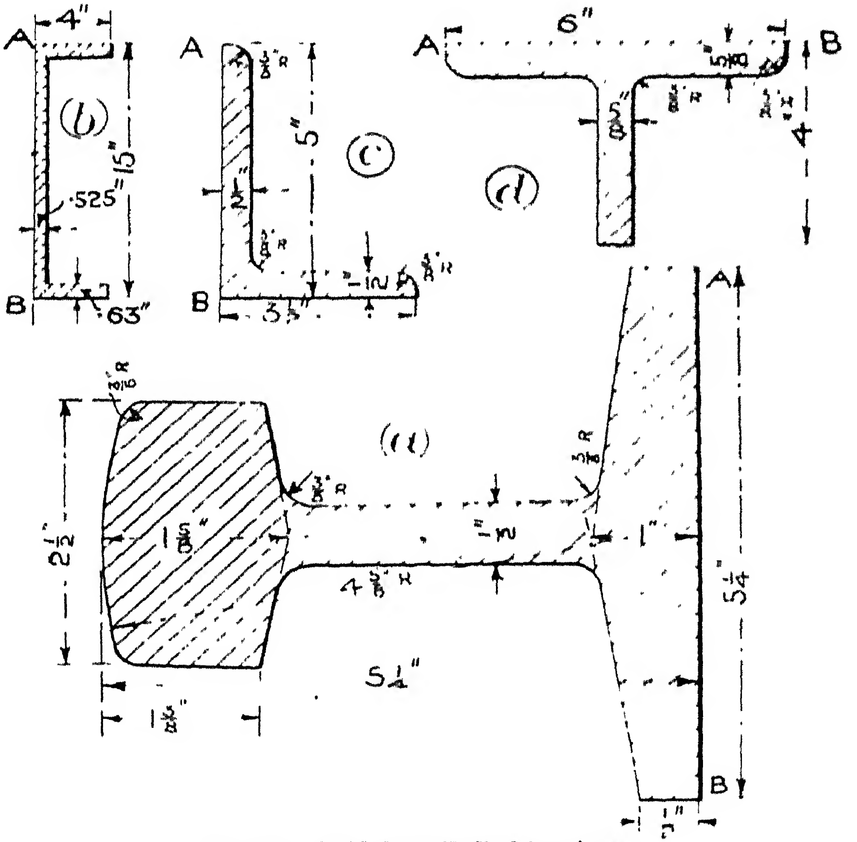


Fig. 12.—Mild Steel Rolled Sections.

Area of Parallelogram and Rhombus. From the three sided figure one progresses to that having four sides, such being spoken of generally as a *quadrilateral*.

Of the regular quadrilaterals reference has already been made to the simplest, viz. the rectangle (the square being a particular example), for which the area = length × breadth.

Since the area of a triangle is given by the product, ½ base × height, it follows that—

(*a*) Triangles on the same base and having the same height are equal in area, and

(*b*) Triangles on equal bases and having the same height are equal in area.

Thus, if in Fig. 13 the length FC is made equal to the length ED, the triangles AED and BFC will be equal in area, since the bases are equal and the height is the same. Also it will be seen that the sides AD and BC are parallel, so that the figure ABCD is a parallelogram. Then—

The area of the parallelogram—

$$\begin{aligned}ABCD &= \text{area of figure ABFD} + \text{area of triangle BFC}\\ &= \text{area of figure ABFD} + \text{area of triangle AED}\\ &= \text{area of rectangle ABFE}\\ &= AB \times BF.\end{aligned}$$

This result could be expressed in the general rule, "**Area of a parallelogram = length of one side × the perpendicular distance from that side to the side parallel to it.**"

In the case of the *rhombus* (a quadrilateral having its sides equal but its angles not right angles) one other rule can be added.

Its diagonals intersect at right angles, and hence its area can be expressed as equal to one-half the product of its diagonals; *i. e.*, Area $= \frac{1}{2} \times BD \times AC$, the reference being to the rhombus in Fig. 14.

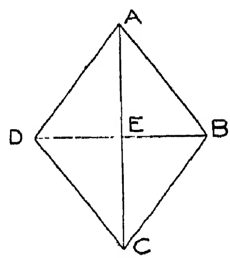


Fig. 14.

This rule should be proved as an example on the $\frac{1}{2}$ base × height rule for the triangle.

Area of Trapezoid.—A trapezoid is a quadrilateral having one pair of sides parallel.

Its area = mean width × perpendicular height.
= $\frac{1}{2}$ (sum of parallel sides) × perpendicular distance between them.

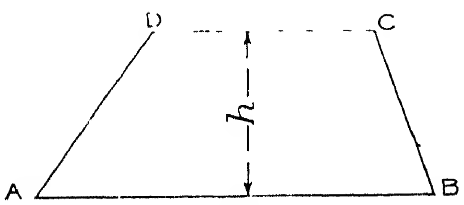


Fig. 15.

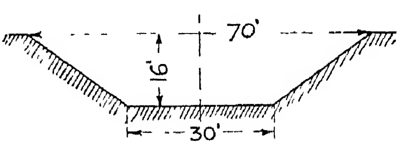


Fig. 16 - Cross Section of a Cutting

In Fig. 15, AB and CD are the parallel sides, and the area

$$= \tfrac{1}{2}\{AB + CD\} \times h$$

Example 4.—Calculate the area of the cross section of a cutting, having dimensions as shown in Fig. 16.

$$\text{Area} = \tfrac{1}{2}\{70 + 30\} \times 16 \text{ sq. ft.}$$
$$= \underline{800 \text{ sq. ft.}}$$

Example 5.—The kathode, or deposit plate, of a copper voltameter has the form shown in Fig. 17. Calculate, approximately, the area and hence the current density (*i. e.,* amperes per sq. in. of surface) if 1·42 amperes are passing.

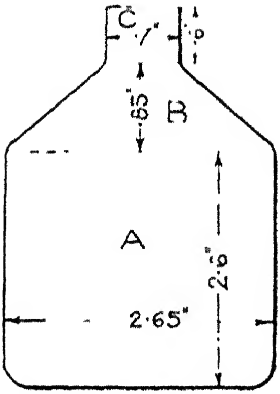


Fig. 17.

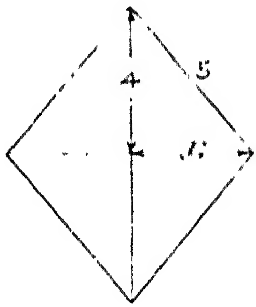


Fig. 18.

We may divide the surface of the plate into three parts, A, B, and C.

$$\text{Area of A} = 2{\cdot}6 \times 2{\cdot}65 \qquad = 6{\cdot}9 \text{ sq. ins.}$$
$$\text{Area of B} = \left(\frac{{\cdot}7 + 2{\cdot}65}{2}\right) \times {\cdot}85 = 1{\cdot}42 \;\; \text{,,} \;\; \text{,,}$$
$$\text{Area of C} = {\cdot}6 \times {\cdot}7 \qquad {\cdot}42 \;\; \text{,,} \;\; \text{,,}$$
$$\text{Total area of one side} \quad 8{\cdot}74 \;\; \text{,,} \;\; \text{,,}$$

This is the area of one side; but the deposit would be on both sides—

$$\therefore \qquad \text{total area} = 2 \times 8{\cdot}74 = 17{\cdot}48 \text{ sq. ins.}$$

$$\text{and current density} = \frac{1{\cdot}42}{17{\cdot}48} \quad \underline{{\cdot}0812 \text{ amp. per sq. in.}}$$

or 1 amp. for 12·3 sq. ins. of surface.

Example 6.—Find the area of the rhombus, one side of which measures 5″ and one diagonal 8″.

Let $2x$ = length of other diagonal in inches (Fig. 18).
Then, by the right-angled triangle rule,

$$x^2 = 5^2 - 4^2 = 9$$
$$\therefore \qquad x = 3 \text{ and } \quad 2x = 6.$$
$$\text{Area} = \tfrac{1}{2}\text{ (product of diagonals)} = \tfrac{1}{2} \times 8 \times 6 = \underline{24 \text{ sq. ins.}}$$

This example could also have been worked as an exercise on the "*s*" rule, the sides of the triangle being 5, 5 and 8 respectively.

Areas of Irregular Quadrilateral and Irregular Polygons.—Having dealt with the regular and the "semi" regular quadrilaterals, attention must now be directed to the irregular ones. No simple rule can be given that will apply to all cases of irregular quadrilaterals: the figure must be divided up into two triangles and the areas of these triangles found in the ordinary way.

This method applies also to *irregular polygons* (many-sided figures) having more than four sides; but these figures split into more than two triangles.

Example 7.—Find the area of the quadrilateral ABCD, Fig. 19, in which AD = 17 ft., DC = 15 ft., BC = 19 ft., AC = 26 ft., and the angle ABC is a right angle.

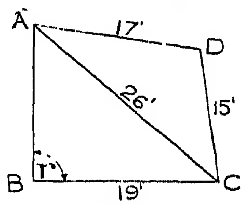


Fig 19.

It will be necessary to find the length of AB.
By the rule for the right-angled triangle,

$$(AB)^2 = 26^2 - 19^2 = 7 \times 45 = 315$$
$$\therefore \quad AB = 17{\cdot}76 \text{ ft.}$$

The quadrilateral ABCD = Triangle ADC + Triangle ABC.

Dealing first with the triangle ADC, its area must be found by the "*s*" rule.

$$s = \frac{17 + 15 + 26}{2} = 29$$
$$\therefore \quad \Delta = \sqrt{29 \times 12 \times 14 \times 3}$$
$$= 120{\cdot}9 \text{ sq. ft.}$$

$$\text{Area of triangle ABC} = \tfrac{1}{2} \times 17{\cdot}76 \times 19 = 169 \text{ sq. ft.}$$
$$\therefore \quad \text{Area of quadrilateral ABCD} = 121 + 169$$
$$= \underline{290 \text{ sq. ft.}}$$

Example 8.—Find the area of the plot of land represented in Fig. 20 (being the result of a chain survey).

Some of the dimensions are given in chains: it is worth while to remind ourselves of the magnitude of a chain.

Surveyors' Measure

1 chain	= 22 yards = 66 feet.
1 chain	= 100 links (1 link = 7·92".)
10 chains	= 1 furlong.
80 chains	= 1 mile.
1 sq. chain	= 22^2 = 484 sq. yards = $\frac{1}{10}$ of an acre.
or 10 sq. chains	= 1 acre.
10 sq. chains	= 100,000 sq. links.
1 acre	= 100,000 sq. links.

The given figure is divided by the "offsets" into triangles and trapezoids; the offsets being at right angles to the main chain lines.

It will be convenient to work in feet.

Dealing with the separate portions.

Area ACJ	$= \frac{1}{2} \times 198 \times 264$	$= 26136$	sq. ft.
ACB	$= \frac{1}{2} \times 198 \times 24$	$= 2376$	,,
CDB	$= \frac{1}{2} \times 16 \times 24$	$= 192$	,,
CDEF	$= \frac{1}{2}(20 + 16) \times 64$	$= 1152$	,,
FEGH	$= \frac{1}{2}(20 + 8) \times 80$	$= 1120$	,,
HGJ	$= \frac{1}{2} \times 8 \times 120$	$= 480$	,,
JKL	$= \frac{1}{2} \times 100 \times 13$	$= 650$	,,
LKMN	$= \frac{1}{2}(13 + 15) \times 200$	$= 2800$	,,
NMA	$= \frac{1}{2} \times 15 \times 30$	$= 225$	,,
		35131	sq. ft.

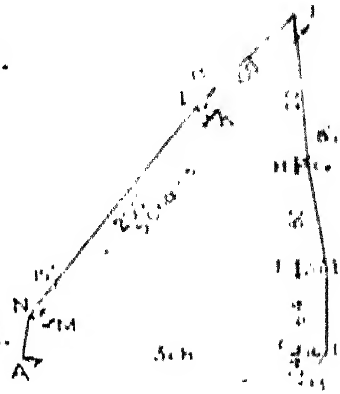


Fig. 20.

$\therefore$ Total area $= 35131$ sq. ft.

$$= \frac{35131}{66 \times 66} = 8{\cdot}07 \text{ sq. chns.}$$

$$= \underline{{\cdot}807 \text{ acre.}}$$

Areas of Regular Polygons. Regular polygons should be divided up into equal isosceles triangles, and there will be as many of these as the figure has sides. The areas of the triangles are best found (at this stage) by drawing to scale, and as an aid to this the following rule should be borne in mind.

The angle of a regular polygon of n sides

$$= \left(\frac{2n - 4}{n}\right) \times 90 \text{ degrees.}$$

Thus, for a regular pentagon (a five sided figure) $n = 5$ and the

$$\text{angle} = \left[\frac{(2 \times 5) - 4}{5}\right] \times 90 = 108^\circ$$

Alternatively, the following construction may be used. Suppose that the area of a regular heptagon, *i. e.*, a seven sided figure, is required, the length of side being $1\frac{1}{2}''$; and we wish to find its area by drawing to scale. Set out on any base line (Fig. 21) a semicircle with A as centre and radius $1\frac{1}{2}''$ (the length of side). Divide the semi circumference into seven (the number of sides) equal parts, giving the points a, B, c, d, e, f, G (this division to be done by trial). Through the *second* of these divisions, viz. B, draw the line AB; drawing also lines Ac, Ad, etc., radiating from A. With centre B and radius $1\frac{1}{2}''$ strike an arc cutting Ac in C; then BC is a side of the heptagon. This process can be repeated until the figure ABCDEFG is completed.

To find the area of ABCDEFG. Bisect AG and GF at right angles and note the point of intersection O of these bisectors; this being the geometrical centre of the figure. Measure OH (it is found to be $1 \cdot 56''$).

Then area of AOG—

$= \frac{1}{2} AG \times OH = \frac{1}{2} \times 1 \cdot 5 \times 1 \cdot 56$

$= 1 \cdot 17$ sq. ins.

$\therefore$ Area of ABCDEFG—

$= 7 \times \Delta AOG = 7 \times 1 \cdot 17$

$= \underline{8 \cdot 19 \text{ sq. ins.}}$

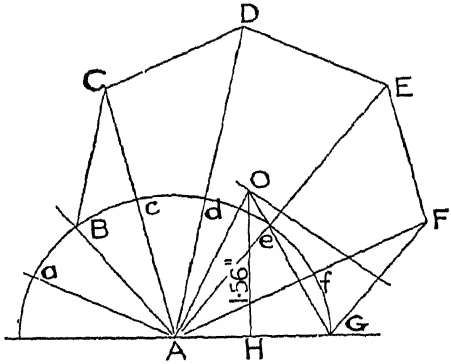


Fig. 21.—Area of Polygon.

Exercises 11.—On Areas of Quadrilaterals and Polygons.

1. The central horizontal section of a hook is in the form of a trapezoid $2\frac{1}{2}''$ deep, the inner width being $2''$ and the outer width $\frac{3}{4}''$. Find the area of the section.

2. The diagonals of a rhombus are $19 \cdot 74''$ and $5 \cdot 28''$ respectively. Find the length of side and the area.

3. Find the area of the quadrilateral ABCD shown at (*a*), Fig. 22. What is the height of a triangle of area equal to that of ABCD, the base being $5''$ long?

4. A field in the form of a quadrilateral ABCD has the following dimensions in yards: CD = 38, DA = 29, AC = 54, BE (perpendicular from B on to AC) = 23. Find its area in acres.

5. Reproduce (*b*), Fig. 22, to scale, and hence calculate the area of ABCDEF.

6. Find the area, in acres, of the field shown at (*c*), Fig. 22.

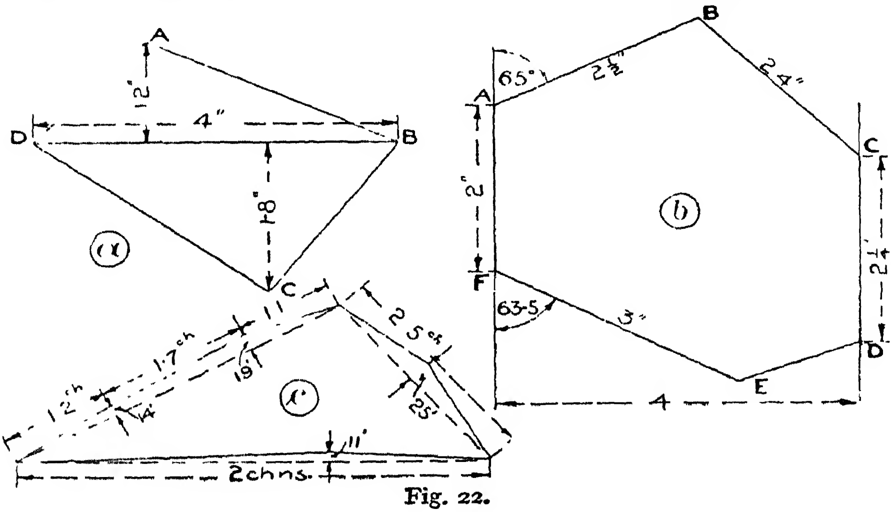


Fig. 22.

7. A retaining wall has a width of 4 ft. at base and 2′-6″ at top. The face of the wall has a batter of 1 in 12, and the back of wall is vertical. Find the area of section and also the length along the face.

8. The side slopes of a canal (for ordinary soil) are $1\frac{1}{2}$ horizontal to 1 vertical. If the width of the base is 20 ft. and the depth of water is 5 ft. find the "area of flow" when the canal is full.

9. Find the hydraulic mean depth $\left(i.e., \dfrac{\text{Area of flow}}{\text{Wetted perimeter}}\right)$ for the canal section for which the dimensions are given in Question 8.

10. The end of a bunker is in the form of a trapezoid. Find its area if the parallel sides are 9′-5″, and 15′-11″ respectively, the slant side being 24′-8″, while the other side is perpendicular to the parallel sides.

11. A regular octagon circumscribes a circle of 2″ radius. Find its area.

12. Find the area of a regular hexagon whose side is 4·28″.

13. The "end fixing moment" for the end A of the built in girder, Fig. 22*a*, is found by making the area ABEF equal to the area ABCD. Find this moment, *i. e.*, find the length AF.

14. A plate having the shape of a regular hexagon of side $2\frac{1}{2}$″ is to be plated with a layer of copper on each of its faces. Find the current required for this, allowing 1·6 amperes per 100 sq. ins.

15. An irregular pentagon of area 59·08 sq. ins. is made up of an equilateral triangle with a square on one of its sides. Find the length of side.

16. Neglecting the radii at the corners, find the approximate area of the rail section shown at (*a*), Fig. 12.

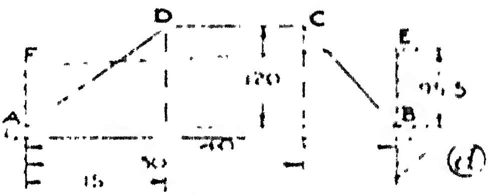


Fig. 22*a*.

Circumference and Area of Circle. When *n*, the number of sides of a polygon, is increased without limit, the sides merge into one outline and the polygon becomes a *circle*.

A circle is a plane figure bounded by one line, called the *circumference* and is such that all lines, called *radii*, drawn to meet the circumference from a fixed point within it, termed the *centre*, are equal to one another.

The meanings of the terms applied to parts of the circle will best be understood by reference to Fig. 23 and Fig. 24.

If a piece of thread be wrapped tightly round a cylinder for, say, five turns and the length then measured and divided by 5, the length of the circumference may be found. Comparing this with the diameter, as measured by callipers, it would be found to be about $3\frac{1}{7}$ times as long.

Repeating for cylinders of various sizes, the same ratio of these lengths would be found. The Greek symbol π (*pi*) always denotes

this ratio of circumference to diameter, which is invariable; but its exact value cannot be found. It has been calculated to a large number of decimal places, of which only the first four are of use to the practical man. For considerable exactness π can be taken as 3·1416: however, $\frac{22}{7}$ or 3·1428 is quite good enough for general use, the error only being about 12 in 30,000 or about ·04%. Even $\frac{22}{7}$ need not be remembered if a slide rule be handy, for a marking will be found to represent π (see Fig. 1, p. 17).

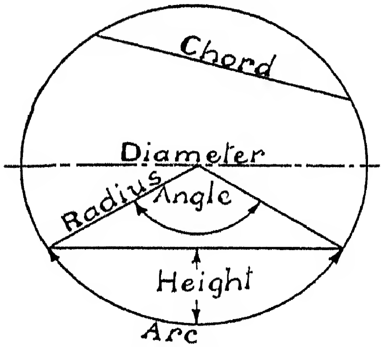


Fig. 23.

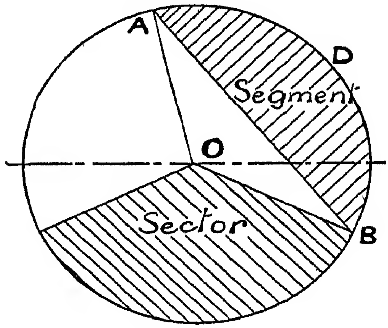


Fig. 24.

Then— **circumference** $= \pi \times$ **diameter** $= \pi d$ **or** $2\pi r$
where d = diam. and r = radius.

Also— **Area** $= \pi r^2$ **or** $\frac{\pi}{4}d^2$

$\frac{\pi}{4} = \cdot 7854$: a marking on the slide rule indicating this.

$\left(\text{The mark M on the slide rule is } \frac{1}{\pi}\right)$

It is sometimes necessary to convert from the circumference to the area; thus—

$$\text{Area} = \pi r^2 = \frac{4\pi^2 r^2}{4\pi} = \frac{(2\pi r)^2}{4\pi} = \frac{(\odot\text{ce})^2}{4\pi}$$

[$\odot$ stands for circle and $\odot$ce for circumference.]

Example 9.—Find the diameter of the driving wheel of a locomotive which in a distance of 3 miles makes 1010 revolutions (assuming no slipping).

In one revolution, the distance covered = $\odot$ce.

$$\therefore \quad \odot\text{ce} = \frac{\text{total distance}}{\text{number of revolutions}} = \frac{3 \times 5280}{1010}\text{ ft.}$$

and $\quad \text{diam.} = \frac{\odot\text{ce}}{\pi} = \frac{3 \times 5280}{\pi \times 1010} = \underline{5\text{ ft.}}$

Example 10.—Find the area of the cross section of shafting, $3\frac{1}{2}''$ diam.

$$\text{Area} = \frac{\pi}{4} \times 3{\cdot}5^2 = \underline{9{\cdot}62 \text{ sq. ins.}}$$

Notice that $\frac{\pi}{4}$ or ·7854 is in the neighbourhood of ·75 or $\frac{3}{4}$; therefore, for approximation purposes, square the diameter (to the nearest round figure) and take $\frac{3}{4}$ of the number so obtained.

In this case, $(3{\cdot}5)^2 = 12$ approximately,

and $\frac{3}{4}$ of $12 = 9$.

Areas of circles can most readily be obtained by the use of the slide rule, the method being as follows:—

Set one of the C's (marked on the C scale) (refer Fig. 1, p. 17) level with the diameter on the D scale, place the cursor over 1 on the B scale, then the area is read off on the A scale; the approximation being as before. This method is of the greatest utility, and several examples should be worked by means of it for the sake of practice.

Examples.—

Dia.	Area	Approximation
4·8	18·1	$\frac{3}{4} \times 25 = 18$
79·5	4965	$\frac{3}{4} \times 6400 = 4800$
·65	·332	$\frac{3}{4} \times {\cdot}50 = {\cdot}36$

If the C's are not indicated on the C scale of the slide rule, markings should be made for them at 1·128 and 3·568 respectively. The reason for these markings may be explained as follows.

The area of a circle $= \frac{\pi}{4}d^2$, or, as it might be written, $d^2 \div \frac{4}{\pi}$.

Now $\frac{4}{\pi} = 1{\cdot}286$, of which the square root is 1·128. A marking is thus placed at 1·128, so that when this mark is set level with the diameter on the D scale, the reading on the D scale opposite the index of the C scale gives the value of $d \div \sqrt{\frac{4}{\pi}}$. By reading the figure on the A scale level with the index on the B scale, the square of $d \div \sqrt{\frac{4}{\pi}}$ or $d^2 \div \frac{4}{\pi}$ is found; this being the area of a circle of diameter d.

For convenience in handling the rule a marking is made at

3·569 on the C scale also; this figure being obtained by extracting the square root of 12·86 instead of that of 1·286.

Some slide rules are supplied with a three-line cursor. If the centre line is placed over the dia. on the D scale then the left hand line is over the area on the A scale.

Example 11.—Find the connection between circumferential pitch and diametral pitch (as applied to toothed wheels).

The circumferential pitch—

$$p_c = \frac{\odot\text{ce of pitch circle}}{\text{number of teeth}} = \frac{\pi d}{N}$$

The diametral pitch $p_d = \dfrac{\text{number of teeth}}{\text{diam. of pitch circle}} = \dfrac{N}{d}$

Hence, $$p_c = \pi \times \frac{1}{\frac{N}{d}} = \frac{\pi}{p_d}$$

$$\textbf{circumferential pitch} = \frac{\pi}{\textbf{diametral pitch}}$$

E.g., if p_c is $\frac{3}{8}''$, then $\frac{\pi}{p_d} = \cdot 375$ or $p_d = \frac{\pi}{\cdot 375} = \underline{8\cdot 37''}$.

To find the area of an *Annulus*, i.e., the area between two concentric circles.

It is evident that the area will be :— Area of outer − area of inner circle = $\pi R^2 - \pi r^2$. (Fig. 25.)

This can be put into a form rather more convenient for computation, thus—

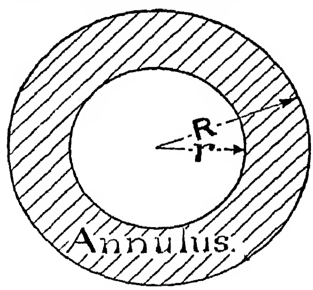


Fig 25.

Area of annulus $= \pi(R^2 - r^2)$ or $\frac{\pi}{4}(D^2 - d^2)$

or, in a form more easily applied—

$$\textbf{Area of annulus} = \pi(\mathbf{R} - r)(\mathbf{R} + r) \quad \textbf{or } \frac{\pi}{4}(\mathbf{D} - d)(\mathbf{D} + d).$$

This rule can be written in a form useful when dealing with tubes, thus—

$$\begin{aligned} \text{Area} = \pi(R - r)(R + r) &= 2\pi(R - r)\left(\frac{R + 2r - r}{2}\right) \\ &= 2\pi(R - r)\left(r + \frac{R - r}{2}\right) \\ &= 2\pi \times t \times \text{average radius} \\ &= \pi \times \text{average diameter} \times t \end{aligned}$$

where t is the thickness of the metal of the tube.

Example 12.—What is the area of a piece of packing in the form of a circular ring, of outside diameter $9\frac{1}{2}''$ and width $3\frac{1}{4}''$?

Here, $R = 4\cdot75''$, $r = 4\cdot75 - 3\cdot25 = 1\cdot5''$

Hence the area $= \pi(R - r)(R + r) = \pi(4\cdot75 + 1\cdot5)(4\cdot75 - 1\cdot5)$
$= \pi \times 6\cdot25 \times 3\cdot25$
$= \underline{63\cdot7 \text{ sq. ins.}}$

Example 13.—A hollow shaft, $5''$ internal diam., is to have the same sectional area as a solid shaft of $11''$ diam. Find its external diam.

$$\text{Area of solid shaft} = \frac{\pi}{4} \times 11^2 = \frac{\pi}{4} \times 121.$$

Let D = outside diam. required.

$$\text{Area of hollow shaft} = \frac{\pi}{4}(D^2 - 25)$$

The two areas are to be the same; equating the expressions found for these—

$$\frac{\pi}{4}(D^2 - 25) = \frac{\pi}{4} \times 121$$

whence $D^2 - 25 = 121$

and $D^2 = 146$

$\therefore \underline{D = 12\cdot07''.}$

Products, etc., of π.—Certain relations occurring frequently are here stated for reference purposes.

$\pi = 3\cdot142$ $\qquad \frac{1}{\pi} = \cdot318$ $\qquad \frac{1}{4\pi} = \cdot0795$

$\pi^2 = 9\cdot87$ $\qquad 4\pi^2 = 39\cdot48$, say $39\cdot5$ $\qquad \frac{4}{3}\pi = 4\cdot19$

$\frac{\pi}{6} = \cdot5236$ $\qquad \frac{\pi}{4} = \cdot7854$ $\qquad \frac{4\pi}{10} = 1\cdot256$ (often taken as $\frac{5}{4}$).

$\log \pi = \cdot4972$

Exercises 12.—On Circumference and Area of Circles.

1. Find the circumference of a circle whose diam. is $7\cdot13''$.

2. The semi-circumference of a circle is $91\cdot4$ ft. What is its radius?

3. Find the area of a circle of diam. $14'$-$3\frac{1}{2}''$.

4. The following figures give the girth of a tree at various points along its length. Find the corresponding areas of cross sections: $4\cdot28$, $5\cdot19$, $6\cdot47$, $2\cdot10$, $\cdot87$.

{Suggestion: area $= \frac{(\text{⊙ce})^2}{4\pi}$; first find value of constant multiplier $\frac{1}{4\pi}$ (approx. $\cdot08$). Keep the index of the slide-rule B scale fixed at this;

place cursor over ⊙cc on the C scale and read off result on A scale; the squaring and the multiplying are thus performed automatically.}

5. If the circumferential pitch of a wheel is $1\frac{1}{4}''$, find the diametral pitch. (See Example 11, p. 93.)

6. A packing ring, for a cylinder $12''$ diam., before being cut is $12 \cdot 5''$ diam. How much must be taken out of its circumference so that it will just fit the cylinder?

7. A circular grate burning 10 lbs. of coal per sq. ft. of grate per hour burns 66 lbs. of coal in an hour. Find the diameter of the grate.

8. Assuming that cast-iron pulleys should never run at a greater circumferential speed than 1 mile per min., what will be the largest diam. of pulley to run at 1120 revs. per min.?

9. The wheel of a turbine is $30''$ diam. and runs at 10600 R.P.M. What is the velocity of a point on its circumference?

Note.—The rule used in questions such as this is $v = 2\pi r N$, where v = velocity in feet per min., r = radius of wheel in feet, and N = number of revolutions per minute.

10. A piece, $4''$ long, is cut out of an elastic packing ring, fitted to a cylinder of $30''$ diam., so that the ends are now $\frac{1}{8}''$ apart. Find the diam. of the ring before being cut.

11. Find the diameter of an armature punching, round the circumference of which are 40 slots and the same number of teeth. The width of the teeth and also of the slots (at circumference) is $\cdot 35''$.

12. While the load on a screw jack was raised a distance equal to the pitch of the screw ($\frac{1}{2}''$), the effort was exerted through an amount corresponding to 1 turn of a wheel $10 \cdot 51''$ in diam. Find the Velocity-Ratio of the machine $\left\{ \text{V.R.} = \dfrac{\text{distance moved by effort}}{\text{distance moved by load}} \right\}$

13. The stress f in a flywheel rim due to centrifugal action is given by $f = \dfrac{12wv^2}{g}$, where w = weight of rim in lbs. per cu. in., v = circumferential speed of rim in ft per sec., and $g = 32 \cdot 2$. Find the revs. per min. if $f = 12 \times 2240$ when $w = \cdot 28$ and diam. = 10 ft.

14. Find the bending stress in a locomotive connecting rod revolving at n revs. per sec. from the equation—

$$\text{stress} = \frac{\rho}{8} \times \frac{4\pi^2 n^2 y r l^2}{k^2 g} \text{ where } y = \tfrac{1}{4},\ \rho = \frac{480}{1728}$$
$$r = 12,\ l = 120,\ k^2 = 3, \text{ and } g = 32 \cdot 2.$$

The driving wheels are 6 ft. in diam., and the locomotive travels at 40 miles per hour.

15. Find the area of the section of a column, the circumference of which is $18 \cdot 47''$.

16. Calculate the diameter of a circular plate whose weight would be the same as that of a rectangular plate measuring $2'$-$6''$ by $3'$-$2''$, both plates being of the same thickness and material.

17. Find the area of the section of a rod of $\cdot 498''$ diam.

18. If there is a stress of 48000 lbs. per sq. in. on a rod of $\cdot 566''$ diam., what is the load causing it?

19. Find the total pressure on a piston $9''$ diam., when the other

side of the piston is under a back pressure of 3 lbs. per sq. in. above a vacuum.

Gauge pressure (pressure above atmosphere) = 64 lbs. per sq. in.
1 atmosphere = 14·7 lbs. per sq. in.

20. The driving wheel of a locomotive, 5 ft. in diameter, made 10000 revolutions in a journey of 26 miles. What distance was lost owing to slipping on the rails?

21. The total pressure on a piston was 8462 lbs. If the gauge registered 51 lbs. per sq. in. (*i.e.*, pressure above atmosphere) and there was no back pressure, what was the diameter of the piston?

22. Find the area of section of a hollow shaft of external diam. $5\frac{1}{2}''$ and internal diam. $3''$.

23. A circular plot of land is to be surrounded by a fencing, the distance between the edge of the plot and the fencing being the same all round, viz. 6 ft. The length of the fencing required is 187 ft. Find the area of the space between the plot and the fencing.

24. Find the resistance of 60·5 cms. length of copper wire of diam. ·068 cm. from

$$R = \frac{kl}{a}$$

where a = area in sq. cms., l = length in cms., and k = resistivity = ·0000018 ohm per centimetre cube.

25. The buckling load P on a circular rod is given by —

$$P = \frac{Af_c}{1 + c\left(\frac{L}{K}\right)^2} \quad \text{where A = area of section and } K = \frac{\text{diameter}}{4}$$

Find the diameter when —

$$P = 188500, \; f_c = 67000, \; c = \frac{1}{5000}, \text{ and } L = 50.$$

26. A pair of spur wheels with pitch of teeth $1\frac{1}{2}''$ is to be used to transmit power from a shaft running at 120 R.P.M. to a counter shaft running at 220 R.P.M. The distance between the centres of the shafts is to be 24″ as nearly as possible.

If the diameters of the pitch circles are inversely as the R.P.M., find the true distance between the centres and the number of teeth on each wheel.

27. Calculate the area of the zero circle (the circle of no registration of the wheel), the radius of which is BD, for the planimeter shown in outline in Fig. 26.

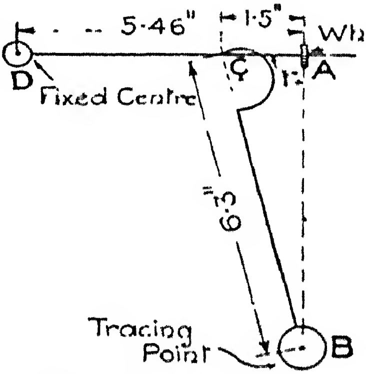


Fig. 26.—Amsler Planimeter.

28. The resistance of 1 mile of copper wire is found from—

$$R = \frac{\cdot 04232}{\text{area in sq. ins.}}$$

Find the resistance of 1 mile of wire of No. 22 B.W.G. (diam. = ·03″).

Length of Chord and Maximum Height of Arc.—In Fig. 27 let h = maximum height of the arc, $2a$ = length of the chord, and r = radius.

Then, by the right-angled triangle rule, applied to the triangle ACO—

$$r^2 = (r-h)^2+a^2 = r^2+h^2-2rh+a^2 \quad . \ . \ . \ . \ . \ (1)$$

Transposing terms—

$$a^2 = 2rh-h^2$$

whence—

$$a = \sqrt{2rh-h^2}$$

or **length of chord**

$$= 2\sqrt{2rh-h^2}$$

a rule giving the length of chord when the radius and maximum height of arc are given.

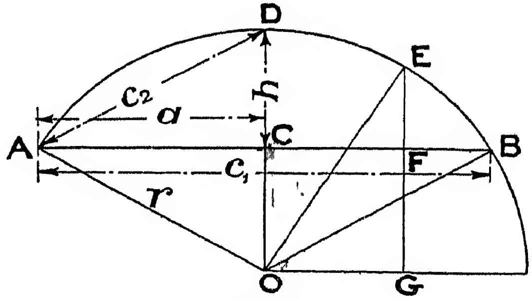


Fig. 27.

If h is expressed as a fraction of the radius, say $h = fr$, the rule for the length of chord becomes—

$$\text{length of chord} = 2r\sqrt{2f-f^2}.$$

From equation (1) $\qquad 2rh = a^2+h^2$

$$\therefore \quad r = \frac{a^2+h^2}{2h}$$

a rule giving the radius when the chord and the maximum height of arc are known.

From (1) also, $h^2-2rh+a^2 = 0$, and from this, by solution of the quadratic—

$$h = \frac{2r \pm \sqrt{4r^2-4a^2}}{2}$$

or

$$h = r \pm \sqrt{r^2-a^2}$$

giving two values for h (*i. e.*, for the arc less than, and the arc greater than, the semi-circumference) when the radius and length of the chord are known.

If two chords intersect, either within or without the circle, the rectangles formed with their segments as sides are equal in area, *Euc.* III, 35 and 36. Thus, in Fig. 28, at both (*a*) and (*b*)—

$$AP.PB = CP.PD$$

If C and D coincide as at (*c*), Fig. 28, then—

$$(PT)^2 = AP \cdot PB$$

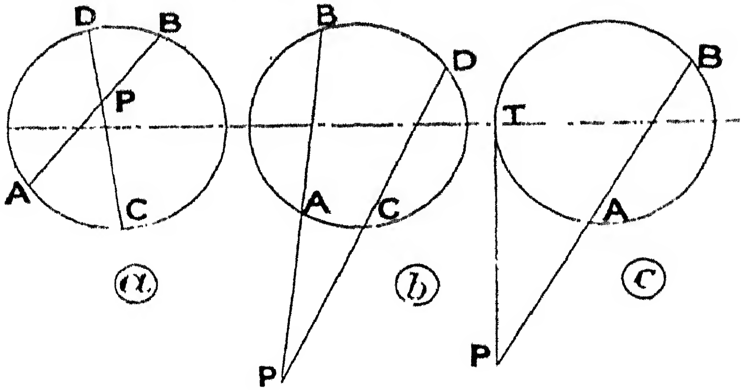


Fig. 28.

Example 14. The hardness number of a specimen, according to Brinell's test, is given by $\dfrac{\text{load}}{\text{curved area of depression}}$.

Express this as a formula.

The curved area (of segment of sphere) $= 2\pi rh$ (see p. 120).

r is radius of the ball making the indentation.

D diameter of depression.

Then $\dfrac{D}{2}$ corresponds to a in the foregoing formulae,

$$\text{so that } h = r - \sqrt{r^2 - \frac{D^2}{4}}$$

$$\text{and hardness number} = \frac{P}{2\pi r\left(r - \sqrt{r^2 - \dfrac{D^2}{4}}\right)}$$

Length of Arc. *Exact Rule.* The length of the arc depends on the angle it subtends at the centre of the circle: the total angle at the centre is 360°, this being subtended by the circumference.

An angle of 36° would be opposite an arc equal to one tenth of the circumference, whilst if the arc was $= \frac{1}{3}$ of ⊙ce, the angle at the centre would be 120°.

$$\text{In general—}\quad \frac{\text{arc}}{\odot\text{ce}} = \frac{\text{angle in degrees}}{360}$$

$$\text{or, arc} = \frac{2\pi r \times \text{angle in degrees}}{360} = \frac{\text{angle in degrees} \times \text{radius}}{57 \cdot 3}$$

If the arc is exactly equal in length to the radius, the angle then subtended ought to serve as a useful unit of measurement,

for one always expresses the circumference in terms of the radius. This angle is known as a *radian*.

If the chord were equal to the radius, the central angle would then be 60°, so that when the arc is involved in the same way the angle must be slightly less than 60°.

Actually, the radius is contained 2π times in the ⊙ce, hence

$$2\pi \text{ radians} = 360°, \textit{ i.e.}, 1 \text{ radian} = \frac{360}{2\pi} = 57{\cdot}3°.$$

Therefore, to convert from degrees to radians divide by 57·3.

Thus— $$273° = \frac{273}{57{\cdot}3} = 4{\cdot}76 \text{ radians.}$$

Radian or *circular measure* is the most natural system of angular measurement; all angles being expressed in radians in the higher branches of the subject.

A simple rule for the length of an arc can now be established.

$$\begin{aligned}\text{Length of arc} &= \frac{2\pi r \times \text{angle (degrees)}}{360} \\ &= \frac{2\pi r \times \text{angle (radian)}}{2\pi} \qquad \left\{\begin{matrix}\text{since } 360° \\ = 2\pi \text{ radians}\end{matrix}\right\} \\ &= r \times \text{angle subtended by the arc, expressed in radians.}\end{aligned}$$

Now it is usual to represent the measurement of an angle in radians by θ, and when in degrees by α. Thus the angle AOB in Fig. 27, subtended at the centre of the circle by the arc ADB would be expressed as θ, if in radians; or α, if in degrees.

Hence, **length of arc** $= \dfrac{\pi r \alpha}{180}$ **or** $r\theta$

Example 15.—A belt passing over a pulley 10″ diam. has an angle of lap of 115°: find the length of belt in contact with the pulley.

In this case $r = 5$ and $a = 115$

$$\therefore \text{ length in contact} = \text{length of arc} = \frac{\pi \times 5 \times 115}{180} = \underline{10{\cdot}03''}.$$

Example 16.—What angle is subtended at the centre of a circle of 14·8 ft. diam. by an arc of 37·4 ft.?

$$\text{Arc} = r\theta$$

$$\therefore \theta = \frac{\text{arc}}{r} = \frac{37{\cdot}4}{14{\cdot}8} \times 2 = 5{\cdot}05$$

Thus the angle required is 5·05 radians or 290 degrees.

It may be found of advantage to scratch a mark on the C scale of the slide rule at 57·3, so that the conversions from degrees to radians can be performed without any further tax on the memory.

Example 17.—It is required to find the diameter of the broken eccentric strap shown in the sketch (Fig. 29).

Here— $a = 2''$, and $h = 1{\cdot}2''$.

Then— $r = \dfrac{a^2 + h^2}{2h} = \dfrac{4 + 1{\cdot}44}{2{\cdot}4}$

$= \dfrac{5{\cdot}44}{2{\cdot}4}$

$= 2{\cdot}265$.

$\therefore$ diam. $= \underline{4{\cdot}53''}$ (probably $4\frac{1}{2}''$).

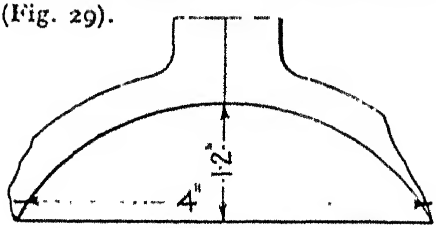


Fig. 29.

An approximate rule for the length of arc is that known as Huyghens'; viz.—

$$\textbf{Length of arc} = \frac{8c_2 - c_1}{3}$$

where c_2 and c_1 represent the chord of half the arc and the chord of the arc respectively ($i.e.$, $c_1 = 2a$). (See Fig. 27.)

To find the Height of an Arc from any Point in the Chord.

It is required to find the height EF (see Fig. 27) of the arc ADB, the length of chord AB, the maximum height CD of the arc ADB and the distance CF being given.

If O is the centre of the circle, OE is a radius and its length can be found from $r = \dfrac{a^2 + h^2}{2h}$

Then— $(OE)^2 = (EG)^2 + (GO)^2$

$= (EG)^2 + (CF)^2$

and of these lengths, OE and CF are known; therefore EG is found.

But FG is known, since $FG = OC = r - h$.

$\therefore$ the height EF, which $=$ EG $-$ FG, is known also.

A numerical example will demonstrate this more clearly.

Example 18.—A circular arc of radius 15″ stands on a base of 24″. Find its maximum height, and also its height at a point along the base 5″ from its extremity. (Deal only with the arc less than a semicircle.) (See Fig. 30.)

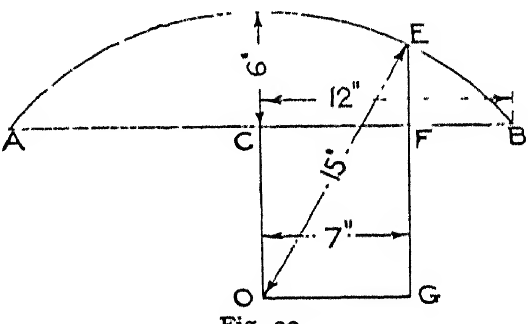


Fig. 30.

To find h. We know that $r = 15''$, and $a = 12''$

$$\text{hence } h = r \pm \sqrt{r^2 - a^2}$$
$$= 15 \pm \sqrt{225 - 144}$$
$$= 15 \pm 9 = 6'' \text{ or } 24''.$$

According to the condition stated in brackets h must be taken as $6''$, *i. e.*, the maximum height of the arc is $\underline{6''}$.

Then—
$$15^2 = EG^2 + 7^2$$
$$EG^2 = 15^2 - 7^2 = 22 \times 8 = 176$$
or
$$EG = 13{\cdot}26''$$
$$CO = r - h = 15 - 6 = 9''$$
$$\therefore \; EF = 13{\cdot}26 - 9 = 4{\cdot}26''$$

or the height of the arc at the $5''$ mark is $\underline{4{\cdot}26''}$.

Area of Sector.—A sector is a portion of a circle bounded by two radii and the arc joining their extremities; it is thus a form of triangle, with a curved base (see Fig. 24). Its area is given by a rule similar to that for the area of a triangle, viz., $\frac{1}{2}$ base $\times$ height, but in this case the base is the arc and the height is the radius (the radius being always perpendicular to the circumference).

Hence— **Area of sector = $\frac{1}{2}$ arc $\times$ radius,**

or, in terms of the radius, and angle at the centre (in radians).

Area $= \frac{1}{2}r^2\theta$, since for the arc we may write $r\theta$.

The area of the sector bears the same relation to the area of the circle as the length of arc does to the $\odot$ce, *i. e.*—

$$\frac{\text{Area of sector}}{\text{area of } \odot} = \frac{\text{angle (in degrees)}}{360}$$

$$\therefore \; \textbf{Area of sector} = \frac{a}{360} \times \pi r^2$$

Area of Segment.—The area of the segment, being the area between the chord and the arc (see Fig. 24), can be obtained by subtracting the area of the triangle from that of the sector. Thus, in Fig. 24—

Area of segment ADB = area of sector OADB — area of triangle OAB.

In place of this exact rule, we may use an approximate one, viz.—

$$\textbf{Area of segment} = \frac{2h}{3}\left\{\frac{7 \text{ chord} + 3 \text{ arc}}{10}\right\}$$

where h = maximum height of segment.

When the arc is very flat the chord and arc become sensibly the same, so that—

$$\textbf{Area of segment} = \frac{2h}{3}\left\{\frac{10 \text{ chord}}{10}\right\}$$

$$= \frac{2}{3} \times h \times \text{chord}$$

$$= \frac{2}{3} \times \text{maximum height} \times \text{width.}$$

The area of a segment may also be found from the approximate rule—

$$\textbf{Area of segment} = \frac{4}{3}h^2\sqrt{\frac{d}{h} - \cdot 608}$$

where d = diam. of circle, and h = maximum height of segment.

Example 19.—The *Hydraulic Mean Depth* (H.M.D.)—a factor of great importance in connection with the flow of liquids through pipes or channels—is equal to the section of flow divided by the wetted perimeter.

Find this for the case illustrated in Fig. 31.

Here, section of flow = area of segment ACB

= area of sector OACB − area of triangle OAB

$$= \frac{90}{360} \times \pi \times 6^2 - \tfrac{1}{2} \times 6 \times 6$$

$$= 9\pi - 18$$

$$= 10\cdot 3 \text{ sq. ins.}$$

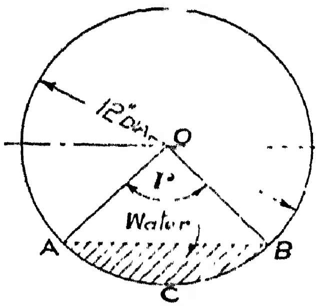


Fig. 31.

Wetted perimeter

$$= \text{arc ACB} = \frac{90}{360} \times 2 \times \pi \times 6 = 3\pi$$

$$= 9\cdot 42''$$

∴ H.M.D. (usually denoted by m)

$$= \frac{10\cdot 3}{9\cdot 42} = \underline{1\cdot 094''}.$$

Note that for a pipe running full bore the H.M.D.—

$$= \frac{\frac{\pi}{4}d^2}{\pi d} = \underline{\frac{d}{4}}$$

Exercises 13.—On Arcs, Chords, Sectors and Segments of Circles.

In Exercises 1 to 5, the letters have the following meanings as in Fig 27, r = radius, c_1 = chord of arc, c_2 = chord of half arc, h = maximum height of arc, and a = angle subtended at the centre of the circle by the arc.

1. $r = 8''$, $c_2 = 2{\cdot}4''$, find c_1 and h.

2. $c_1 = 80''$, $r = 50''$; find c_2 and h.

3. $c_1 = 49''$, $c_2 = 25''$; find r and h.

4. $c_1 = 6''$, $r = 9''$; find arc and area of segment.

5. $c_1 = 10''$, $h = 1{\cdot}34''$; find area of segment and a.

6. A circular arc is of 10 ft. base and 2 ft. maximum height. Find the height at a point on the base 1′-6″ distant from the end, and also the distance of the point on the base from the centre at which the height is 1 ft.

7. A circular arc has a base of 3″ and maximum height $\frac{5}{8}''$. Find (*a*) radius, (*b*) length of arc, (*c*) area of segment, (*d*) height of arc at a point on the base 1″ distant from its end.

8. A crank is revolving at 125 R.P.M. Find its angular velocity (*i. e.*, number of radians per sec).

9. If the angular velocity of a flywheel of 12′-6″ diam. is 4·5, find the speed of a point on the rim in feet per minute.

10. Find the area of a sector of a circle of 9·7″ diam., the arc of the sector being 12·3″ long.

11. One nautical mile subtends an angle of 1 minute at the centre of the earth; assuming a mean radius of 20,890,000 feet, find the number of feet in 1 nautical mile.

12. Find the difference between the apparent and true levels (*i e*, CE), if AC = 1500 yards and R = 3958 miles. [See (*a*), Fig. 32.]

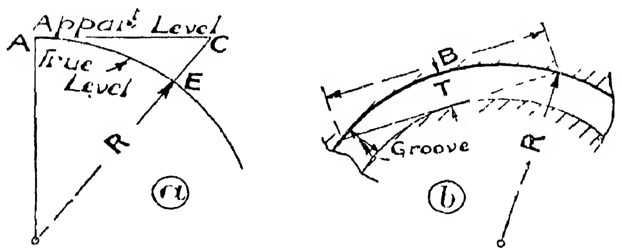


Fig. 32.

13. (*b*), Fig. 32 (which is not drawn to scale), shows a portion of a curve on a tramway track. If R = radius of quickest curve allowable (in feet), T = width of groove in rail (in inches), and B = greatest permissible wheel base (in feet) for this curve, find an expression for B in terms of R and T.

14. A circle of 2·4″ diam. rolls without slipping on the circumference of another circle of 6·14″ diam. What angle at the centre is swept out in 1 complete revolution of the rolling circle ?

15. A railway curve of $\frac{1}{2}$ mile radius is to be set out by "1 chain" steps. Find the "deflection angle" for this, *i. e.*, the angle to which the theodolite must be set to fix the position of the end of the chain. The deflection angle is the angle between the tangent and the chord.

16. Fig. 33 shows a hob used for cutting serrations on a gauge. It was found that the depth of tooth cut when the cutting edge was along AB was not sufficiently great. Find how far back the cutter must be ground so that the depth of serration is increased from ·025″ to ·027″, *i. e.*, find x when AB = ·025″ and CD = ·027″.

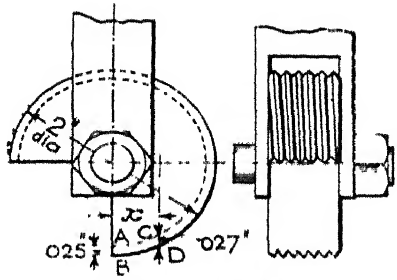


Fig. 33.—Gauge Hob.

The Ellipse.—The ellipse is the locus of a point which moves in such a way that the sum of its distances from two fixed points, called the *foci* is constant. This constant length is the length of the longer or major axis.

In Fig. 34, if P is any point on the ellipse, $PF + PF^1 =$ constant $= AA^1$, F and F^1 being the foci.

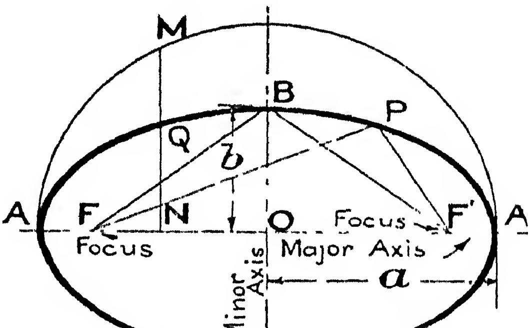


Fig. 34. The Ellipse.

Let major axis $= 2a$, and minor axis $= 2b$.
Then from the definition, $FB = F^1B = a$.

In the triangle FOB, $(FB)^2 = (FO)^2 + (OB)^2$

$$\text{or} \qquad a^2 = (FO)^2 + (b)^2$$

$$\therefore \qquad FO = \sqrt{a^2 - b^2}$$

so that if the lengths of the axes are given the foci are located.

Area $= \pi ab$. (Compare with the circle, where area $= \pi rr$.)

The *perimeter* of the ellipse can only be found very approximately as the expression for its absolute value involves the sum of an

infinite series. Various approximate rules have been given, and of these the most common are, **perimeter** $= \pi(a+b)$, or $\pi\sqrt{2(a^2+b^2)}$; the second of which might be written in the more convenient form $4{\cdot}443\sqrt{a^2+b^2}$. These rules, however, do not give good results when the ellipse is flat. A rule which appears to give uniformly good results is that of Boussinesq, viz.—

$$\textbf{perimeter} = \pi\{1{\cdot}5(a+b)-\sqrt{ab}\}.$$

The *height of the arc* above the major axis at any point can most easily be found by multiplying the corresponding height of the semi-circle described on the major axis as diameter by $\frac{b}{a}$, *e. g.*, referring to Fig. 34, $QN = \frac{b}{a} \times MN$.

Example 20.—The axes of an ellipse are 4·8″ and 7·4″. Find its perimeter and its area.

According to our notation, viz. as in Fig. 34, $2a = 7{\cdot}4,\ a = 3{\cdot}7$
$2b = 4{\cdot}8,\ b = 2{\cdot}4.$

$$\left.\begin{aligned}\text{Then the perimeter} &= \pi(a+b) = \pi(6{\cdot}1) = \underline{19{\cdot}15''}\\ \text{or}\quad &\pi\sqrt{2(a^2+b^2)} = \pi\sqrt{2(19{\cdot}45)} = \underline{19{\cdot}58''}\\ \text{or}\quad &\pi\{1{\cdot}5(a+b)-\sqrt{ab}\} = \pi\{1{\cdot}5(6{\cdot}1)-\sqrt{3{\cdot}7\times 2{\cdot}4}\}\\ &= \underline{19{\cdot}36''}\end{aligned}\right\}$$

$$\text{Area} = \pi ab = \pi \times 3{\cdot}7 \times 2{\cdot}4 = \underline{27{\cdot}85 \text{ sq. ins.}}$$

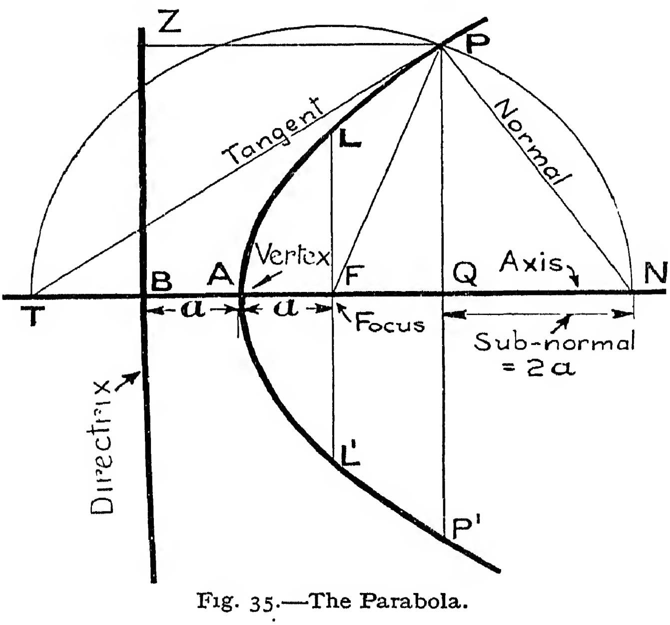


Fig. 35.—The Parabola.

The Parabola.—The parabola is the locus of a point which moves in such a way that its distance from a fixed straight line, called the *directrix*, is always equal to its distance from a fixed point called the *focus*.

Referring to Fig. 35, PZ = PF, where F is the focus, and P is any point on the curve.

The distance BA, which is equal to AF, is always denoted by a.

The chord LL^1 through the focus, perpendicular to the axis, is called the *latus rectum*, and from the definition it will be seen to be equal to $4a$. The latus rectum, in fact, determines the proportions of the parabola just as the diameter does the size of the circle.

If $PQ = y$ and $AQ = x$, then $FQ = AQ - AF = (x - a)$ and $PF = PZ = BQ = x + a$.

Then in the triangle FPQ,

$$(FP)^2 = (PQ)^2 + (FQ)^2$$

$$(x + a)^2 = y^2 + (x - a)^2$$

or $\quad x^2 + a^2 + 2ax = y^2 + x^2 + a^2 - 2ax$

whence $\quad y^2 = 4ax$

or $\quad (\frac{1}{2}\text{ width})^2 =$ latus rectum × distance along axis from vertex,

e. g., $\quad (MR)^2 = 4a \times AR$ in Fig. 36.

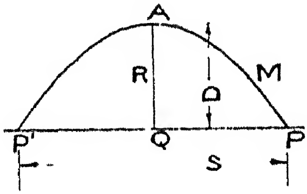


Fig. 36.

If a semi-circle be drawn with F as centre and with FP as radius, to cut the axis of the parabola in T and N, PT is the tangent at P and PN is the normal. (Fig. 35.)

The distance along the axis, under the normal, *i. e.*, QN in Fig. 35, is spoken of as the *sub-normal*. For the parabola, the length of the sub-normal is constant, being equal to $2a$, *i. e.*, $\frac{1}{2}$ latus rectum.

Use is made of this property in the design of governors. If the balls are guided into a parabolic path, the speed will be the same for all heights, for it is found that the speed depends on the sub-normal of the parabola, and as this is constant so also must the speed be constant.

The area of a parabolic segment $= \frac{2}{3}$ **of surrounding rectangle,** *i. e.*, area of P^1AP (Fig. 36) $= \frac{2}{3} \times PP^1 \times AQ$. **Length of parabolic arc** $= S + \frac{8}{3}\frac{D^2}{S}$ approximately, where S = span and D = droop or sag, as indicated in Fig. 36.

Circular and other arcs are often treated as parabolic when the question of the areas of segments arises; and if the arcs are very flat no serious error is made by so doing. The rule for the

area of a parabolic segment is so simple and so easily remembered that one is tempted to use it in place of the more accurate but more complicated ones which may be more applicable.

Take, for example, the case of the ordinary stress-strain diagram, as in Fig. 37. To find the work done on the specimen up to fracture it is necessary to measure the area ABCDE. Replacing the irregular curve (that obtained during the plastic stage) by a portion of a parabola BF, and neglecting the small area ABG, we can say that area ABCDE = rectangle AGHE + parabolic segment BFH

$$= Le + \frac{2}{3}e(M-L) = \frac{e}{3}\{2M+L\}.$$

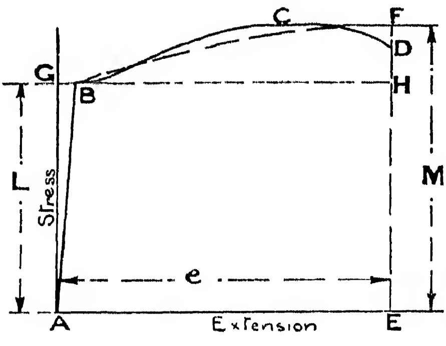


Fig. 37.—Stress-strain Diagram.

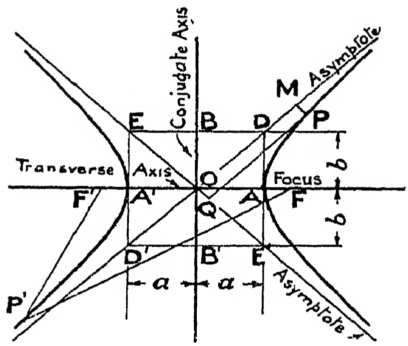


Fig. 38.—The Hyperbola.

If the ratio $\frac{L}{M}$ is denoted by r, then the result may be written—

$$\text{area ABCDE} = \frac{eM}{3}\left(2+\frac{L}{M}\right) = \frac{Me}{3}(2+r),$$

which is Kennedy's rule.

So, also, in questions on calculations of weights, circular segments are often treated as parabolic.

Example 21.—The bending moment diagram for a beam 28 feet long, simply supported at its ends, is in the form of a parabola, the maximum bending moment, that at the centre being 49 tons feet. Find the area of the bending moment diagram, and find also the bending moment at 6 feet from one end (this being given by the height of the arc at D, Fig. 39).

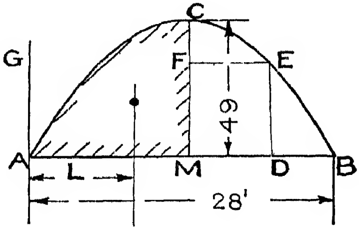


Fig 39.

Area of parabolic segment ACB $= \frac{2}{3} \times 49 \times 28 = \underline{915 \text{ units}}$.

These units are tons feet × feet or tons feet².

Now it can be shown that the moment of one-half the bending moment area (viz. AMC), taken round AG determines the deflection at A and also at B.

Actually, the maximum deflection (at A or B) $= \frac{1}{EI} \times$ area of AMC $\times$ L where E = Young's modulus for the material of the beam and I = second moment of its section. Since E would be expressed in tons per sq. foot and I in $(\text{feet})^4$ the deflection would be expressed in

$$\frac{\text{feet}^2 \times \text{feet}^2 \text{ tons} \times \text{feet}}{\text{tons} \times \text{feet}^4} \quad i.\,e., \text{ in feet.}$$

To find the height ED—

$$(MB)^2 = 4a \times MC \quad \text{from definition}$$

$$\therefore \quad 4a = \frac{(MB)^2}{MC} = \frac{14^2}{49} = 4$$

$$EF^2 = 4a \times CF$$

$$\therefore \quad CF = \frac{(EF)^2}{4a} = \frac{8^2}{4} = 16$$

$$DE = MC - CF = 49 - 16 = 33.$$

$\therefore$ Bending moment 6 feet from end = 33 foot tons.

Example 22.—Find the length of the sub-normal of the parabola

$$y^2 - 6y - 16x - 23 = 0.$$

The equation might be written—

$$(y^2 - 6y + 9) - 16x - 32 = 0$$

$$\text{or} \qquad (y - 3)^2 = 16(x + 2).$$

This is of the form — $\quad Y^2 = 4aX$

$$\text{where } Y = y - 3, \quad X = x + 2, \quad a = 4$$

$\therefore$ Length of subnormal $= 2a =$ 8 units; and is a constant.

The Hyperbola.—The hyperbola is the locus of a point which moves in such a way that the difference of its distances from two fixed points, called the foci, is constant. There are two branches to this curve, which is drawn in Fig. 38. If P^1 is any point on the curve, then $P^1F - P^1F^1 = AA^1 = 2a$.

AA^1 = transverse axis, and BB^1 = conjugate axis $= 2b$.

DOD^1 and EOE^1 are called *asymptotes*, i. e., the curve approaches these, but does not meet them produced: they are, as it were, its boundaries.

PM and PQ are parallel to EOE^1 and DOD^1 respectively: then a most important property of this curve is that the product PM $\times$ PQ is constant for all positions of P.

If $BB^1 = AA^1$, the asymptotes are at right angles and the

hyperbola is rectangular: *e.g.*, the curve representing Boyle's law (the case of isothermal expansion) is a rectangular hyperbola, the constant product being denoted by C in the formula, $pv = C$.

Exercises 14.—On the Ellipse and the Parabola.

1. A parabolic arc (as in Fig. 36) stands on a base of 12″. The latus rectum of the parabola being 8″, find—

(*a*) Maximum height of arc; (*b*) area of segment; (*c*) width at point midway between the base and the vertex.

2. A parabola of latus rectum 5″ stands on a base of 6″, find—

(*a*) Maximum height of arc; (*b*) height at a point on the base 2″ from the centre of the base; (*c*) area of segment; (*d*) position of focus.

3. A parabolic segment of area 24 sq. ins. stands on a base of 12″. Find the height of the arc at a point $2\frac{1}{2}''$ from the centre of the base and also the latus rectum.

4. The axes of an ellipse are 10″ and 6″ respectively. Find—

(*a*) The area; (*b*) distance between foci; (*c*) height of arc at a point on the major axis 4″ from the centre, (*d*) perimeter by the 3 rules.

5. The lengths of the axes of an ellipse can be found from $a^2 = 30$, $b^2 = 15$, where a and b have their usual meanings (see Fig. 34). Find—

(*a*) Area of ellipse; (*b*) distance of foci from centre; (*c*) perimeter by the three given rules.

6. A manhole is in the form of an ellipse, 21″ by 13″. Find, approximately, the area of plate required to cover it, allowing a margin of 2″ all the way round and assuming that the outer curve is an ellipse.

7. A cantilever is loaded with a uniform load of 15 cwts. per foot run. The bending-moment diagram is a parabola having its vertex at the free end, and its maximum ordinate (at the fixed end) is $\frac{wl^2}{2}$, where w = load per foot run, and l is the span which is 18 ft. Find the bending moment at the centre, and at a point 3 ft. from the free end.

8. It is required to lay out a plot of land in the form of an ellipse. The area is to be 6 acres and the ratio of the axes 3 : 2. Find the amount of fencing required for this plot.

9. There are 60 teeth in an elliptical gear wheel, for which the pitch is ·235″. If the major axis of the pitch periphery is twice its minor axis, find the lengths of these axes.

10. Find the number of feet per ton of oval electrical conduit tubing, the internal dimensions being $\frac{29}{32}'' \times \frac{3}{8}''$ and the thickness being ·042″ (No. 19 B.W.G.). Weight of material = ·296 lb. per cu. in.

The Prism and Cylinder.—A straight line moving parallel to itself, its extremities travelling round the outlines of plane figures generates the solid known as the *prism*. If the line is always at right angles to the plane figures at its extremities the prism is known as a *right prism*. If the plane figures are circles the prism becomes a *cylinder*.

A particular case of the prism is the *cuboid*, in which all the faces are rectangular, *i. e.*, the plane figures at the extremities of the revolving line are rectangles.

For all prisms, right or oblique —

Volume = area of base × perpendicular height.

The lateral or side surface of a right prism —

= perimeter of base × height.

Total surface = lateral surface + areas of ends.

Applying to the Cuboid.

$$\begin{aligned} \text{Volume} &= \text{area of base} \times \text{height} \\ &= ac \times b = a \times b \times c. \quad \text{(Fig. 40.)} \\ \text{Lateral surface} &= 2ab + 2bc \\ \text{Total surface} &= 2ab + 2bc + 2ac \\ &= 2(ab + bc + ca). \end{aligned}$$

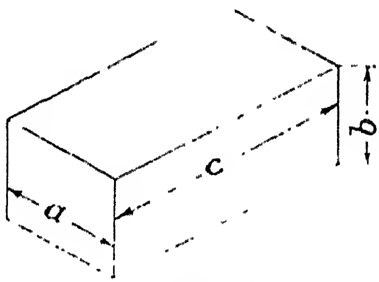


Fig. 40.

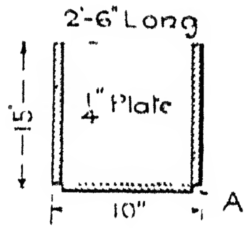


Fig. 41.

If $a = b = c$, the cuboid becomes a cube,

and then vol. $= a^3$

and total surface $= 2(a^2 + a^2 + a^2) = 6a^2$

If the diagonal of a cuboid is required it can be found from, diagonal $= \sqrt{a^2 + b^2 + c^2}$; whilst for the cube, diagonal $= a\sqrt{3}$.

Example 23. An open tank, made of material $\frac{1}{8}''$ thick, is $2'\ 6''$ long, $10''$ wide and $15''$ deep (these being the outside dimensions). Find the amount of sheet metal required in its construction if the plates are prepared for acetylene welding, and find also the capacity of the tank.

If the plates are to be joined by acetylene welding no allowance must be made for lap; the plates would be left as shown in the sketch at A, Fig. 41.

$$\text{Total Surface} = 2 \times (15 - \tfrac{1}{4})[10 - (2 \times \tfrac{1}{4})]$$
$$+ 2 \times (15 - \tfrac{1}{4})[30 - (2 \times \tfrac{1}{4})] + [30 - (2 \times \tfrac{1}{4})][10 - (2 \times \tfrac{1}{4})]$$
$$= 280 + 870 + 280 \text{ sq. ins.} = 1430 \text{ sq. ins.} = \underline{9 \cdot 94 \text{ sq. ft.}}$$
$$\text{Volume} = (30 - \tfrac{1}{2}) \times (10 - \tfrac{1}{2}) \times (15 - \tfrac{1}{4})$$
$$= \frac{29 \cdot 5 \times 9 \cdot 5 \times 14 \cdot 75}{1728} \text{ cu. ft.}$$
$$\text{Capacity} = \frac{29 \cdot 5 \times 9 \cdot 5 \times 14 \cdot 75 \times 6 \cdot 25}{1728} \text{ gallons}$$
$$= \underline{14 \cdot 9 \text{ gallons.}}$$

If the weight of water contained is required—

$$\text{Weight} = 14 \cdot 9 \times 10 = \underline{149 \text{ lbs.}}$$

Note.—1 cu. ft. of fresh water weighs 62·4 lbs.
1 cu. ft. of salt water weighs 64 lbs.
1 gallon of fresh water weighs 10 lbs.
$6\frac{1}{4}$ gallons occupy 1 cu. ft.
1 cu. cm. of water weighs 1 grm.

Applying the foregoing rules to the cylinder.

$$\text{Vol.} = \text{area of base} \times \text{height}$$

i. e., **Volume of cylinder** $= \pi r^2 \times h = \pi r^2 h$ **or** $\frac{\pi}{4} d^2 h$, where r = radius of base, d = diam. of base, h = height or length.

Lateral surface $= 2\pi r h$.

$$\text{Total surface} = 2\pi r h + 2\pi r^2$$
$$= 2\pi r(h + r)$$

Volumes of cylinders can most readily be obtained by the use of the slide rule, adopting an extension of the rule mentioned on p. 92.

It is repeated here with the necessary extension :—

Place one of the C's, on the C scale of the rule, opposite the diameter on the D scale: then place the cursor over the length on the B scale, and the volume is read off on the A scale.

Rough approximation, Vol. $= \frac{3}{4} d^2 h$.

E.g., Diam. $= 4 \cdot 63''$
Length $= 18 \cdot 75''$.

Vol. (by approximation) $= \frac{3}{4} \times 20 \times 20 = 300$ cu. ins.
Vol. (by slide rule) $= \underline{316 \text{ cu. ins.}}$

Exercises—

Dia	Length	Vol
·23	300	12·45
47·3	2·8	4945

Example 24. Find the weight, in lbs. and grms., per metre of copper wire of diam. ·045 cm. (Copper weighs ·32 lb. per cu. in.)

Note — 2·54 cms. = 1 in.
453·6 grms. = 1 lb.

Then—
$$1 \text{ cu. cm.} = \frac{1}{(2\cdot54)^3} \text{ cu. in.}$$

$$\therefore \text{ Weight of 1 cu. cm. of copper} = \frac{\cdot32}{(2\cdot54)^3} \text{ lb.}$$

$$\text{Vol. of 1 metre of wire} = \frac{\pi}{4} \times (\cdot045)^2 \times 100 \text{ cu. cms.}$$
$$= \cdot159 \text{ cu. cm.}$$

$$\therefore \text{ Weight} = \frac{\cdot159 \times \cdot32}{(2\cdot54)^3} = \underline{\cdot00311 \text{ lb.}}$$

$$\text{or weight} = \cdot00311 \times 453\cdot6$$
$$= \underline{1\cdot409 \text{ grms.}}$$

Example 25. A boiler contains 480 tubes, each 6 ft. long and 2⅜ ins. external diameter. Find the heating surface due to these.

The heating surface will be the surface in contact with the water, *i. e.*, the outside surface of the tubes.

$$\text{Lateral surface} = \pi d \times \text{length} \times \text{no. of tubes}$$
$$= \pi \times \frac{11}{48} \times 6 \times 480$$
$$= \underline{2070 \text{ sq. ft.}}$$

Exercises 15. On Prisms and Cylinders.

Prisms

1. A room 22 ft. long by 15′ 10″ wide is 9′ 5″ high. Find the volume of oxygen in it, if air contains 21 % of oxygen and 79 % of nitrogen by volume.

2. A block of wrought iron 15″ × 9″ × ⅜″ weighs 14·2 lbs. Find the density of W.I. (lbs. per cu. in.) and also its specific gravity if 1 cu. ft. of water weighs 62·4 lbs.

3. The weight of a brass plate of uniform thickness, of length 6′-5″ and breadth 11″ was found to be 70·4 lbs. If brass weighs ·3 lb. per cu. in., find the thickness of this plate.

4. The sectional area of a ship at its water-line is 5040 sq. ft.; how many tons of coal would be needed to sink her 1 ft? (35 cu. ft. of sea water weigh 1 ton.)

5. The coefficient of displacement of a ship
$$= \frac{\text{volume of immersed hull of ship}}{\text{volume of rectangular block of same dimensions}}$$
If the displacement is 4000 tons and the hull can be considered to have the dimensions 320′×35′×15′ find the coefficient of displacement.

6. The ends of a right prism 8′-4″ long are triangles having sides, 19″, 27·2″ and 11·4″ respectively. Find the volume of this prism.

7. Water is flowing along a channel at the rate of 6·5 ft. per sec. The depth of the channel is 9″, the width at base 14″, and the side slopes are 1 horizontal to 3 vertical. Find the discharge—

(*a*) In cu. ft. per sec.; (*b*) in lbs. per min.

8. A tightly-stretched telephone cable, 76 ft. long, connects up two buildings on opposite sides of the road. The points of attachment of the ends are 38 and 64 ft. above the ground respectively, one being 37 ft. further along the road than the other, and the buildings each standing 10 ft. back from the roadway. Find the width of the road.

9. The section of an underground airway is as shown in Fig. 42. Air is passing along the airway at 10·5 ft. per sec.; find the number of cu. ft. of air passing per minute.

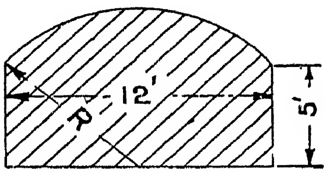


Fig 42.

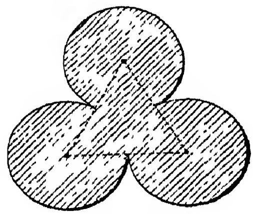


Fig 42*a*.

10. Find the volume of stone in a pillar 20 ft. high, the cross-section being based on an equilateral triangle of 1 foot side, having three circular sectors described from the angular points as centres, and meeting at the mid points of the sides. Find also its weight at 140 lbs. per cu. foot. (Fig. 42*a*.)

Cylinders

11. The diameter of a cylinder is 38·7″, and its length is 28·3″. Find its curved surface, its total surface and its volume.

12. Find the ratio of total heating surface to grate area in the case of a Caledonian Railway locomotive. The heating surface in the firebox is 119 sq. ft., the grate area is 20·63 sq. ft., and there are 275 tubes, of $1\frac{3}{4}$″ external diameter, the length between the tube plates being 10′-7″.

13. A current of ·6 ampere at 100 volts was passed through the two field coils of a motor. If the diam. of the coils was 4″ and the length $4\frac{1}{2}$″, find the number of watts per sq. in. of surface. (Curved surface only is required.)

14. Find the weight of 5 miles of copper wire of ·02″ diam., when copper weighs ·32 lb. per cu. in.

15. Find the weight of a hollow steel pillar, 10 ft. long, whose external diam. is 5″ and internal diam. is 4″ (1 cu. ft. of steel weighs 499 lbs.). (See Area of Annulus, p. 93.)

16. Water flows at the rate of 288 lbs. per min. through a pipe of $1\frac{1}{4}$″ diam. Find the velocity of flow in feet per sec.

17. Find the heating surface of a locomotive due to 177 tubes of $1\frac{3}{4}$″ diam., the length between the tube plates being 10′-6″.

18. A piston is moving under the action of a mean effective pressure of 38·2 lbs. per sq. in. at a speed of 400 ft. per min. If the horse-power developed is 70, find the diam. of the piston.

$$\left[\text{H.P.} = \frac{\text{Feet per min.} \times \text{total pressure in lbs.}}{33000}\right]$$

19. In a ten-coupled locomotive there were 404 tubes of 2" diam and the heating surface due to these was 3280 sq. ft. Find the length of each tube.

20. The diameter of a hydraulic accumulator is 12" and the stroke is 6 ft. Find the work stored per stroke if a constant pressure of 750 lbs. per sq. in. be assumed.

21. In calculating the indicated horse-power of an engine at various loads it was found that a saving of time was effected if an "engine constant" was found.

$$\text{If the engine constant} = \frac{\text{Vol. of cylinder}}{12 \times 33000}$$

find this, if diam. = 5·5" and stroke = 10".

22. The weight of a casting is to be made up from 4·14 lbs. to 4·16 lbs. by drilling a $\frac{5}{16}$" diam. hole and plugging with lead. To what depth must the hole be drilled if the weights of lead and cast iron are ·41 and ·26 lb. per cu. in. respectively?

23. The conductivity of copper wire can be expressed by its resistance per gramme metre. Find the "conductivity" of a wire 5 metres long and of ·762 cm. diam. (No. 1 S.W.G.) if the Resistance is given by $\cdot 0000017 \times \frac{\text{length}}{\text{area}}$; the units being cms. and the weight of 1 cu. cm. of copper being 8·91 grms.

24. Find the weight, in lbs. per 100 feet, of electrical conduit tubing of external diam. 2" and internal diam. 1·872", the weight of the material being ·206 lb. per cu. in.

25. A 10" length of 1" diam. steel rod is to be forged to give a bar $1\frac{1}{4}$" wide and $\frac{1}{2}$" thick. Assuming no loss in the forging, find the length of this bar.

26. Find the capacity (in gallons) of an oil-drum 9" dia. and $13\frac{1}{2}$" high

Pyramid and Cone.—If a straight line of variable length moves in such a way that one extremity is always on the boundary of a plane figure, called the base, whilst the other is at a fixed point, called the vertex, the solid generated is termed a *pyramid*. If the line joining the vertex to the geometrical centre of the base is at right angles to the base, then the pyramid is spoken of as a *right pyramid*.

When the base is circular the figure is termed a cone; *right circular* cones being those most frequently met with. These are cones for which all sections at right angles to the axis are circles.

The lateral surface of a right pyramid will evidently be the sum of the areas of the triangular faces.

Consider the case of a "square" pyramid, *i. e.*, where the base is square [see (*a*), Fig. 43]. The triangular faces are equal in area.

Area of each $= \frac{1}{2}$ base $\times$ height
$= \frac{1}{2} \times AB \times VL$ [see (*a*), Fig. 43]

where VL is spoken of as the ***slant height*** of the pyramid; its value being found from—

$$VL = \sqrt{VO^2 + OL^2} \quad \text{[see (}b\text{), Fig. 43]}$$

LO being $\frac{1}{2}$ side of base.

$$\therefore \quad \text{Total lateral surface} = 4 \times \tfrac{1}{2}AB \times VL$$
$$= 2 . AB \times VL$$

or **lateral surface of pyramid = $\frac{1}{2}$ perimeter of base $\times$ slant height.**

This rule will hold for all cases in which the base is regular.

[Note that if the base is rectangular, there will be two distinct slant heights.]

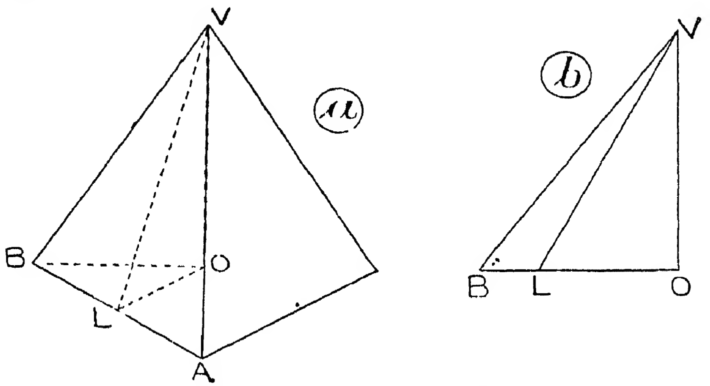


Fig 43 —Square Pyramid.

Length of edge of pyramid $= VB = \sqrt{VO^2 + OB^2}$ [see (*b*), Fig 43], where $OB = \frac{1}{2}$ diagonal of base.

The three lengths or heights should be clearly distinguished.

VO = perpendicular height, or more shortly, the height
VL = slant height, and VB = length of edge.

Volume of pyramid is one-third of that of the corresponding prism (*i. e.*, the prism on the same base and of same vertical height).

$\therefore$ **Vol. of pyramid = $\frac{1}{3}$ $\times$ area of base $\times$ perpendicular height.**

Example 26.—A flagstaff, 15 ft. high, is kept in position by four equal ropes, one end of each being attached to the top of the staff, whilst the other ends are fastened to the corners of a square of 6 ft. side. Find the length of each rope.

Diagonal of base $= 6\sqrt{2}$ (the diagonal of a square always $=$ $\sqrt{2} \times$ side). The length required is the length of the edges, viz. VB [see (*b*) Fig. 43].

Now VO $= 15$, OB $= 3\sqrt{2}$, hence $\text{VB} = \sqrt{(3\sqrt{2})^2 + (15)^2} = \sqrt{18 + 225}$
$$= \sqrt{243}$$

$$\therefore \quad \text{VB} = \text{length of each rope} = \underline{15 \cdot 6 \text{ ft.}}$$

Applying to the Cone.—If the lateral surface of the cone is developed, *i. e.*, laid out into one plane, a sector of a circle results, the radius being the slant height l, and the arc being the circumference of the base of the cone or $2\pi r$ (see Fig. 44).

Now area of sector of circle $= \frac{1}{2}$ arc $\times$ radius $= \frac{1}{2} \times 2\pi r \times l$
$$= \pi r l$$

***i. e.*, area of curved surface of a cone $= \pi r l$.**

Notice that this agrees with the result obtained from the rule for the pyramid, viz. $\frac{1}{2}$ perimeter of base $\times$ slant height.

If the development of the cone were actually required it would be necessary to find the angle α (Fig. 44).

Now—

$$\frac{\alpha}{360} = \frac{\text{arc}}{\text{circle}} = \frac{2\pi r}{2\pi l} = \frac{r}{l}$$

$$\therefore \quad \alpha = \frac{360 r}{l}$$

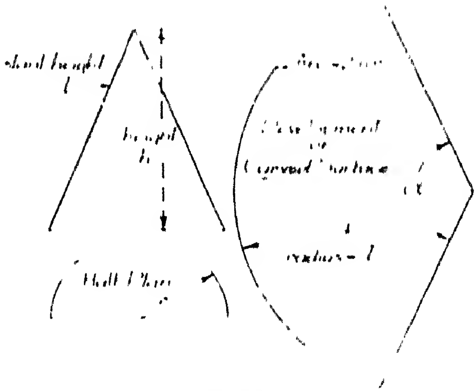


Fig. 44.

Lateral surface, then, $= \pi r l$
Total surface $= \pi r l + \pi r^2 = \pi r(l + r)$.

As the cone is a special form of pyramid its volume will be one-third that of the cylinder on the same base and of the same height.

$$\therefore \quad \textbf{Vol. of cone} = \frac{1}{3}\pi r^2 h \quad \text{or} \quad \frac{\pi}{12} d^2 h \quad \text{or} \quad \cdot 2618 d^2 h$$

d being the diameter of the base and h the perpendicular height.

The approximation for the volume is $\frac{1}{4} \times (\text{diam.})^2 \times$ height.

Example 27.—A projectile is cylindrical with a conical point (see Fig. 45). Find its volume.

As the cone is on the same base as the cylinder its volume can be accounted for by adding $\frac{1}{3}$ of its length to that of the cylinder, and treating the whole as one cylinder.

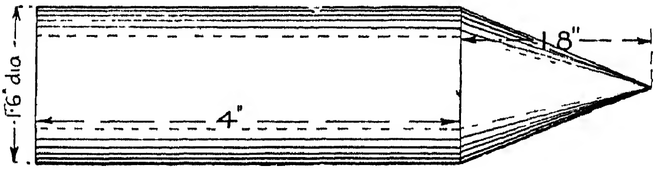


Fig. 45.

$$\text{Hence, net length} = 4'' + (\tfrac{1}{3} \times 1{\cdot}8'') = 4{\cdot}6''$$

$$\therefore \text{ total vol.} = \frac{\pi}{4} \times (1{\cdot}6)^2 \times 4{\cdot}6$$

$$= \underline{9{\cdot}26 \text{ cu. ins.}}$$

Frusta.—If the pyramid or cone be cut by a plane parallel to its base the portion of the solid between that plane and the base is known as a *frustum* of the pyramid or cone.

The lateral surface and the volume can be found by subtracting that of the top cone from that of the whole cone or by the following rules, which give the results of this procedure in a more advanced form.

Lateral surface of frustum of pyramid or cone

$$= \tfrac{1}{2}\,\{\text{sum of perimeters of ends}\} \times \text{slant thickness.}$$

$$\textbf{Vol. of frustum of pyramid or cone} = \frac{h}{3}\{\mathbf{A} + \mathbf{B} + \sqrt{\mathbf{AB}}\}$$

where A and B are the areas of the ends, and h is the perpendicular height or thickness of the frustum. (The proofs of these rules are given on p. 123.)

For the frustum of a cone these rules may be expressed in rather simpler fashion—

$$\textbf{Lateral surface of frustum of cone} = \pi l(\mathbf{R} + r)$$

l being the slant height of the frustum.

$$\textbf{Volume of frustum of cone} = \frac{\pi h}{3}\{\mathbf{R}^2 + r^2 + \mathbf{R}r\}$$

where R and r are radii of ends, and h is the thickness of the frustum.

Example 28.—A friction clutch is in the form of the frustum of a cone, the diameters of the end being $6\frac{1}{2}''$ and $4\frac{1}{2}''$, and length $3\frac{1}{2}''$. Find its bearing surface and its volume (see Fig. 46).

The slant height must first be found

$$l^2 = (3\tfrac{1}{2})^2 + (1\tfrac{1}{8})^2$$
$$= 13{\cdot}51$$
$$\therefore\ l = 3{\cdot}68''.$$

Now $R = 3{\cdot}25$, and $r = 2{\cdot}13$.

$\therefore$ Lateral surface

$$= \pi \times 3{\cdot}68(3{\cdot}25 + 2{\cdot}13)$$
$$= \pi \times 3{\cdot}68 \times 5{\cdot}38 = \underline{62{\cdot}2 \text{ sq. ins.}}$$

Also—

$$\text{Volume} = \frac{\pi h}{3}\{R^2 + r^2 + Rr\}$$
$$= \frac{\pi \times 3{\cdot}5}{3}\{10{\cdot}54 + 4{\cdot}53 + 6{\cdot}92\}$$
$$= \frac{\pi \times 3{\cdot}5 \times 21{\cdot}00}{3} = \underline{80{\cdot}5 \text{ cu. ins.}}$$

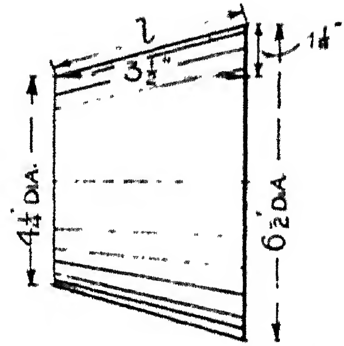


Fig. 46.—Friction Clutch.

Exercises 16. On Pyramids, Cones and Frusta.

1. The sides of the base of a square pyramid are each $13{\cdot}7''$ and the height of the pyramid is $9{\cdot}5''$. Find (*a*) the volume, (*b*) the lateral surface, (*c*) the length of the slant edge.

2. The volume of a pyramid, whose base is an equilateral triangle of $5{\cdot}2''$ side, is $79{\cdot}6$ cu. ins. Find its height.

3. Find the total area of slating on the roof shown at (*a*) Fig. 47.

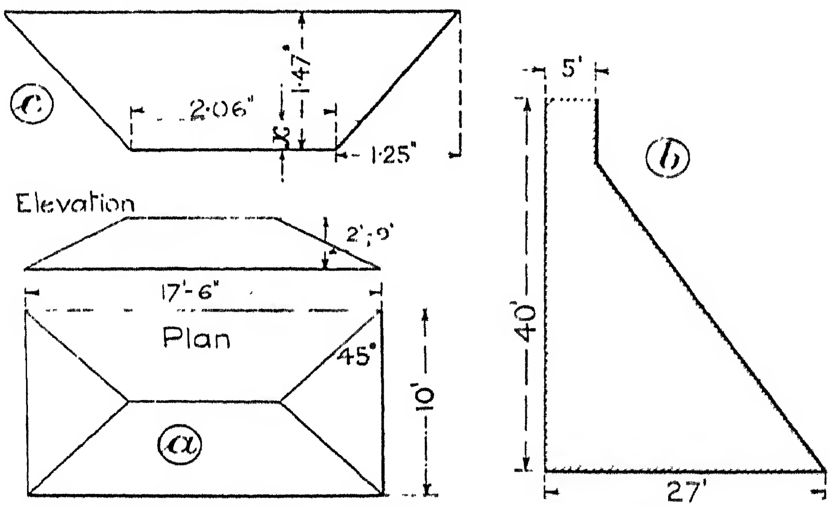


Fig. 47.

4. Find the volume of a hexagonal pyramid, of height $5{\cdot}12''$, the base being a regular hexagon of $1{\cdot}74''$ side.

5. A square pyramid of height 5 ft., the sides of the base being each 2 ft., is immersed in a tank in such a way that the base of the

pyramid is along the surface of the water. Find the total pressure on the faces of the pyramid if the average intensity of pressure is the intensity at a depth of 1′-3″ below the surface; the weight of 1 cu. ft. of water being 62·4 lbs.

6. A turret is in the form of a hexagonal pyramid, the height being 25 ft. and the distance across the corners of the hexagon being 15 ft. Find the true length of the hip (*i. e.*, the length of a slant edge), and also the lateral surface.

Cones.

7. The curved surface of a right circular cone when developed was the sector of a circle of 11·42″ radius, the angle of the sector being 127°. Find the radius of the base of the cone, and also its height. (Refer p. 116.)

8. A piece in the form of a sector (angle at centre 66°) is cut away from a circular sheet of metal of 9″ diam., and the remainder is made into a funnel. Find the capacity of this funnel.

9. A right circular cone is generated by the revolution of a right-angled triangle about one of its sides. If the length of this side is 32·4 ft. and that of the hypotenuse is 55·9 ft., find the total surface and the volume of the cone.

10. A vessel is in the form of a right circular cone, the circumference of the top being 19·74 ft. and the full depth of the vessel being 12 ft. Find the capacity in gallons. Find also the weight of water contained when the vessel is filled to one-half its height.

11. A conical cap is to be fitted to the top of a chimney The cap is to be of 7″ height and the diam. of the base is 12″. Find the amount of sheet metal required for this.

If this surface be developed, forming a sector of a circle, what will be the angle of the sector?

Frusta of Pyramids and Cones.

12. A pier is in the form of a frustum of a square pyramid. Its ends are squares, of side 3 ft. and 8′-6″ respectively, and its height is 6 ft. Find its volume and its weight at 140 lbs. per cu ft.

13. A circular brick chimney is 100 ft. high and has an internal diam. of 5 ft. throughout. The external diam. at base is 11 ft and at the top 7 ft, the thickness being uniformly reduced from bottom to top. Find its weight at 120 lbs. per cu. ft.

14. Find the lift h of the valve shown in Fig. 48, given that BC = $1\frac{3}{8}$″ and AD = $1\frac{5}{8}$″. It is necessary that the area of the lateral surface of ABCD should be 1·375□″.

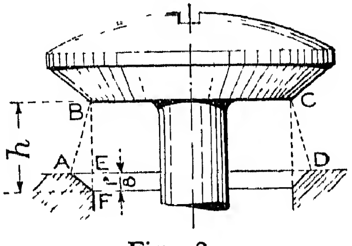


Fig 48

15. One of a set of weights had the form of a frustum of a cone, the thickness being $4\frac{1}{2}$″, the diam. at the top being 10″, and the diam. at the bottom being $2\frac{1}{2}$″. Find its volume and its weight at ·26 lb. per cu. in.

16. A square pyramid of height 9″ and side of base 15″ is cut into two parts by a plane parallel to the base and distant 4″ from it. Find the volume of the frustum so formed, and also its lateral surface.

17. A cone 12″ high is cut at 8″ from the vertex to form a frustum of volume 190 cu. ins. Find the radius of the base of the cone.

18. The parallel faces of a frustum of a pyramid are squares on sides of 3″ and 5″ respectively, and its volume is $32\frac{2}{3}$ cu. ins. Find its altitude and the height and lateral edge of the pyramid from which it is cut.

19. A conical lamp-shade is $2\frac{1}{2}$″ diam. at the top and $8\frac{1}{2}$″ diam. at the bottom. The shortest distance between these ends is 5″. Find the area of material required for this, allowing 4 % extra for lapping. By drawing to scale, find the area of the rectangular piece from which the shade would be cut.

20. A pyramid, having a square base of side 18″, and a height of 34″, is cut by a plane distant 11″ from the base and parallel to it. Find the total surface of the frustum so formed, and also its volume.

The Sphere.—If a semi circle revolves about its diameter as axis it sweeps out the solid known as the sphere.

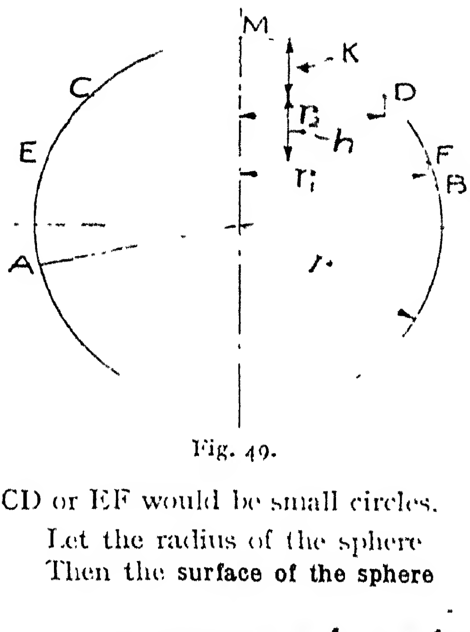


Fig. 49.

The portion of the sphere between two parallel cutting planes is known as a *zone*: thus CDFE in Fig. 49 is a zone.

The portion included between two planes meeting along a diameter is known as a *lune*.

A plane section through the centre is called a *great circle*: any other planes will cut the sphere in *small circles*.

Thus, the section on AB (Fig. 49) would be a great circle, and the sections on CD or EF would be small circles. The portion CMD is a *segment*.

Let the radius of the sphere r, and diam. d.

Then the **surface of the sphere** $= 4 \times$ area of a great ⊙

$$= 4 \times \pi r^2 = 4\pi r^2 \text{ or } \pi d^2$$

$$\textbf{Vol. of sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3}\cdot\frac{\pi}{8}d^3 = \frac{\pi d^3}{6} \text{ or } \cdot 5236 d^3$$

Surface of a zone = curved surface of circumscribing cylinder
$$= 2\pi r h$$

(*h* being the distance between the parallel planes).

$$\textbf{Vol. of zone} = \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}$$

[The proof of these two rules will be found in Vol. II of *Mathematics for Engineers.*]

The zone may be regarded as a form of frustum, r_1 and r_2 being the radii of the ends and h being the thickness.

If $r_1 = 0$, the zone becomes a segment, and then—

$$\textbf{Vol. of segment} = \frac{\pi k}{6}\{3r_2^2 + k^2\}$$

k being the height of the segment.

A relation that exists between the volumes of the cone, sphere and cylinder should be noted. Consider a sphere, of radius r; its circumscribing cylinder (*i. e.*, a cylinder with diam. of base $= 2r$ and height $= 2r$), and the cone on the same base and of the same height.

$$\text{Then, Vol. of the sphere} = \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3 \times 2$$

$$\text{Vol. of the cylinder} = \pi r^2 \times 2r = \frac{2}{3}\pi r^3 \times 3$$

$$\text{Vol. of the cone} = \frac{\pi r^2}{3} \times 2r = \frac{2}{3}\pi r^3 \times 1.$$

Hence the respective volumes of the cone, sphere and cylinder of equal heights and diameters are in the proportion 1 : 2 : 3.

Example 29.—A disc of lead 14″ diam. and ·8″ thick is melted down and cast into shot of (*a*) $\frac{1}{8}$″ diam., (*b*) $\frac{1}{4}$″ diam. How many shot can be made in each case, supposing no loss?

Case (*a*).—

$$\text{Vol. of disc} = \frac{\pi}{4} \times 14^2 \times \cdot 8 \text{ cu. in.}$$

$$= 39 \cdot 2\pi \text{ cu. ins.}$$

$$\text{Vol. of 1 shot} = \frac{\pi}{6} \times \left(\frac{1}{8}\right)^3 = \frac{\pi}{6 \times 512}$$

$$\therefore \text{No. of shot} = \frac{39 \cdot 2\pi \times 6 \times 512}{\pi}$$

$$= \underline{120{,}300.}$$

Case (*b*).—The diam. is twice that of Case (*a*), therefore the vol. of 1 shot is 2^3, *i. e.*, 8 times as great.

$$\therefore \text{No. of shot} = \frac{120{,}300}{8} = \underline{15{,}038.}$$

Example 30.—Find an expression for the weight in lbs. of a sphere of any material, having given that the weight of a cu. in. of copper is ·318 lb. (approx.).

Weight of a copper sphere of diam. D—

$$= \text{volume} \times \text{density}$$

$$= \frac{\pi}{6}D^3 \times \cdot 318$$

$$= \frac{D^3}{6} \text{ lbs}$$

Hence the weight of a sphere of any material, its diameter being D—

$$= \frac{D^3 \times \text{specific gravity of solid}}{6 \times \text{specific gravity of copper}}$$

Example 31.—Find the total surface of a hemispherical dome, of inside diam. $5\frac{1}{2}''$ and outside diam. $7\cdot4''$.

Outside surface $= \frac{1}{2} \times 4\pi \times (3\cdot7)^2 = 85\cdot6$ sq. ins.
Inside surface $= \frac{1}{2} \times 4\pi \times (2\cdot75)^2 = 47\cdot5$,,
Area of base $= \pi(3\cdot7^2 - 2\cdot75^2) = 19\cdot2$,,

$\therefore$ Total surface area $\underline{152\cdot3 \text{ sq. ins.}}$

Similar Figures. Similar figures are those having the same shape: thus a field and its representation on a drawing-board are similar figures. Triangles, whose angles are equal, each to each, are similar figures.

On every hand one comes across instances of the application of similar figures; and in connection with these, three rules should be remembered.

(1) **Corresponding lines or sides of similar figures are proportional.** (*Euclid*, VI. 4.)

(2) **Corresponding areas or surfaces are proportional to the squares of their linear dimensions.** (*Euclid*, VI. 20.)

(3) **Volumes or weights of similar solids are proportional to the cubes of their linear dimensions.**

E.g., consider two exactly similar cones, the height of one being three times that of the other.

Then (1) the radius and hence the circumference of the base of the first are three times the radius and circumference of the second respectively.

(2) The curved surface of the first $= 3^2 \times$ that of the second.

(3) The volume or weight of the first $= 3^3 \times$ volume or weight of the second.

To generalise, using the symbols L, S, and V for side, surfaces and volumes respectively—

If the ratio of the linear dimensions of two similar figures is represented by $\frac{L_1}{L_2}$, then $\frac{S_1}{S_2} = \left(\frac{L_1}{L_2}\right)^2$ . . . . (1)

and $\frac{V_1}{V_2} = \left(\frac{L_1}{L_2}\right)^3$ . . . . (2)

If it is desired to connect up volumes with surfaces—

By cubing equation (1) $\left(\frac{S_1}{S_2}\right)^3 = \left(\frac{L_1}{L_2}\right)^6$

By squaring equation (2) $\left(\frac{V_1}{V_2}\right)^2 = \left(\frac{L_1}{L_2}\right)^6$

Hence— $\left(\frac{V_1}{V_2}\right)^2 = \left(\frac{S_1}{S_2}\right)^3$

$$\text{or} \quad \left(\frac{V_1}{V_2}\right) = \left(\frac{S_1}{S_2}\right)^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Example 32.—A conical lamp-shade has the dimensions shown in Fig. 50. Find the height of the cone of which it is a part.

Let x inches be the height of the top triangle, viz. ABC.

Then ABC and ADE are similar triangles, hence the ratio $\frac{\text{height}}{\text{base}}$ is the same for both.

i. e., $\frac{x}{6}$ for the small triangle must $=$ $\frac{x+4}{10}$ for the large triangle.

Then, by multiplying across—

$$10x = 6x + 24$$
$$4x = 24$$
$$x = 6''$$

$\therefore$ Total height of cone $= 6 + 4 = \underline{10''}$.

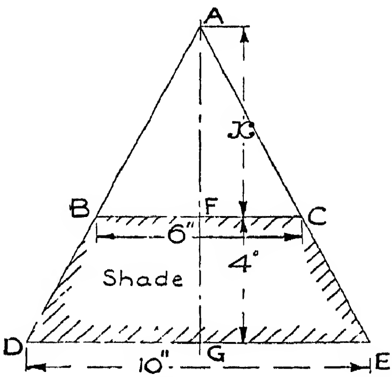


Fig. 50.

It is convenient at this stage to insert the proofs of the rules for the lateral surface and the volume of a frustum, given on p. 117.

In Fig. 50 let the height or thickness FG of the frustum BCED be denoted by h; let A be the area of the end DE and let B be the area of the end BC. [*Note.*—The figure is taken in these proofs to be the elevation of a *pyramid*, so that the proofs may be perfectly general.]

Then, from the similarity of the triangles ABC and ADE—

$$\frac{\text{perimeter of end DE}}{\text{perimeter of end BC}} = \frac{AD}{AB} = \frac{AB + BD}{AB} = 1 + \frac{BD}{AB}$$

whence $\frac{\text{p. of DE}}{\text{p. of BC}} - 1 = \frac{BD}{AB}$ (p. being written to denote perimeter)

$$\text{or} \quad \frac{\text{p. of DE} - \text{p. of BC}}{\text{p. of BC}} = \frac{BD}{AB} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Lateral surface of frustum BCED = lateral surface of pyramid ADE — lateral surface of pyramid ABC

$= \frac{1}{2}(\text{p. of DE} \times \text{AD}) - \frac{1}{2}(\text{p. of BC} \times \text{AB})$

$= \frac{1}{2}[(\text{p. of DE} \times \text{AB}) + (\text{p. of DE} \times \text{BD}) - (\text{p. of BC} \times \text{AB})]$

$= \frac{1}{2}[\text{AB}(\text{p. of DE} - \text{p. of BC}) + (\text{p. of DE} \times \text{BD})]$

Substituting from equation (1) $= \frac{1}{2}[(\text{p. of BC} \times \text{BD}) + (\text{p. of DE} \times \text{BD})]$

$= \frac{1}{2} \times \text{BD} \times \text{sum of perimeters of ends}$

$= \frac{1}{2}$ sum of perimeters of ends $\times$ slant thickness.

Again, since ABC and ADE are similar solids, the areas of their respective bases are proportional to the squares of their respective heights —

$$\text{or} \quad \frac{A}{B} = \frac{(AG)^2}{(AF)^2}$$

By transposition—
$$B = A \times \frac{(AF)^2}{(AG)^2} \quad \ldots \ldots \ldots \ldots \quad (2)$$

Also by extraction of the square root

$$\frac{\sqrt{A}}{\sqrt{B}} = \frac{AG}{AF} \quad \ldots \ldots \ldots \ldots \quad (3)$$

Volume of frustum BCED

$$= \text{vol. of pyramid ADE} - \text{vol. of pyramid ABC}$$
$$= \tfrac{1}{3} \times A \times AG - \tfrac{1}{3} \times B \times AF$$

By substitution from equation (2)

$$= \tfrac{1}{3} \times A \times AG - \tfrac{1}{3} \times A \times \frac{(AF)^2}{(AG)^2} \times AF$$
$$= \tfrac{1}{3}A\left[\frac{(AG)^3 - (AF)^3}{(AG)^2}\right]$$

Factorising the numerator (see p. 53)—

$$= \frac{\tfrac{1}{3}A[(AG-AF)][(AG)^2 + (AG \times AF) + (AF)^2]}{(AG)^2}$$

$$[AG - AF = h] \quad = \tfrac{1}{3}h\left[\frac{A \times (AG)^2}{(AG)^2} + \frac{A \times AG \times AF}{(AG)^2} + \frac{A \times (AF)^2}{(AG)^2}\right]$$

Substituting from equation (3)—

$$= \tfrac{1}{3}h\left[A+\left(A\times\frac{\sqrt{B}}{\sqrt{A}}\right)+\left(\frac{B}{A}\times A\right)\right]$$

$$= \tfrac{1}{3}h[A+\sqrt{AB}+B].$$

Example 33.—A surveyor's chain line is to be continued across a river. Describe a method by which the line may be prolonged and show how the required distance may be deduced.

Suppose C is a point on the line : select some station A on the opposite bank (Fig. 51) and put A, B and C in line. Set off BD and CE as offsets at right angles, so that E, D and A are in a straight line.

Then—
$$\frac{AB}{BD} = \frac{DF}{FE} = \frac{BC}{CE - BD}$$

$$\textit{i. e.,}\ \underline{AB = \frac{BC \times BD}{CE - BD}} \quad \text{or AB is found.}$$

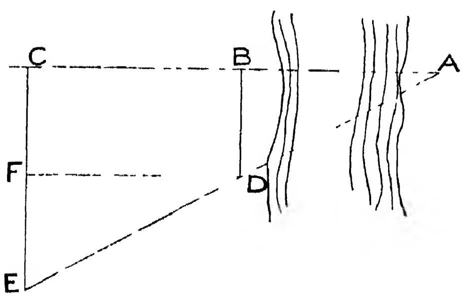


Fig. 51.

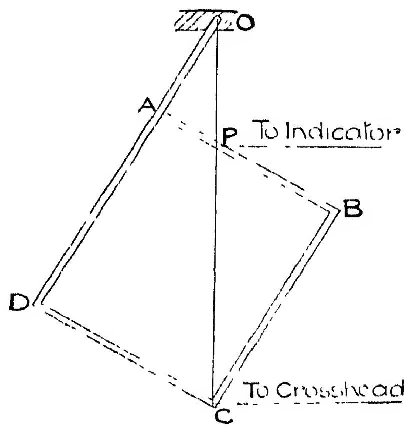


Fig 52.

Example 34.—The actual area of a field is 5 acres : on the plan it is represented by an area of 50 sq. ins. To what scale is the plan drawn ?

We are told that 50 sq. ins. represent 5 acres or 50 sq. chains.

Hence— 1 sq in. represents 1 sq. chain

or 1″ represents 1 chain.

So that the scale is 1″ to a chain, or the representative fraction

$$= \frac{1}{22 \times 36} = \underline{\frac{1}{792}}$$

Example 35. The heating surfaces of two exactly similar boilers are 850 and 996 sq. ft. respectively. The capacity of the second being 750 gallons, what is the capacity of the first?

It is not necessary to determine the ratio of the linear dimensions, for statement (3) on p. 123 can be used, since the capacities are proportional to the volumes.

Now— $S_1 = 850, \quad S_2 = 996, \quad V_2 = 750,$ and V_1 is required.

$$\frac{V_1}{V_2} = \left(\frac{S_1}{S_2}\right)^{\frac{3}{2}} = \left(\frac{850}{996}\right)^{\frac{3}{2}}$$

$$\text{or} \quad V_1 = 750 \times \left(\frac{850}{996}\right)^{\frac{3}{2}}$$

$$\begin{aligned}\log V_1 &= \log 750 + 1{\cdot}5(\log 850 - \log 996)\\ &= 2{\cdot}8751 + 1{\cdot}5(2{\cdot}9294 - 2{\cdot}9983)\\ &= 2{\cdot}8751 - 1{\cdot}5 \times {\cdot}0689\\ &= 2{\cdot}7717\end{aligned}$$

$$\therefore V_1 = \underline{591 \text{ gallons.}}$$

An application of similar figures is found in the engraving machine and in the reducing gear used in connection with indicators. In Fig. 52 such a gear is represented. The movement of the crosshead is reduced, the ratio of reduction being—

$$\frac{\text{movement of crosshead}}{\text{movement of pencil}} = \frac{OC}{OP} \text{ or } \frac{DC}{AP}$$

The performance of large ships can be investigated by comparing with that of small models. Here, again, the laws of similarity are of great importance.

Suppose the model is built to a scale of $\frac{1}{50}$, *i. e.*, any length on the ship is fifty times the corresponding length on the model.

Then its wetted surface is $\frac{1}{2500}$ of that of the ship; while its displacement is $\frac{1}{125000}$ $\left(i.e., \frac{1}{50^3}\right)$ of the ship's displacement. Also the resistance to motion of the ship would be 50^3 times that of the model.

An instance of the use of the rules for similar figures is seen in the following:—

If the circumference of a circle of 3″ diam. is 9·426″ and its area is 7·069 sq. ins., then the circumference of a circle of 30″ diam. will be $9{\cdot}426 \times 10$, *i. e.*, 94·26″, and its area $= 7{\cdot}069 \times 10^2 = 706{\cdot}9$ sq. ins.

Hence one can form a most useful table, to be used for all sizes of circles.

Diam.	Circumference.	Area.
1	3·142	·785
2	6·283	3·142
3	9·426	7·069
4	12·566	12·566
5	15·708	19·635
6	18·850	28·274
7	21·991	38·485
8	25·133	50·265
9	28·274	63·617

Suppose the circumference of a circle of ·375″ is required.

$\odot$ce of circle of ·3″ diam. $= \frac{1}{10}$ of $\odot$ce of $\odot$ of 3″ diam. $=$ ·9426
$\odot$ce of circle of ·07″ diam. $= \frac{1}{100}$ of $\odot$ce of $\odot$ of 7″ diam. $=$ ·2199
$\odot$ce of circle of ·005″ diam. $= \frac{1}{1000}$ of $\odot$ce of $\odot$ of 5″ diam. $=$ ·0157

$\therefore$ $\odot$ce (·375″ diam) $=$ 1·1782

Again, the area of a circle of ·8″ diam.

$$= \frac{1}{10^2} \times \text{area of circle of } 8'' \text{ diam.}$$
$$= \cdot 503 \text{ sq. in.}$$

Exercises 17.—On Spheres.

1. Find the surface and volume of a sphere of 7·14″ diam.

2. A sphere of 8″ diam. is weighed in air and its weight is found to be 80 lbs. Its weight in water is 70·35 lbs. If Specific Gravity $= \frac{\text{weight of solid}}{\text{weight of equal vol. of water}}$ and loss of weight $=$ weight of water displaced, find the specific gravity of the material of which this sphere is composed and the weight of 1 cu. ft. of it.

3. Find the volume of a spherical shell whose external diam. is 4·92″, the thickness of the metal being $\frac{1}{8}''$.

4. A storage tank, in the form of a cylinder with hemispherical ends, is $23\frac{1}{4}$ ft. long over all and 4 ft. in diam. (these being the internal measurements). Calculate the weight of water contained when the tank is half full.

5. A sphere of diameter 22 cms. is charged with 157 coulombs of electricity. Find the surface density (coulombs per sq. cm.), which is given by $\frac{\text{quantity in coulombs}}{\text{area in sq cms.}}$.

6. The volume of a sphere is 84·2 cu. cms. : find its diam

7. Find the surface and volume of the zone of a sphere of radius 8″ if the thickness of the zone is 2″ and the radius of its larger end is 6″.

8. The weight of a hollow sphere of gun-metal of external diam. 6″ was found to be 22·3 lbs. Find the internal diam., if the gun-metal weighs ·3 lb. per cu. in.

9. In a Brinell hardness test a steel ball of diam. 10 mm. was pressed on to a plate, and the diam. of the impression was measured to be 3·15 mm. Find the hardness number for the material of the plate if the load applied was 5000 kgrms. and hardness number $= \dfrac{\text{load}}{\text{curved area of depression}}$. (Compare Example 14, p. 98.)

On Similar Figures.

10. Find the area of section of the masonry dam shown at (*b*), Fig. 47.

11. The symmetrical template shown at (*c*), Fig. 47, was cut too short along the bottom edge; the length dimensioned as 2·06″ should be 2·22″. Find the amount x to be cut off in order to bring the edge to the required length.

12. A plan is drawn to a scale of $\frac{1}{80}$. The area on the paper is 428 $\square''$. What is the actual area of the plot represented?

13. Find the diam. of the small end of the conical roller for a bearing shown in Fig. 53.

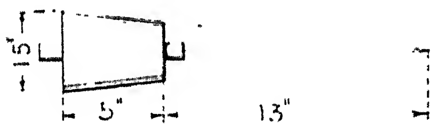


Fig. 53.

14. The wetted surface of a ship of 6500 tons displacement is 26000 $\square'$. What will be the wetted surface of a similar vessel whose displacement is 3000 tons?

15. One side of a triangle is 12″. Where must a point be taken in it so that a parallel to the base through it will be cut off a triangle whose area is $\frac{3}{8}$ that of the original triangle?

16. The parallel sides of a trapezoid are 10″ and 16″, and the other sides are 5″ and 7″. Find the area of the total triangle obtained by producing the non-parallel sides.

17. The surface of one sphere is 6 times that of another. What is the ratio of their volumes? Find also the ratio of their diameters.

18. The area of a field was calculated, from actual measurements taken, to be 52·7 acres. The chain with which the lines were measured was tested immediately after the survey and found to be 100·8 links long. Find the true area of the field (1 chain = 100 links and 10 sq. chains = 1 acre).

19. A plank of uniform thickness is in the form of a trapezoid where one end is perpendicular to the parallel sides and is 12 ft. long. The parallel sides are 12″ and 9″ respectively. At what distance from the narrower end must the plank be cut (the cut being parallel to the 12″ and 9″ sides) so that the weights of the two portions shall be the same?

20. A trapezoid has its parallel sides 24″ and 14″ and the other sides each 8″. Find the areas of the 4 triangles formed by the diagonals.

21. The length of a model of a ship was 10·75 ft., whilst that of the ship itself was 430 ft. If the displacement of the ship was 11600 tons, what was the displacement of the model?

22. To ascertain the height of a tower a post is fixed upright 27 ft. from the base of the tower, with its top 12 ft. above the ground. The

observer's eye is 5′-4″ above the ground and at 3 ft. from the post when the tops of the tower and post are in line with the eye. Find the height of the tower.

23. What should be the diameter of a pipe to receive the discharge of three pipes each $\frac{3}{4}''$ diam.?

The Rules of Guldinus.—These deal with surfaces and volumes of solids of revolution.

A *solid of revolution* is a solid generated by the revolution of a plane figure about some axis; *e. g.*, a right-angled triangle revolving about one of its perpendicular sides traces out a right circular cone; and a hyperbola rotating about either of its axes generates a hyperboloid of revolution.

For the cases with which we deal here the axis must not cut the revolving section, and all sections perpendicular to the axis of revolution must be circular.

The rules are—

Surface of solid of revolution = perimeter of revolving figure × path of its centroid.

Volume of solid of revolution = Area of revolving figure × path of its centroid.

The *centroid* of a plane figure is the *centre of gravity* of an extremely thin plate of the same shape as the figure. The motion of the centroid may be taken to be the mean of the motions of all the little elements of the curve or area.

These rules are of great value in dealing with awkward solids; *e. g.*, suppose the volume of the nose of a projectile is required, it being generated by the revolution of a curved area round the axis of the projectile (see Fig. 54).

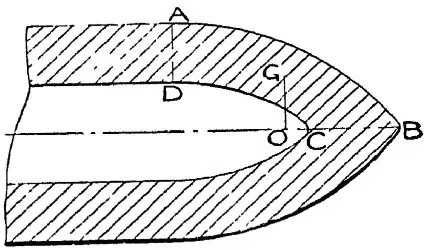


Fig. 54.

The area of ABCD and the position of its centroid G can be found by rules to be detailed later, and then—

$$\text{Vol. of nose} = \text{area of revolving figure} \times \text{path of its centroid}$$
$$= (\text{ABCD}) \times (2\pi \times \text{OG})$$

A simpler example is that of a flywheel rim.

Example 36.—Find the weight of the rim of a cast-iron flywheel of 5 ft. outside diam.; the rim being rectangular, 8″ across the face and 4″ thick radially. (C.I. weighs ·26 lb. per cu. in.)

Here, area of revolving figure $= 8 \times 4$
also the mean diam. $= 56''$
whence path of centroid $= \pi \times 56$
and vol. of rim $= \pi \times 56 \times 32$ cu. ins.
$\therefore$ Weight of rim $= \pi \times 56 \times 32 \times \cdot 26$ lb.
$= \underline{1{,}460 \text{ lbs.}}$

The positions of the centroids (G) for a few of the simple figures is here given (Fig. 55).

Triangular area (1) . . . . . OG $= \frac{1}{3}h$, GD $= \frac{1}{3}$BD $\left\{ \begin{array}{l} \text{BD is the median,} \\ \textit{i.e.}, \text{AD} = \text{DC} \end{array} \right\}$

Semicircular arc (2) . . . . . OG $= \dfrac{2r}{\pi} = \cdot 637\, r$

Semicircular area (2) . . . . $\text{OG}_1 = \dfrac{4r}{3\pi} = \cdot 424\, r$

Semicircular perimeter (2) . . $\text{OG}_2 = \dfrac{2r}{2+\pi} = \cdot 389\, r$
(*i. e.*, arc + diameter).

Parabolic segment (3) . . . . OG $= \frac{3}{5}h$

Semi-parabolic segment (4) . . OQ $= \frac{3}{5}h$, QG $= \frac{3}{8}b$

Area over parabolic curve (5) . OG $= \cdot 3h$; GP $= \dfrac{b}{4}$

Area of quadrant of circle (6) . OG $=$ GP $= \cdot 424\, r$

Area over circular arc (*quadrant*) *or Fillet* (7). OG $=$ GP $= \cdot 223\, r$

Trapezoid (8). Bisect AB at E and DC at F. Join EF. Set off BM $=$ DC and DN $=$ AB. Intersection of MN and EF is at G, or, by calculation, OG $= \dfrac{h}{3}\left(\dfrac{2a+b}{a+b}\right)$

Quadrilateral (9). Bisect AC at F and BD at E.

Make OP $= \frac{2}{3}$OE and OQ $= \frac{2}{3}$OF

Through Q draw a parallel to BD and through P, a parallel to AC. The intersection of these gives G, the centroid of ABCD.

Exercises 18.—On Guldinus' Rules.

1. An isosceles triangle, each of whose equal sides is 4 ft. and whose altitude is 3 ft., revolves about an axis through its vertex parallel to its base. Find the surface and volume of the solid generated.

2. Find the surface and volume of the anchor-ring described by a circle of 3″ diam. revolving round a line 4″ from the nearest point on the circle.

3. Find the surface and volume described by the revolution of a semicircle of 4″ diam. about an axis parallel to its base and 5″ distant from it.

4. An equilateral triangle of 5″ side revolves about its base as axis. Find the surface and volume of the double cone thus generated.

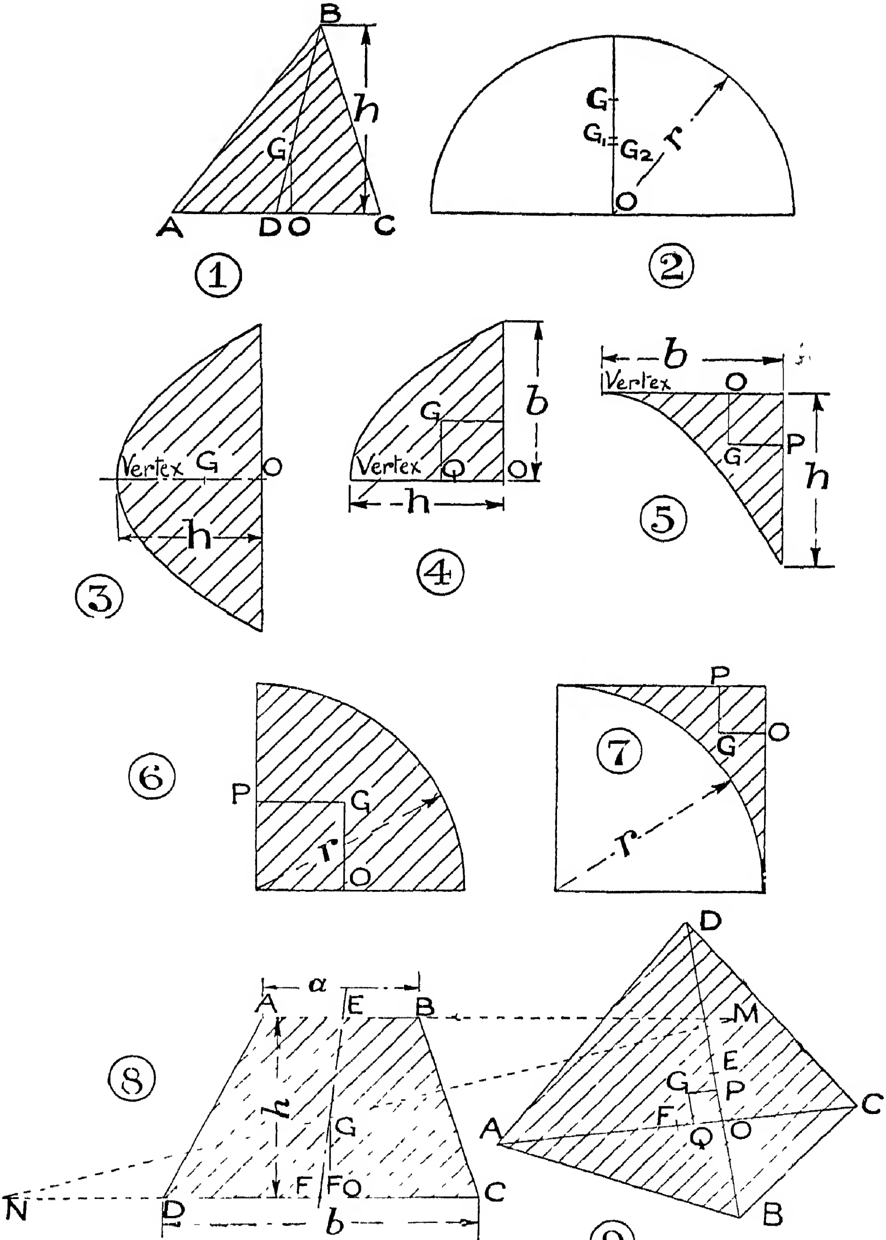


Fig. 55.—Positions of Centroids (G) for Simple Figures.

5. A parabola revolves about its axis. Compare the volume of the paraboloid thus generated with that of the circumscribing cylinder.

6. At (*a*), Fig. 56, is shown in section the winding of the secondary wire of an induction coil. Find the volume of the winding.

7. Calculate the weight, in mild steel weighing ·287 lb. per cu. in., of the spindle weight for a spring compressor shown at (*b*), Fig. 56.

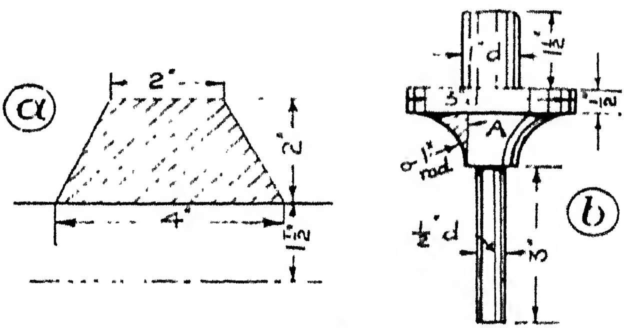


Fig. 56.

[*Hints.* **Area of a fillet,** as at A, **·215r^2** where r is the radius of the circular arc.

For the position of the centroid of a fillet refer to (7), in Fig. 55, and also to p. 130.]

Application to Calculation of Weights. – When calculating weights two rules should be borne in mind in addition to the foregoing.

(*a*) The solid should be broken up into simple parts, *i. e.*, those whose volumes can be found by the rules already given; and (*b*) suitable approximations should be made wherever possible. Circular segments may be replaced by parabolic segments if the rules for the latter are easier, the rounding of corners may be neglected, unless very large, mean widths may be estimated, etc.

For purposes of reference the table of weights of materials and other useful data are inserted here; but the values given must be considered as average values.

WEIGHTS AND DENSITIES OF METALS.

METAL.	Weight in lbs. per cu. in.	Weight in lbs. per cu. ft.	Specific Gravity (grms. per cu. cm.).
Cast iron	·26	450	7·21
Wrought iron	·28	485	7·76
Steel	·29	500	8·04
Brass or Gun-metal	·30	518	8·31
Copper (Cu)	·32	553	8·87
Lead (Pb)	·41	710	11·34
Tin (Sn)	·27	465	7·48
Aluminium (Al)	·0932	161	2·58
Zinc (Zn)	·26	450	7·21

WEIGHTS AND DENSITIES OF EARTH, SOIL, ETC.

MATERIAL.	Slate.	Granite.	Sandstone.	Chalk.	Clay.	Gravel	Mud.
WEIGHT (cwt. per cu. yd.) .	43	42	39	36	31	30	25

Useful Data.—Wrought-iron plate weighs about 10 lbs., and steel 10·4 lbs. per sq. ft. of area per $\frac{1}{4}''$ of thickness, *i. e.*, 8 sq. ft. of W.I. plate $\frac{3}{4}''$ thick would weigh $10 \times 8 \times 3 = 240$ lbs.

Wrought-iron bar or rod weighs about 10 lbs., and steel 10·4 lbs. per yard for every sq. in. of section.

Wrought-iron bar or rod, 1″ diam., weighs 8 lbs. and steel 8·2 lbs. per yard: also the weight is proportional to the diameter squared; thus, a yard of steel bar 2″ in diam. would weigh $2^2 \times 8{\cdot}2$ or 32·8 lbs.

Four hundred cu. ins. of wrought iron, 430 cu. ins. of cast iron, 390 cu ins. of steel, each weigh about 1 cwt.

A few examples are here worked out to give some idea of the method of treatment.

Example 37.—Calculate the weight, in cast iron, of the D slide valve shown in Fig. 57.

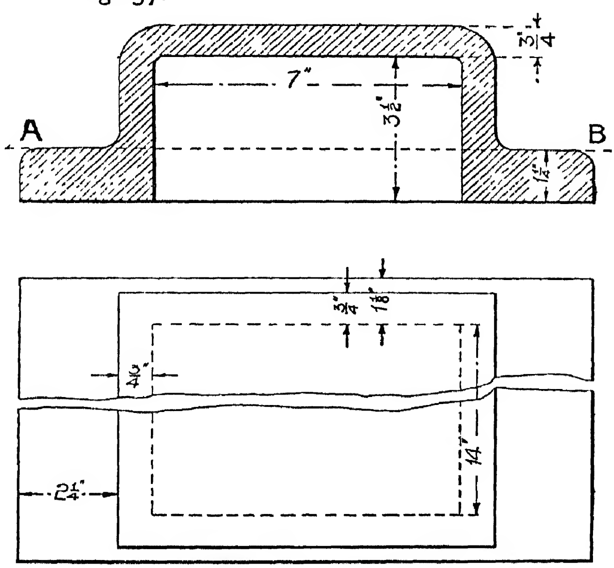


Fig. 57.—D Slide Valve.

In many cases where the solid is partially hollowed it is best to treat first as a solid and then subtract the volume cut away.

First, considering as a solid —

Vol. above AB $= 15{\cdot}5 \times 8{\cdot}5 \times 3$ cu. ins. $= 395$ cu. ins.

Vol. below AB $= 16{\cdot}25 \times 13 \times 1{\cdot}25 \qquad = 264$ „

∴ Total vol. (as a solid) $\qquad = 659$ „

To be subtracted —

Vol. of cavity $= 14 \times 7 \times 3{\cdot}5 \qquad = 343$ „

∴ Net vol. $\qquad = 316$ „

and weight $= 316 \times {\cdot}26 = \underline{82{\cdot}1 \text{ lbs.}}$

Example 38. Find the weight of a plate for a cast-iron tank. The plate (see Fig. 58) is 24″ square and $\frac{3}{8}$″ thick; there are 20 ribs, each $\frac{1}{2}$″ × $1\frac{1}{4}$″ × $1\frac{1}{4}$″, and 24 bolt-holes in the flanges, each $\frac{5}{8}$″ square; also the flanges are $23\frac{1}{4}$″ × $\frac{1}{2}$″ × $1\frac{1}{4}$″.

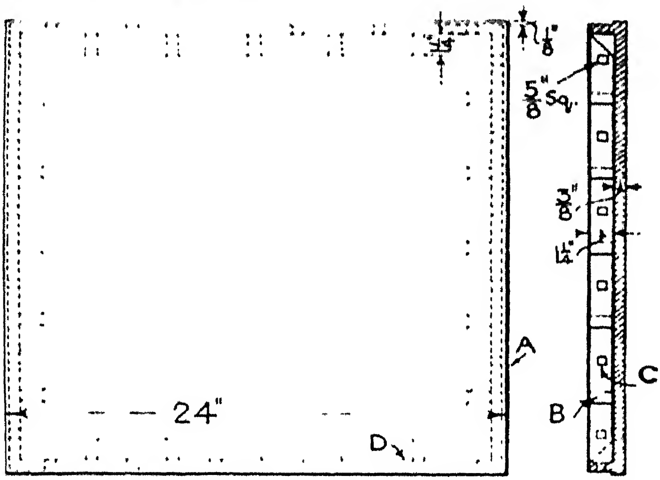


Fig. 58.—Plate for Tank.

Dealing with the separate portions :—

Flat Plate (A).

Vol. $= 24 \times 24 \times \frac{3}{8}$ . . . . 216 cu. ins.

Flanges (D).

Length $= (2 \times 22\frac{3}{4}″) + (2 \times 23\frac{1}{4}″) = 93″$

∴ Vol. $= 93 \times \frac{5}{4} \times \frac{1}{2}$ . . . . $= 58{\cdot}1$ „

Ribs (B).

Area of face of one $= \frac{1}{2} \times \frac{5}{4} \times \frac{5}{4}$

∴ Vol. of 20 each $\frac{1}{2}$″ thick $= \frac{1}{2} \times \frac{5}{4} \times \frac{5}{4} \times \frac{1}{2} \times 20 = \underline{7{\cdot}8}$ „

Gross vol. $= 281{\cdot}9$ „

Subtract for 24 bolt-holes (C); $24 \times \frac{5}{8} \times \frac{5}{8} \times \frac{1}{2} = \underline{4{\cdot}7}$ „

Net vol. $= 277{\cdot}2$ „

∴ Weight $= 277{\cdot}2 \times {\cdot}26 = \underline{72 \text{ lbs.}}$

Example 39.—Find the weight of the wrought-iron stampings for a dynamo armature as shown in Fig. 59, 14″ diam. and 10″ long, 10% of the length being taken off by ventilation and insulation. There are 3 ventilating ducts, each 6″ internal diam. and 1″ thick, the gaps between these being $1\frac{1}{2}$″ long; and also 60 slots, each $\frac{7}{8}$″ by $\frac{3}{8}$″. The shaft is 3″ diam.

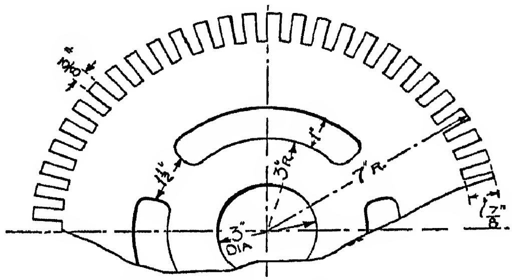


Fig 59.—Stamping for Dynamo Armature.

Note.—The stampings are only thin and are separated one from the other by some insulator; also there would be a small gap for ventilation purposes, and hence the actual length of the stampings is less than 10″; in this case it is to be taken as 90% of 10″, *i. e.*, 9″.

Area of face of stamping $= \frac{\pi}{4} \times 14^2$ . . . . $= 154$ sq. ins.

To be subtracted—

Area of 60 slots $= 60 \times \frac{7}{8} \times \frac{3}{8}$ . . . . $= 19 \cdot 7$ „

Mean length of ventilating ducts $= (\pi \times 7) - (3 \times 1\frac{1}{2})$

$= 17 \cdot 5$″

$\therefore$ Area $= 17 \cdot 5 \times 1$ . . . . $= 17 \cdot 5$ „

Area of hole for shaft $= \frac{\pi}{4} \times 3^2$ . . . . $= 7 \cdot 1$ „

Thus the total area to be subtracted . . . . $= 44 \cdot 3$ „

or the net area of the stamping . . . . $= 109 \cdot 7$ „

Then the volume $= 109 \cdot 7 \times 9$ cu. ins.

and the weight $= 109 \cdot 7 \times 9 \times \cdot 28$ lbs.

$= \underline{277}$ lbs.

Example 40.—Find the weight of 150 yards of steel chain, the links of which have the form shown in Fig 60.

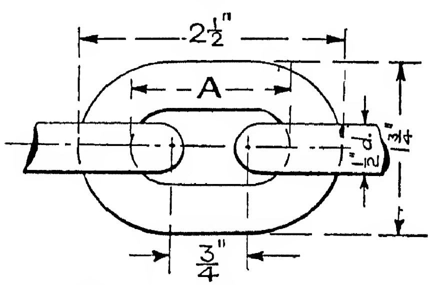


Fig. 60.—Chain Link.

The effective length A of a link is the inside length, provided that a number of yards of chain are being considered. (For small lengths this is not quite correct.)

In this case the effective length of a link $= 1\frac{1}{2}$″, so that in 1 yard of the chain there are $\frac{36}{1\frac{1}{2}}$, *i. e.*, 24 links, or in 150 yards of the chain there are 3600 links.

The mean length of 1 link = ⊙ce of circle of $1\frac{1}{4}$″ diam. $+ (2 \times \frac{3}{4}$″$)$

$= 3 \cdot 93 + 1 \cdot 5 = 5 \cdot 43$″.

Now 1″ diam. steel rod weighs 8·2 lbs. per yard (see p. 133); therefore $\frac{1}{2}$″ diam. steel rod weighs $\frac{8\cdot2}{2^2}$, *i. e.*, 2·05 lbs. per yard.

$$\text{Hence, weight of 1 link} = \frac{5\cdot43}{36} \times 2\cdot05 \text{ lbs.}$$

$$\text{and weight of 3600 links} = \frac{5\cdot43 \times 2\cdot05 \times 3600}{36} \text{ lbs.} = \underline{1115 \text{ lbs.}}$$

Example 41.—Two straight cast-iron pipes, making an angle of 135° with one another, have the centres of their ends 2 ft. apart (in a straight line). They are to be joined by a curved pipe (as in Fig. 61), 4″ external and 3″ internal diam., with flanges 8″ diam. and $\frac{1}{2}$″ thick. Find the weight of the curved pipe if the flanges each have five bolt-holes, of $\frac{5}{8}$″ diam.

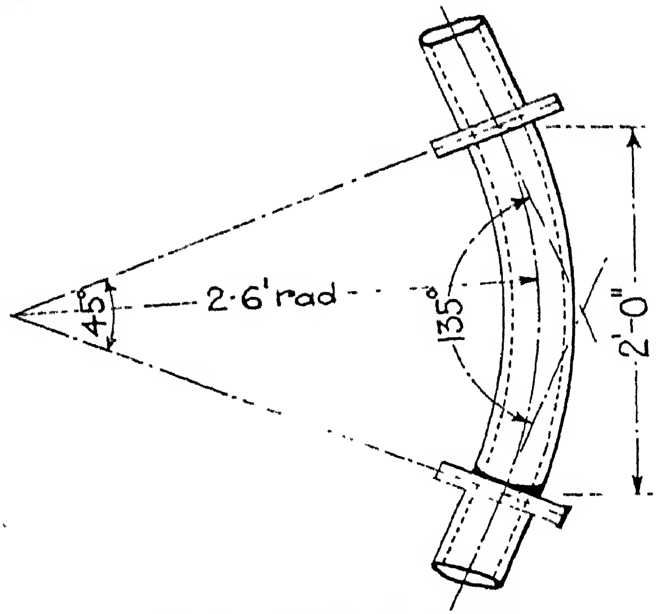


Fig. 61.– Curved Cast-iron Pipe.

This is a useful example on the application of Guldinus' rule.

Path of centroid = arc of circle, which is $\frac{45}{360}$ or $\frac{1}{8}$ of the circumference.

By drawing to scale (or by Trigonometry), the radius is found to be 2·6 ft.

$$\therefore \text{ Path of centroid} = \tfrac{1}{8} \times \pi \times 5\cdot2 = 2\cdot04 \text{ ft.}$$

and length of the path of the centroid between the flanges–

$$= 2\cdot04 \text{ ft.} - (2 \times \tfrac{1}{2}'')$$
$$= 1\cdot96 \text{ ft.} = 23\cdot5''.$$

Area of revolving section $= \left(\frac{\pi}{4}\times 4^2\right) - \left(\frac{\pi}{4}\times 3^2\right) = 5{\cdot}5$ sq. ins.

hence the volume of the solid between the flanges $= 23{\cdot}5 \times 5{\cdot}5$ cu. ins.
$= 129$ cu. ins.

Vol. of 2 flanges, each $\frac{1}{2}''$ thick, 8″ external and 3″ internal diam.

$$= 2\times\tfrac{1}{2}\times\frac{\pi}{4}(8^2-3^2) = \frac{\pi}{4}\times 55 = 43{\cdot}2 \text{ cu. ins.}$$

∴ Gross vol. of bend = 172·2 cu. ins.

To be subtracted—
Vol. of ten $\frac{5}{8}''$ diam. holes :

Diam.	Length.	Vol.
·625	5″	1·5

∴ Net vol. of bend = 170·7 cu. ins.
and weight = 170·7 × ·26 = 44·4 lbs.

Example 42.—Find the weight of the wrought-iron crank shown in Fig. 62, allowing for the horns at the junctions of the web and bosses.

Dealing with the three parts separately :—

Vol. of the upper boss is the difference of the volumes of two cylinders—

Diam	Length.	Vol.
12″	8″	908
6″	8″	227

net volume = 681 cu. ins.

Similarly, vol. of the lower boss—

Diam	Length.	Vol.
15″	7·25	1282
9″	7·25	462

net volume = 820 cu. ins.

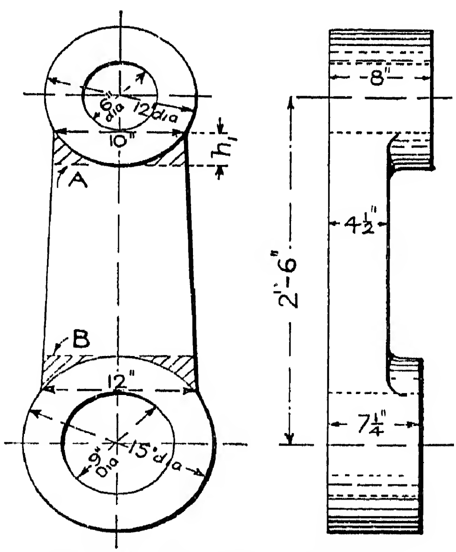


Fig. 62.—Wrought-iron Crank.

The horns can be allowed for by adding $\frac{1}{3}$ of the height of each to the length of the web (*i. e.*, we replace the circular segment by a parabolic segment, because the rule for the area is simpler).

To find the height h_1 of the top horn A, $\{a_1 = 5,\ r_1 = 6\}$.

$$h_1 = r_1 - \sqrt{r_1^2 - a_1^2} = 6 - \sqrt{36-25} = 6 - 3{\cdot}32 = 2{\cdot}68''.$$

Hence add $\frac{1}{3}$ of 2·68″, *i. e.*, ·9″ to the length of the web.

For the lower horn B, $a_2 = 6''$, $r_2 = 7.5''$

$$\therefore \quad h_2 = 7.5 - \sqrt{7.5^2 - 6^2}$$
$$= 3''$$

Hence add on $1''$ to the length of the web.

Thus the effective length of the web—

$$= 30'' + 1'' + .9'' - 7\tfrac{1}{2}'' - 6''$$
$$= 18.4''$$

its mean width— $= 11''$

so that its vol.— $= 18.4 \times 11 \times 4.5 = 910$

$\therefore$ Total vol. of crank $= 2411$ cu. ins.

$\therefore$ Weight $= 2411 \times .28 = \underline{675 \text{ lbs.}}$

Example 43.—Determine the number of $1''$ diam. rivets, as at (*a*) Fig. 63 (*i. e.*, with snap or spherical heads) to weigh 1 cwt. (Given that $d = t + \frac{7}{16}''$ and length $= 2t$.)

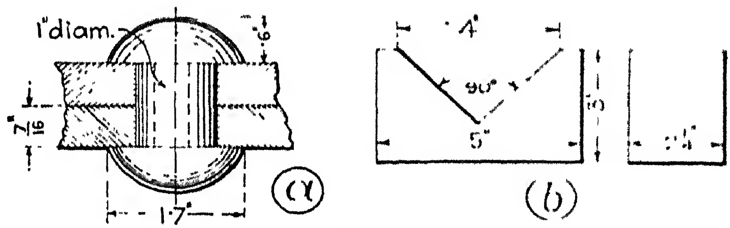


Fig. 63.

If $d = 1''$ then $t = \frac{9}{16}''$ and length $= 1\tfrac{1}{8}''$.

For the heads, a rough approximation is that the two together are one-half the volume of a sphere of diameter $1.8d$, this being the diameter of the sphere of which the heads are segments, but the result will be somewhat more accurate if $.52$ is taken in place of $.5$. (This figure is arrived at by the use of the rule given on p. 121 for the segment of a sphere.)

Then— vol. of heads $= .52 \times \frac{4}{3}\pi \times .9^3 = 1.58$ cu. ins.

vol. of body {Diam. $= 1''$, length $= 1.125''$} $= .88$ " "

or vol. of 1 rivet $= 2.46$ " "

$$\therefore \quad \text{Number of } 1'' \text{ rivets per cwt.} = \frac{112}{2.46 \times .29} = \underline{157}.$$

Example 44.—Find the weight of the cast-iron hanger bearing shown in Fig. 64.

This example illustrates well the method of breaking a solid up into its component parts; the different parts being dealt with according to the letters on the diagram.

Treating first as a solid throughout—

cub. ins.

A. Cuboid, length = 12", breadth = 6·75", thickness = ·75".
Volume = 12×6·75×·75 . . . . = 60·75

B. 4 cylinders, of diam. 1·625" and total length = 5"
Volume (obtained from the slide rule) = 10·35

C. Area of section = semicircle + rectangle—

$$= \left(\frac{\pi}{8}\times 5{\cdot}5^2\right) + (5{\cdot}5\times 2{\cdot}5)$$

$$= 10{\cdot}86 + 13{\cdot}75 = 25{\cdot}61$$

Volume = 25·61×2·75 . . . . = 70·48

D. Cylinder, diam. = 4", length = 4"
Volume . . . . = 50·30

E. Cylinder, diam. = 4·5", length = ·75
Volume . . . . = 11·92

F. 4 cylinders, diam. = 2", total length = 1"
Volume . . . . = 3·14

Gross Volume . . . . = 206·94

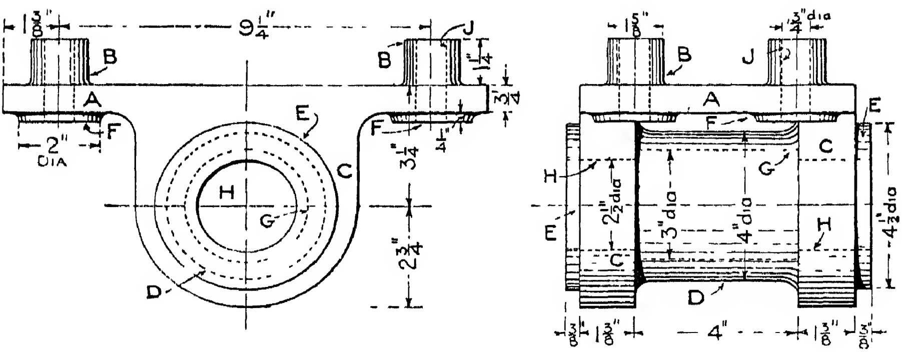


Fig. 64.—Cast-iron Hanger Bearing.

To be subtracted—

cub ins

G. Cylinder, diam. = 3", length = 4"
Volume . . . . = 28 20

H Cylinder, diam. = 2½", length = 3½"
Volume . . . . = 17·15

J. 4 cylinders, diam. = ·75", total length = 9"
Volume . . . . = 3·97

Total volume to be subtracted = 49·32

Net volume . . . . = 157·62

Hence, weight = 157 6 × ·26
= <u>41 lbs.</u>

Exercises 19.—On Calculation of Weights.

1. Find the weight of the cast-iron Vee-block shown at (*b*), Fig. 63.
2. Find the weight in steel of the crank axle shown in Fig. 65

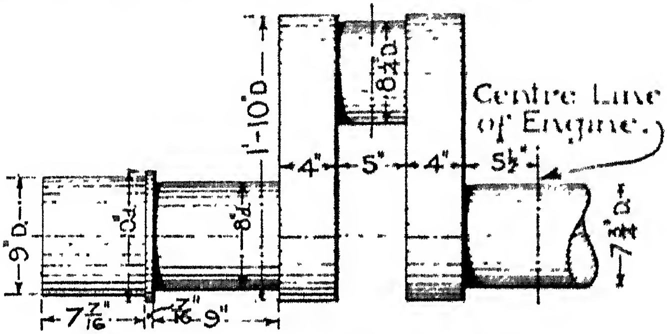


Fig. 65.—Steel Crank Axle.

3. Find the weight of sheet iron in a rectangular measuring tank; the metal being ¼" thick. Inside dimensions of the tank are 4'-6" by 3'-6" by 7'-0" deep. Cut from the sides are openings to accommodate fittings as follows: One rectangular hole 4'-0" by 2", two elliptical holes 4" × 2", two circular holes 4" diam. and eight ⅝"-diam. bolt holes.

4. Determine the weight of a wrought-iron boiler end plate, 8 ft. diameter and $\frac{9}{16}$" thick. There are two flue holes, each 2 ft. diam. and an elliptical manhole 18"×12".

5. Find the weight of 22 yards of iron chain. The links are elliptical and are made of elliptical metal 1"×½", the greatest width of section being at right-angles to the plane of the link. The mean lengths of the axes of the link are 4" and 2½".

6. How many ¾"-diam. snap-headed rivets weigh 1 cwt.? (Compare with *Example* 43, p. 138.)

7. Find the weight in cast iron of the flywheel of a steam engine having a rectangular rim, 7" wide by 4" radial thickness; six straight arms of elliptical section, the axes of the ellipse being 4½" and 2¼"; a boss 7½" wide, 9" diam. and 4½" bore. The outer diameter of the wheel is 7'-9".

8. Required the weight of the cast-iron anchor plate shown in Fig. 66.

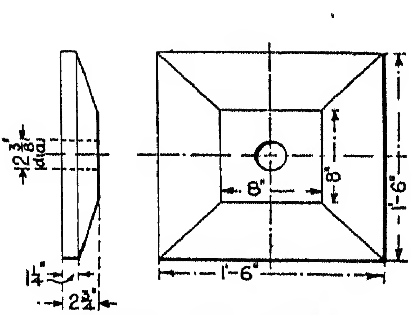


Fig. 66.—Anchor Plate.

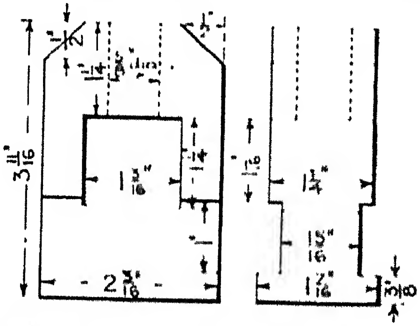


Fig. 67.—Planer Tool Holder.

9. Calculate the weight in cast iron of the tool holder for a planer shown in Fig. 67.

10. Find the weight of the cast-iron roll for a rubber mill as in Fig. 68 (Use the slide rule throughout.)

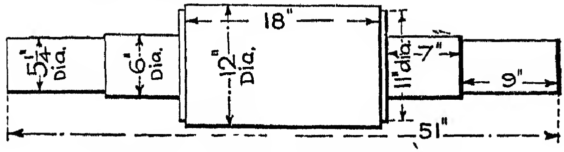
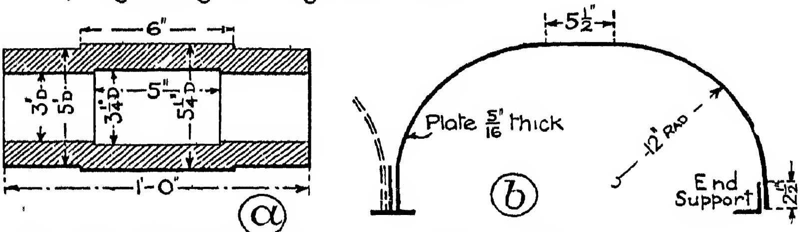


Fig. 68.—Roll for Rubber Mill.

11. A mild steel sleeve coupling for 3″ shaft is shown at (*a*), Fig. 69. Find its weight.

12. The steelwork for Hobson's flooring has the sectional form shown at (*b*), Fig. 69. There are 20 such plates for each span of the bridge, each $\frac{5}{16}$″ thick and 22 ft. long. Find the total weight of the steelwork, neglecting the angle and T-bar.



Mild Steel Sleeve Coupling. Section of Hobson's Flooring.

Fig. 69.

13. Find the weight in cast iron of the simple plummer block shown in Fig. 70.

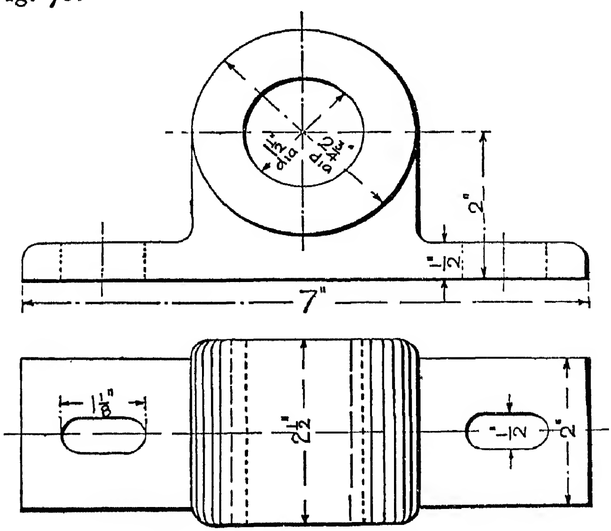


Fig. 70. Plummer Block.

14. Fig. 71 shows the worm shaft for a motor-car rear axle. It is made of nickel steel, weighing ·291 lb. per cu. in. Find its weight.

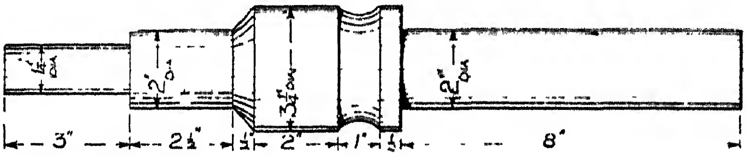


Fig. 71.—Worm Shaft.

15. Calculate the weight in cast iron of the half coupling shown in Fig. 72.

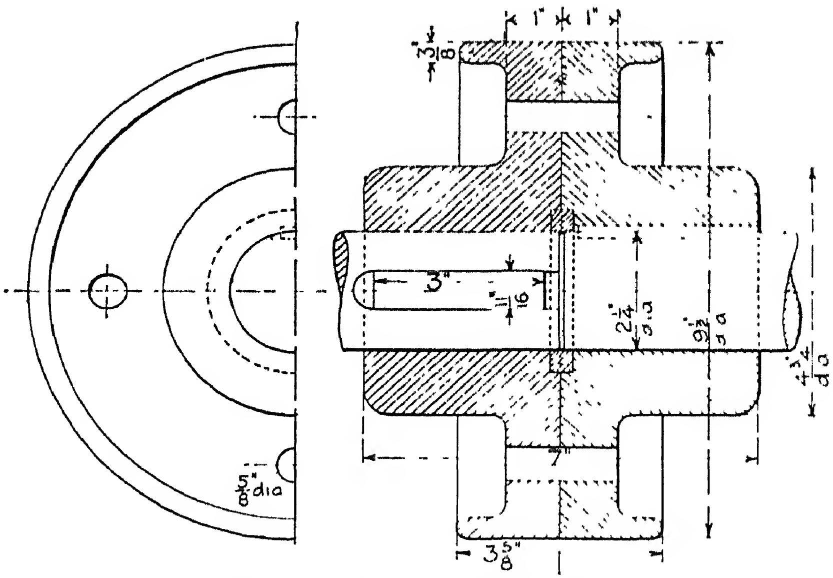


Fig. 72.—Wrought-iron Coupling.

16. Find the weight in cast iron of the cylinder cover shown in Fig. 73.

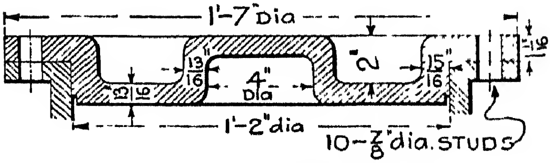


Fig. 73.—C.I. Cylinder Cover.

17. Fig. 74 shows the brasses for the crankshaft of a $6\frac{1}{2}'' \times 6''$ launch engine. Find the weight of one of these in gun metal.

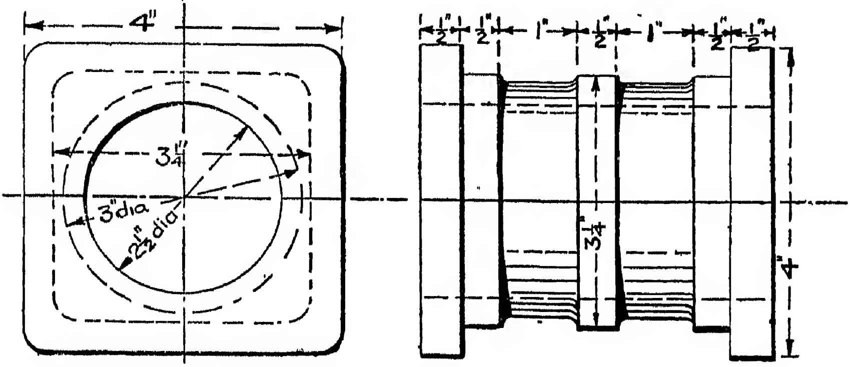


Fig. 74.—Crank Shaft Brasses.

18. The brasses for a thrust block are shown in Fig. 75. Calculate the weight of one of these in gun metal.

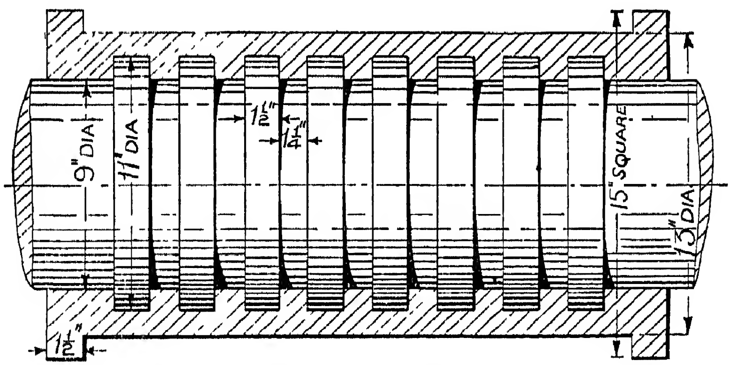


Fig. 75 —Brasses for a Thrust Block.

19. An air vessel is shown in Fig 76. Find its weight in cast iron.

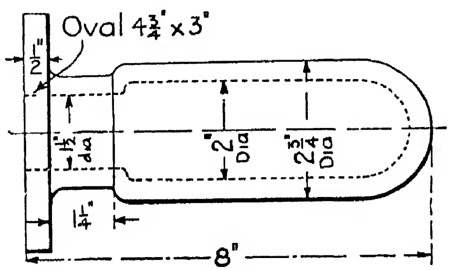


Fig. 76.—Air Vessel for Pump.

TABLE OF AREAS AND CIRCUMFERENCES OF PLANE FIGURES.

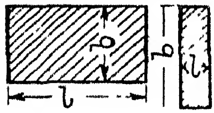
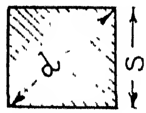
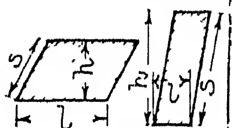
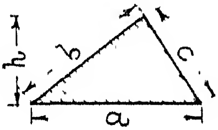
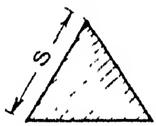
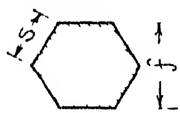
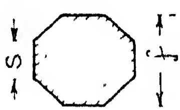
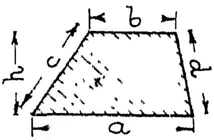
Title.	Figure.	Circumference or Perimeter.	Area.
Rectangle .		$2(l+b)$	lb
Square . .		$4s$	s^2 or $\frac{d^2}{2}$
Rhomboid .		$2(l+s)$	lh
Triangle . .		$a+b+c$ $s=\frac{1}{2}$ perimeter	$\frac{ah}{2}$ or $\sqrt{s(s-a)(s-b)(s-c)}$
Equilateral triangle .		$3s$	$\cdot433s^2$
Hexagon .		$6s$ or $3{\cdot}46f$	$2{\cdot}6s^2$ or $\cdot866f^2$
Octagon . .		$8s$ or $3{\cdot}32f$	$4{\cdot}83s^2$ or $\cdot829f^2$
Trapezoid .		$a+b+c+d$	$h\left(\frac{a+b}{2}\right)$
Irregular quadrilateral or trapezium. .		Sum of all four sides.	Divide into two triangles by either diagonal. Find area of each triangle and add. Or area $=\frac{lh}{2}$
Circle . . .		πd or $2\pi r$	$\frac{\pi}{4}d^2 = \cdot7854d^2$ or $\pi r^2 = 3{\cdot}142r^2$

TABLE OF AREAS AND CIRCUMFERENCES OF PLANE FIGURES (*continued*).

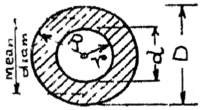
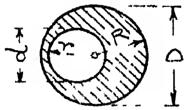
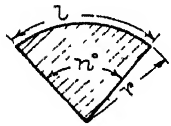
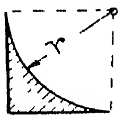
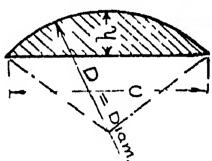
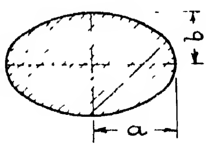
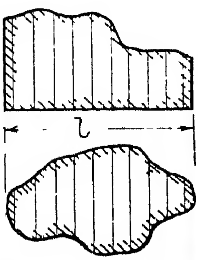
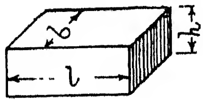
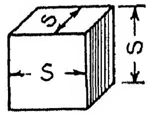
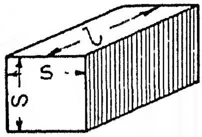
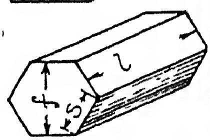
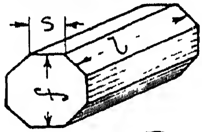
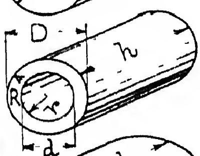
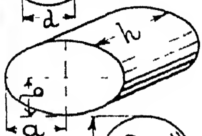
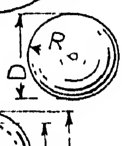
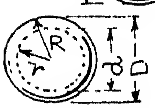
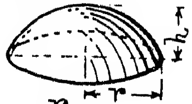
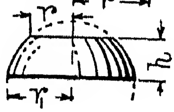
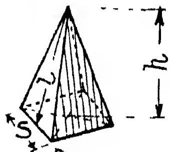
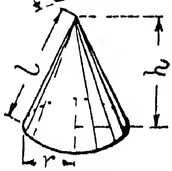
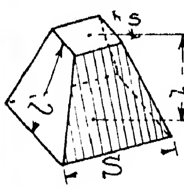
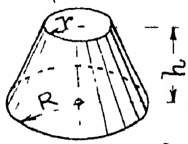
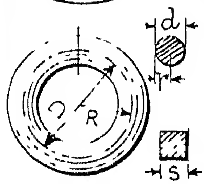
Title.	Figure.	Circumference or Perimeter.	Area.
Hollow circle (annulus) .			$\frac{\pi}{4}(D^2-d^2)=\cdot 7854(D^2-d^2)$ or $\pi(R^2-r^2)$ or $\pi\times$ mean dia. $\times$ thickness
Hollow circle (eccentric)			$\cdot 7854(D^2-d^2)$ or $\pi(R^2-r^2)$
Sector of circle . .		$l=\frac{rn}{57\cdot 3}$	$\frac{\pi n r^2}{360}$ or $\frac{lr}{2}$
Sector of hollow circle .			$\frac{\pi n(R^2-r^2)}{360}$
Fillet . . .			$\cdot 215r^2$ or approx. $\frac{1}{5}r^2$
Segment of circle . .			Area = sector − triangle Various approx. formulæ on p. 102.
Ellipse . .		$\pi(a+b)$ approx or $\pi\{1\cdot 5(a+b)-\sqrt{ab}\}$ more nearly	πab
Irregular figures . .		Step round curved portions in small steps, with dividers; add in any straight pieces.	Divide into narrow strips, measure their mid-ordinates. Then— Area = aver. mid-ordinate $\times$ length l

TABLE OF VOLUMES AND SURFACE AREAS OF SOLIDS.

Title.	Figure.	Volume.	Surface Area.
Any prism .		Area of base × height	Circumference of base × height
Rectangular prism or cuboid . .		lbh	Whole area $= 2(lb+lh+bh)$
Cube . . .		S^3	Whole area $= 6S^2$
Square prism		S^2l	Lateral surface $= 4Sl$ Ends $= 2S^2$ Whole surface $= 2S(2l+S)$
Hexagonal prism . .		$2{\cdot}6S^2l$ or $\cdot 866f^2l$	Lateral $= 6Sl$ or $3{\cdot}46fl$ (For ends see Table on p 144)
Octagonal prism . .		$4{\cdot}83S^2l$ or $\cdot 829f^2l$	Lateral $= 8Sl$ or $3{\cdot}32fl$
Cylinder . .		πr^2h or $\cdot 7854d^2h$	Lateral $= 2\pi rh$ Two ends $= 2\pi r^2$ Whole area $= 2\pi r(h+r)$
Hollow cylinder . . .		$\pi(R^2 - r^2)h$	Outer lateral surface $= 2\pi Rh$ Inner lateral surface $= 2\pi rh$
Elliptical prism . .		πabh	Lateral $= \pi h\{1{\cdot}5(a+b) - \sqrt{ab}\}$ or $\pi(a+b)h$ (less accurate)
Sphere . .		$\frac{4}{3}\pi R^3$ or $\frac{\pi}{6}D^3$ or $\cdot 5236D^3$	$4\pi R^2$
Hollow sphere . .		$\frac{4}{3}\pi(R^3 - r^3)$	$4\pi(R^2 + r^2)$

TABLES OF VOLUMES AND SURFACE AREAS OF SOLIDS (*continued*).

Title.	Figure.	Volume.	Surface Area.
Segment of sphere . .		$\frac{\pi h}{6}(3r^2 + h^2)$ or $\cdot 5236h(3r^2+h^2)$	Curved surface $= 2\pi Rh$ or $2\pi R(R - \sqrt{R^2 - r^2})$ where R=rad. of sphere
Zone of sphere . .		$\frac{\pi h}{6}\left\{3(r^2+r_1^2)+h^2\right\}$	
Any pyramid		$\frac{1}{3}$ area of base $\times$ height	Lateral $= \frac{1}{2}$ circum of base $\times$ slant height
Square pyramid . . .		$\frac{1}{3}S^2h$	Lateral $= 2Sl$
Cone . . .		$\frac{1}{3}\pi r^2h$	Lateral $= \pi rl$
Frustum of any pyramid . . .	h=height of frustum A =area of large end B =area of small end	$\frac{h}{3}(A + B + \sqrt{AB})$	Lateral$=\frac{1}{2}$ mean circum. $\times$ slant height
Frustum of square pyramid .		$\frac{h}{3}(S^2 + s^2 + Ss)$	Lateral $= 2l(S + s)$ (l = slant height)
Frustum of cone . . .		$\frac{\pi h}{3}(R^2 + r^2 + Rr)$	Lateral $= \pi l(R + r)$ (l = slant height)
Anchor ring .		Round section $2\pi^2Rr^2$ Square section πDS^2	$4\pi^2Rr$ $4\pi DS$

These four tables are reproduced from *Arithmetic for Engineers* by kind permission of the author, Mr. Charles B. Clapham.

CHAPTER IV

INTRODUCTION TO GRAPHS

Object and Use of Graphs.—A *graph* is a pictorial statement of a series of values all drawn to scale. Such a diagram will often greatly facilitate the understanding of a problem; for the meaning is more readily transmitted to the brain by the eye than by description or formulæ. When reading a description, one has often to form a mental picture of the scenes before one can grasp and fully appreciate the ideas or facts involved. If, however, the scenes are presented vividly to us, much strain is removed from the brain. A few pages of statistics would have to be studied carefully before their meaning could be seen in all its bearings, whereas if a "graph" or picture were drawn to represent these figures, the variations of their values could be read off at a glance.

To take another example: a set of experiments are carried out with pulley blocks; the results will not be perfect, some readings may be too high, others too low: and to average them from the tabulated list of values would be extremely laborious; whereas the drawing of a graph is itself in the nature of an averaging.

Or, again, a graph shows not only a change in a quantity, but the *rate* at which that change is taking place, this latter being often the more important. On a boiler trial a graph is often drawn to denote the consumption of coal: from which is shown during what period the consumption is uniform, or when the demand has been greater or less than the average, and so on.

A graph, then, is a picture representing some happenings, and is so designed as to bring out all points of significance in connection with those happenings. The full importance and usefulness of graphs can only be appreciated after many applications have been considered.

To commence the study of this branch of our work let us consider an example based on some laboratory experiments.

Example 1.—In some experiments on the flow of water over notches the following figures were actually obtained.

RIGHT-ANGLED V-NOTCH

Head (ft.) H . .	·1888	·2365	·2617	·2878	3065	·3361
Quantity flowing (lbs. per min.) Q	141·5	249·8	323·5	411·4	483·6	608

The flow, in subsequent experiments, was to be gauged by the "head" of water at the notch, so that a good "calibration" curve was desired.

The figures were plotted as shown in Fig. 77, H along a horizontal axis and Q parallel to a vertical axis.

In such plotting as this the following points of detail should be observed.

Select two lines at right angles for the main axes and thicken them in: these lines should be as far over to the left and as low down, respectively, as will permit of the scales being written to the outside of each.

Look to the values to be plotted, noting the "range" in either direction, **the scales for the plotting being selected so that the whole of the available space is utilised:** but care must be taken to select a sensible scale. Generally a decimal scale is to be preferred, *e. g.*, in the present case we take $\frac{1}{2}''$ to represent ·02 ft. of head, horizontally and $\frac{1}{2}''$ to represent 100 lbs. per min. vertically.

Write figures along the axes to indicate the scales adopted, and also indicate clearly which quantity is plotted along the horizontal axis and which along the vertical axis; for attention to such details greatly enhances the value of the graph.

To plot: We wish to illustrate the fact that for each value of H there is a value of Q; which we can do by selecting some value of H, running up the vertical through the marking denoting that value until we meet the horizontal through the given corresponding value of Q, and then making a small mark, *e. g.*, the point denoting that H = ·2878 when Q = 411·4, as shown on the diagram by the point P.

The use of paper ruled in squares will ease matters, although in a good many instances a series of horizontal and vertical lines through points specified in a table of given values will suffice.

When all the points have been plotted, the best average or

smooth curve must be drawn through them: the points above the line should about balance those below it, and any obviously inaccurate values must be disregarded. For good results the curve should be drawn with the aid of either a spline or a French curve.

The curve is now what is called a *calibration curve* for the notch, *i. e.*, for any head within the range for which experiments were carried out, the quantity flowing can be read off.

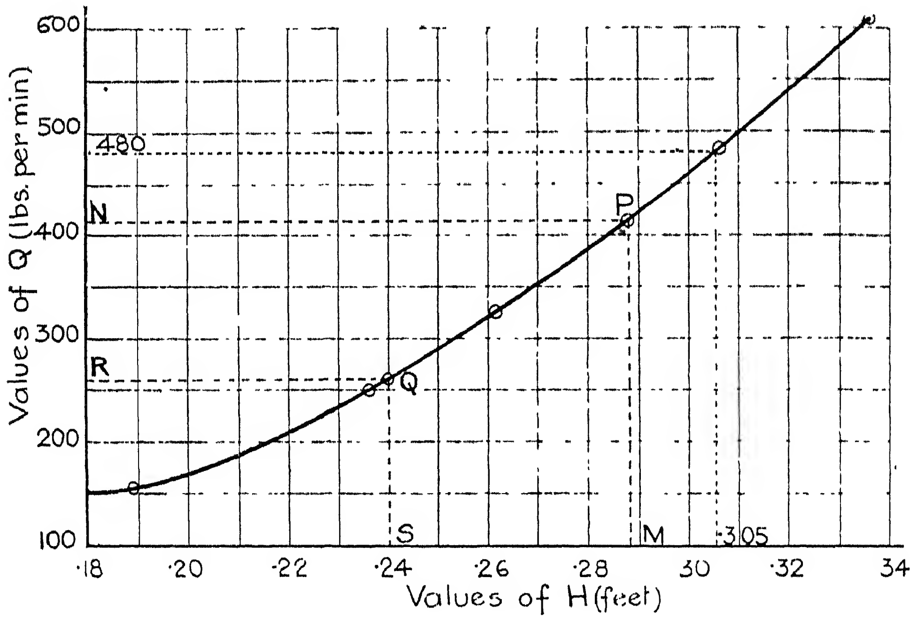


Fig 77.—Calibration Curve for V-notch. (Full size.)

This process of reading off intermediate values is spoken of as "**interpolation.**" Without the graph, for any values not given in the table one would have either to estimate or to repeat the experiment if intermediate values were required. Also one further point should be noticed: even the figures in the table may not be quite the best, and better approximations can be obtained from the curve.

Ex.—To find the quantity when the head is ·24 ft.: erect the perpendicular SQ through ·24 on the scale of head, meeting the curve at Q. Draw QR horizontally to cut the axis of quantity at Q = 260.

Then for a head of ·24 feet, 260 lbs. per min. are flowing.

Ex.—Find the head when Q = 480. From the diagram, H = ·305 ft.

Example 2.—The following figures were obtained in some trials on a gas engine. Draw the efficiency curve, *i. e.*, the curve in which the efficiency is plotted against the output.

I.H.P. (Input)	1·54	3·09	4·58	5·67	6·50
B.H.P. (Output)	0	1·62	3·33	4·71	5·81

The efficiency (to be denoted throughout this book by η, the Greek letter eta) $= \frac{\text{output}}{\text{input}}$ or $\frac{\text{B.H.P.}}{\text{I.H.P.}}$ and could be calculated by taking corresponding values of B and I from the table.

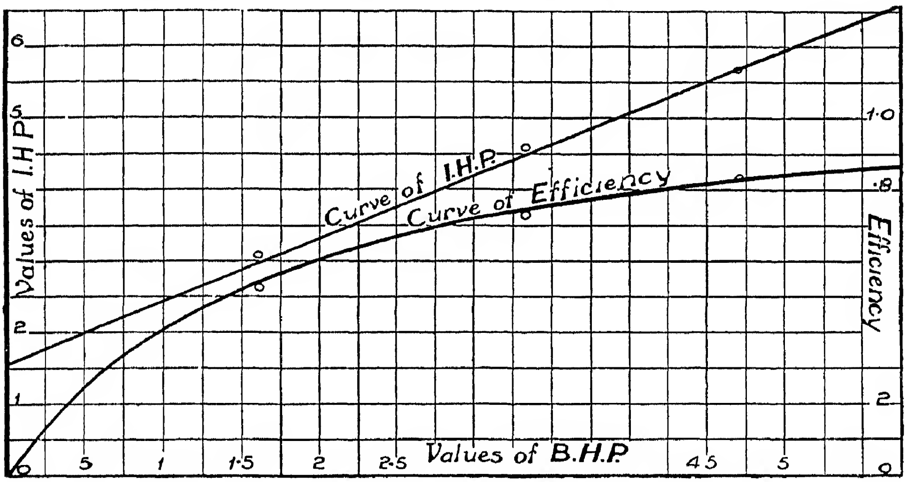


Fig 78.—Test on Gas Engine.

It is better, however, to first plot B.H P. against I.H.P. and average these points by a straight line, which can be drawn with more certainty than a curve (see Fig. 78). The efficiencies at various loads can now be calculated from this "curve" by taking the ratios of $\frac{B}{I}$ for convenient values of B; *e. g*, when B = 1, I = 2·43 and η = ·412.

Plotting the values of the efficiency so obtained to a base of output, a well-defined smooth curve is obtained, as in Fig. 78

The efficiencies worked from the experimental figures are—

B H P. . . .	0	1·62	3·33	4 71	5·81
η	0	·525	·726	·831	·895

If now these values are plotted to a base of B H.P. the points lie fairly equally about the efficiency curve already drawn.

The efficiency-output and the input-output curves now agree, whereas they would not do so in all probability if plotted quite separately.

This derivation of one curve from another is of wide application. To illustrate by another example :—

Example 3.—A test on a Morris-Bastert pulley block gave the following results :—

Load lifted (lbs.) W.	27·5	47·5	67·5	87·5	107·5	127·5	147·5	167·5	187·5
Effort required (lbs.) P_1	2·07	2·5	3·15	4·05	4·52	5·2	5·85	6·4	7·1

The velocity ratio (V.R.) of the machine was 48.

Draw the efficiency curve to a base of loads.

Theoretical effort to raise a weight $W = P = \dfrac{W}{V.R.}$

Actual effort $= P_1$ and efficiency $= \dfrac{P}{P_1}$

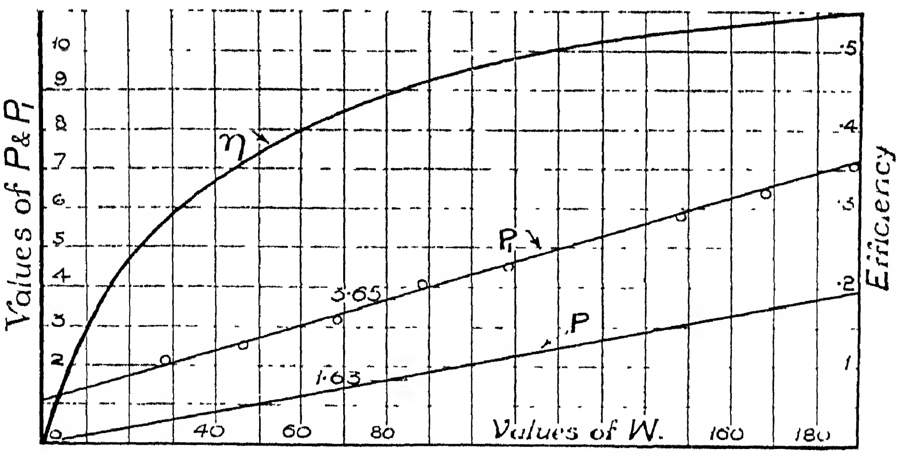


Fig. 79.—Test on Pulley Block.

First plot the given values, W horizontally and P_1 vertically, and draw the straight line which best fits the points (see Fig. 79).

To calculate values of P corresponding to the values of W set 48 on the C scale of the slide rule level with 1 on the D scale. Then the readings on the C scale will correspond to values of W and those on the D scale level with these to values of P; *e. g.*, place the cursor over 27·5 on the C scale and ·572 is read off on the D scale, so that the value of P when W = 27·5, is ·572. All values of P can thus be read off with one, or, at the most, two settings of the rule.

The values of P are ·572, ·99, 1·41, 1·82, 2·24, 2·66, 3·07, 3·49, 3·9. Plotting these to the same scale as chosen for P_1 the lower line in Fig. 79 is obtained.

By division of corresponding ordinates of these lines the efficiency can be calculated for any load, *e. g.*, when W = 80, P = 1·63, P_1 = 3·65 and $\eta = \frac{1 \cdot 63}{3 \cdot 65} = \cdot 447$. A scale must now be chosen for efficiencies, and the curve can then be put in; this will be a smooth curve, because it is obtained from two straight lines.

Example 4.—In some experimental work, only gramme weights were available, whilst for calculation purposes the weights were required in pounds. To save the constant division by 453·6 (the number of grms. equivalent to 1 lb.) a straight line could be drawn from which the required interpolations could be made. To construct such a chart :—

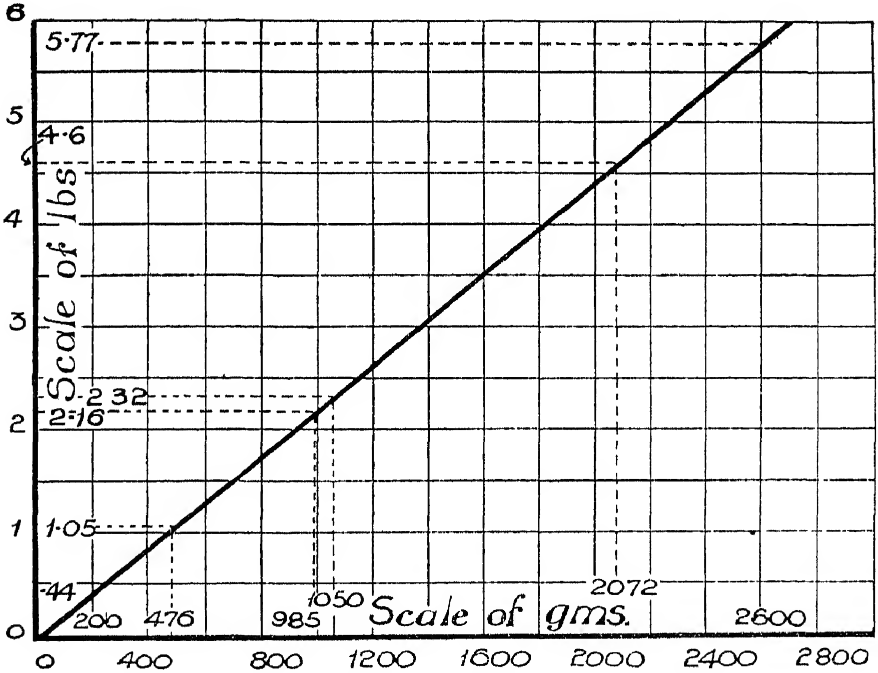


Fig. 80.—Chart to convert grammes to lbs.

Suppose that the readings in grms. were—

200, 476, 985, 1050, 2072, 2600.

Plotting grms. along the horizontal as in Fig. 80, a scale must be chosen to admit of 2600 being shown. Draw a vertical through 453·6 to meet a horizontal through 1 on the " lb. " scale. The line joining this to the origin (*i. e.*, the zero point for both scales) is the conversion

line. The required values can now be quickly read off as in the following table :—

grms.	200	476	985	1050	2072	2600
lbs.	·44	1·05	2·16	2·32	4·6	5·77

One axis might take the place of the two in the above diagram. Along this on one side the graduation would be in lbs. and on the other side, in grms.; thus amounting to putting a scale of lbs. alongside one of grms. Tables of logarithms might be, and in fact are (in Farmer's Log Tables), replaced by a number of lines, graduated in numbers and also in logarithms. For great accuracy a great number of lines are required so that two pages do not suffice as in the case of the tables, this being rather a disadvantage : nevertheless there is much to be said for this method of table construction. There are no differences to add, nor is it necessary to remember when differences have to be subtracted, since for any definite value in the one set of units the corresponding value in the other is read off directly.

Exercises 20.—On Simple Plotting.

1. In a test on Hobson's flooring the following figures were obtained.

Total load (tons)	35	40	50	60	70	80	90	100	110
Deflection (ins.)	$\frac{1}{4}$	$\frac{7}{16}$	$\frac{9}{16}$	$\frac{3}{4}$	$\frac{15}{16}$	$1\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{9}{16}$	2

Plot a graph to give the deflection for any load between 35 and 110 tons; and read off the deflection for a load of 55 tons and also the load causing a deflection of 1″.

2. Plot a curve to show the decrease in the tenacity of copper with increase of heat, from the following table :—

Temperature F.° . . .	212	350	380	400	500	530	580	620	720
Tenacity (lbs per sq. in.)	32000	30000	29500	29000	26500	25500	23500	21500	20000

Read off from your graph : (*a*) the tenacity at 302° F.; (*b*) the temperature at which the tenacity is 21000 lbs. per sq. in.; (*c*) the tenacity at 545° F.

3. Draw the calibration curve for a rectangular notch, given—

Head (foot)	·0871	·1115	·1588	·1838	·2124
Quantity (lbs. per min.)	139·4	199	323·3	406·2	502·8

Find the discharge when the head is ·19 ft.

4. The following figures are given for the working stress allowable on studs and bolts :—

Diam. of stud (ins.) .	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2
Stress (lbs. per sq. in.)	2000	3000	3900	4700	5500	6300	7000

Find the stress allowable on a stud of $\frac{7}{8}''$ diam. and also the stud to be used if the stress is 5100 lbs. per sq. in.

5. Cast-iron pulleys should never run at a greater circumferential speed than 1 mile per minute. In the table the maximum revolutions per minute (R.P.M.) allowable are given for various diameters. Find the R.P.M. for a pulley of 14″. diam. Check this figure by the ordinary rule of mensuration.

Diam. (ins.)	5	6	8	10	12	15	18	20	25	30
R.P.M. . .	4034	3361	2524	2017	1681	1345	1120	1008	807	673

6. Plot a curve to give the diameter of a shaft for any twisting moment from ·7 ton per sq. in. to 360 tons per sq. in.

Equivalent twisting moment (tons per sq. in.	·701	2·367	5·611	10·95	18·94	30·07	44·9	63·9	87·7	152	359
Diam. of shaft (ins.) .	1	1·5	2	2·5	3	3·5	4	4·5	5	6	8

7. The table gives the "time constant" of the coils of an electro-magnet for gaps of various lengths. Represent this variation by a graph.

Distance apart (cms.) .	·125	·5	·75	1	1·5	2	2·5	3
Time constant (secs.) .	2·5	1·7	1·4	1·4	1·1	1·1	·9	·9

8. The relation between pressure p and temperature t of steam shown in the table was found experimentally. Plot a curve to represent this, finding the value of t when p is 105, and the value of p when t is 300.

p lbs. per sq. in. . .	5	10	15	20·5	27	31	36	44	50	60	70	80	90	100	110	120
t (F.°)	235	243	251	260	270	276	282	290	296	306	314	322	329	336	342	348

9. and **10.** Plot curves of Magnetic Induction for (1) Iron, and (2) Cobalt, from the figures given in the tables following.

(**9.** Iron.)

H (magnetising force)	0	5	10	17	25	30	38	45	52	60	65
B (mag. induction density)	0	2400	4500	6000	7100	7800	8300	8500	8600	8600	8700

(**10.** Cobalt.)

H.	0	1·55	3·10	4·65	6.20	7·75	12·40	15·5	23·25	31	38·75	46·5
B.	0	99	268	642	1128	1298	2405	2995	4070	4860	5300	5810

11. Plot a curve to show the variation in the ratio $Q = \left\{\frac{\text{weight of armament and protection}}{\text{load displacement}}\right\}$ as given for a speed of 21 knots, from the following table :—

Load displacement P (tons)	18000	22000	24000	26000	30000	34000	38000	40000
Ratio Q . .	·383	·401	·409	·416	·428	·438	·446	·450

Find the weight of armament and protection when the displacement is 28000 tons.

12. Plot a curve, as for Question 11, but the figures belonging to a speed of 27 knots.

P . .	18000	20000	24000	26000	28000	30000	32000	36000	40000
Q . .	·236	·252	·275	·286	·295	·303	·310	·3[illegible]4	·336

Find the value of Q when P = 34000.

13. The temperature of the field coils of a motor was measured at various times during the passage of a strong current, with the following results :—

Time (mins.)	0	2	5	10	15	20	25	30	35	40	45	50	55	60
Temperature (C.°) . .	14	16	23	32·4	39	43·4	47	50·5	52·5	55	56·8	58·0	59·3	59·4

Find the time that elapses before radiation losses, etc., balance the heating effect of the current, viz. when there is no further sensible rise of temperature; and find also the maximum rise of temperature.

14. Repeat as for Question 13, taking the following results :

Time (mins.)	0	5	10	15	20	25	30	35	40	45	50	55	60	65
Temperature (C.°) . . .	20	26	32·5	41	46	49	52·5	54·5	56·5	58	59·5	61	61·7	62

15. The following figures were obtained by reading spring balances at the ends of a beam on which a weight of 7 lbs. was hung. Plot

curves to give the values of the reactions for any position of the weight. Note their point of intersection.

Distance (ins.) of weight from R.H end	0	2	4	6	8	10	12	14	16	18	24	28	30	32
Left-hand reaction (lbs.)	0	.4	.8	1.25	1.7	2.1	2.55	3	3.45	3.9	5.2	6.05	6.5	7
R.H. reaction (lbs.) . .	7	6.5	6.05	5.6	5.2	4.8	4.3	4.05	3.45	3	1.8	.8	.4	0

In Questions 16 to 19 draw to a base of loads (W) curves whose ordinates gives—

(*a*) Actual effort P_1; (*b*) theoretical effort P; (*c*) efficiency η.

16. Test on a 6 to 1 pulley block, *i. e.*, V.R. = 6.

W	28	48	68	88	108	128	148	168	188	208
P_1	9.75	14.75	20.25	25.75	30.75	35.75	40.25	45.25	49.25	55.25

17. Test on a Single Purchase Crab (V.R. = 27).

W . .	50.1	92.1	137	180	224	266	310	354	394
P_1 . .	3.6	5.35	7.9	9.9	11.8	13.9	14.7	16.9	19.5

18. Test on a Screw Jack (V.R. = 60.5).

W .	34	54	74	94	114	134	154	174	194	214	234
P_1 .	1.73	2.85	3.93	5.17	6.19	7.70	8.95	10	11.3	12	12.9

19. Test on a Weston Pulley Block, when raising (V.R. = 24).

W	25	45	65	85	105	125	145	165	185	205
P_1	4	6.75	8.75	8.75	10	13.5	15	18.75	21	22.5

20. The table gives the current absorbed by a carbon brush at various pressures. Plot, to a base of amperes of current, curves giving resistance and voltage. $\left\{\text{Resistance} = \frac{\text{volts}}{\text{amperes}}\right\}$

The resistance curve should be obtained from that for voltage.

Volts	.35	.65	.88	1	1.3	1.45	1.5	1.65	1.75	1.77	1.8	1.825	1.85
Amps. . . .	4	9	13.5	18.75	21.5	24.5	27.5	32.5	37.5	40.5	42	45.5	47.5

21. To a base of frequency plot curves giving (*a*) voltage, (*b*) current taking the following figures :—

Frequency	40	43.5	47	50	52	54	56	60	64	75	80	88
Current .	5.39	8.75	14.35	18.67	14.73	11.66	9.33	6.83	5.19	3.05	2.64	2.14
Voltage .	52	32	19.5	15	19	24	30	41	54	93	106	131

22. The following figures were obtained in a tensile test on a sample of 25% nickel steel.

Stress (lbs. per sq. in.)	4000	12000	20000	28000	36000	48000	52000	56000	60000
Extension (inches per inch length)	·00015	·00047	·0007	·00111	·00145	·00195	·00213	·00235	·00264

64000	68000	72000	76000	80000	84000	88000	92000	96000	100000	104000
·00365	·0065	·021	·035	·052	·068	·0853	·1025	·134	·171	·201

Plot the "stress-strain" diagram, the stresses being vertical and extensions along the horizontal; also determine the stress at the "yield point," where the sudden change occurs.

23. The voltage supplied to a 4-volt lamp was varied, and the candle-power (C.P.) then measured for various values of the voltage, the results being as follows :—

C.P. .	0	·5	1·0	1·5	2	2·5	3
Volts .	0	3·03	3·57	3·96	4·25	4·44	4·75
Amps. .	0	1·16	1·29	1·36	1·48	1·63	1·71

If watts = volts × amps, plot to a base of C.P. curves whose ordinates represent—

(*a*) volts; (*b*) amps, and by a combination of corresponding ordinates of these—(*c*) watts per C.P.

24. The drop in potential due to a standard resistance of ·3 ohm was measured by a potentiometer, for various currents. The current was also measured on an ammeter.

If current $= \dfrac{\text{volts}}{\text{resistance}}$, calculate the true currents flowing. Also plot a curve of true current against registered current, and hence find the percentage error of the ammeter.

Ammeter reading (Registered current)	1	1·5	2	2·5	3	3·5	4	4·5	4·75
Volts	·3093	·458	·6149	·7629	·92	1·0487	1·204	1·371	1·437

25. From the following figures (taken from a test on a 10 H.P. Diesel engine) plot curves, to a base of B.H.P., to show -

(*a*) I.H.P., from which deduce (*b*) mechanical efficiency; (*c*) oil per hour, and hence (*d*) oil per B.H.P. hour.

B.H.P. . . .	0	3·33	6·71	8·35	9·94
I.H.P. . . .	4·5	7·27	10·66	11·69	12·95
Oil per hour (lbs.)	1·5	2·37	3·63	4·35	5·45

26. From the given figures plot to a base of I.H.P., curves with ordinates to represent (*a*) steam per hour and thence (*b*) steam per I.H.P. hour.

Steam per hour (lbs.)	513	452	436	403	370	327	182
I.H.P.	13·12	10·54	9·83	8·85	8·15	6·57	1·84

27. Results of an efficiency test on a small motor gave the following :—

Output (watts) . . .	6·46	24·2	33·8	37·5	40	55·3	61·5	64·9	77·1	92	117
Input (watts) . . .	57·6	82·4	102	104·2	107·2	138	142·2	141·1	162·4	187·5	228

To a base of output plot curves giving (*a*) input and thence (*b*) efficiency. $\left(\text{Efficiency} = \frac{\text{output}}{\text{input}}\right)$

28. The voltage of an accumulator, when discharging, fell according to the following :—At 2 o'clock voltage = 2·15, at 2.30 o'clock and also at 3.30 voltage = 2·06, at 6.30 voltage = 1·87 and at 9 o'clock voltage = 1·72. Another cell was charged at a uniform rate from 2 o'clock to 7 o'clock, the voltage rising from 1·75 to 2·38. Assuming that the discharge was uniform, find the time at which the cells had the same voltage.

Co-ordinates.—So far, in these graph problems, we have been concerned with positive quantities only; the question now is, How to deal with negative quantities? If the plotting "movement" has been in a certain direction for the positive, then clearly for a negative the motion must be reversed. The convention adopted is that to the right and upwards are positive directions for the horizontal and vertical axes respectively; and therefore to the left and downwards will be the corresponding negative directions. These are indicated in the diagram (Fig. 81). To admit of all arrangements of signs the paper must be divided into four parts or quadrants as shown, the point of intersection of the axes being termed the *origin*, viz. the point O

The points A_1 A_2 A_3 and A_4 are all distant 4 units from the vertical axis and 3 units from the horizontal, so that to distinguish between them we must make some mention of the *quadrant* in which each is placed by affixing the correct signs.

The distances from the axes together are spoken of as *co-ordinates*, that along the horizontal being usually called the *abscissa*, while vertical distances are called *ordinates*. In representing a point by its co-ordinates the abscissa is always stated first.

Point A_1 is thus	$+4$ and $+3$	or	more shortly	$(4, 3)$
A_2 is	-4 and $+3$	or	more shortly	$(-4, 3)$
A_3 is	-4 and -3	or	more shortly	$(-4, -3)$
A_4 is	$+4$ and -3	or	more shortly	$(4, -3)$.

Note that $(-4, -3)$ does not imply -7, but a movement of 4 units to the left of the vertical axis and then 3 units down from the horizontal axis.

E.g.,

Point B is $(1\cdot5, -1)$
Point C is $(-3\cdot2, 0)$
Point D is $(-1\cdot4, 2\cdot3)$.

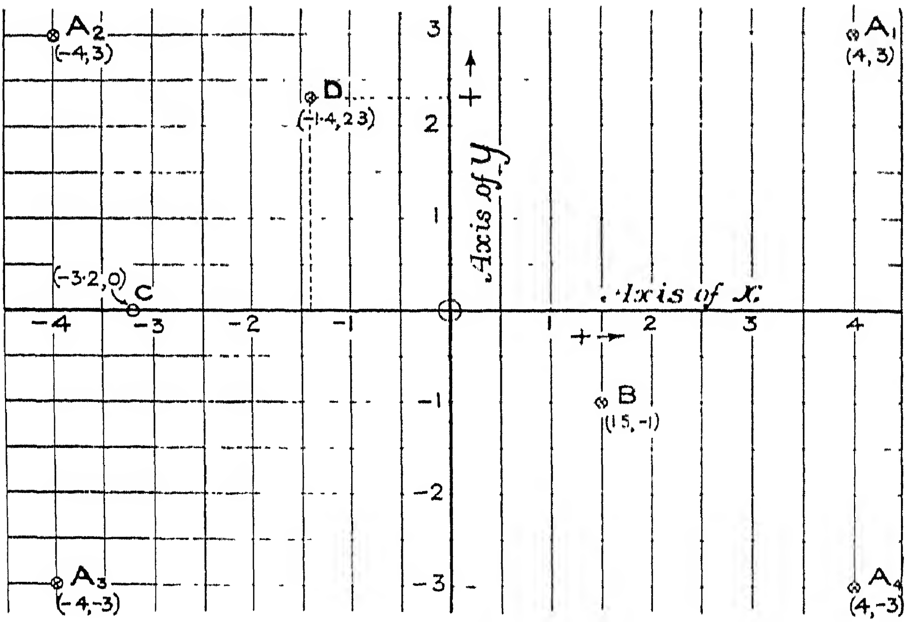


Fig. 81. Co ordinates of Points.

To fix the position of a point in space it would be necessary to state the three co-ordinates, viz. the distances from three axes mutually at right angles. For example, a gas light in a room would be referred to two walls and the floor to give its position in the air.

Representation of an Equation by a Graph. If two quantities x and y depend in a perfectly definite way, the one upon the other, the relation between them may be illustrated by a graph which will take the form of a straight line or a smooth curve. From this curve much information can be gleaned to assist in the study of the function as it is called. [*Explanation.*—If $y = 2x + 5$,

y is said to be a *function* of x, for y depends for its value on that given to x; if $y = 4z^2+7z^3-8 \log z$, y is a function of z or, as it would be expressed more shortly, $y = f(z)$, meaning that y has a definite value for every value ascribed to z: *e. g.*, in the case first considered, $y = f(x) = 2x+5$, then $f(3)$ would indicate the value of y when 3 was written in place of x, i. e., $f(3) = (2\times3)+5 = 11$.]

Dealing first with the simplest type of graph, viz. the straight line, whenever the equation giving the connection between the variables is of the first degree as regards the variables, *i. e.*, it contains the first power only of the variables, a straight line will result when the equation is plotted.

Example 5.—Plot a graph to represent the equation $y = 5x-9$.

In all cases of calculation for plotting purposes it is best to tabulate in the first instance; for any error can thus be readily detected, and in any case some system must be adopted to reduce the mental labour and the time involved.

The general plan in these plotting questions is to select various values for one of the variables, which we can speak of as the "*independent variable*" (I.V.), and then to calculate the corresponding values of the other, which may be spoken of as the "*dependent variable.*" In questions where x and y are involved it is customary to make x the I.V., and to plot its values along the horizontal axis.

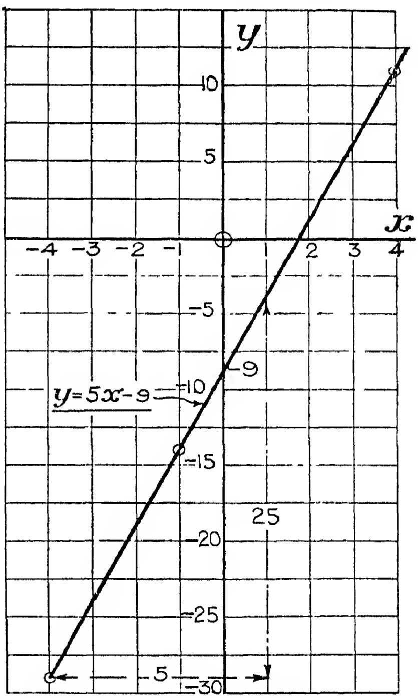


Fig. 82.—Curve of $y = 5x - 9$.

We may take whatever values for x we please, since nothing is said in the question about the range. Let us suppose that x varies from -4 to $+4$. The table, showing values of y corresponding to values of x would be as follows :—

x	$5x - 9$	y
* -4	$-20-9$	-29
-3	$-15-9$	-24
-2	$-10-9$	-19
-1	$-5-9$	-14
0	$0-9$	-9
1	$5-9$	-4
2	$10-9$	1
3	$15-9$	6
4	$20-9$	11

* *i. e.*, $5x = 5\times(-4) = -20$.

When we come to the plotting we see that it is advisable to select different scales for x and y, since the range of x is 8 and that of y is 40. On plotting the above values a straight line passes through them all (Fig. 82).

A straight line would be definitely fixed if one knew its slope or inclination and some point through which it passes. As regards the slope, a line sloping upwards towards the right has a positive slope, because the increase in the value of x is accompanied by an increase in the value of y, and the slope is measured by $\frac{\text{change of } y}{\text{change of } x}$. In measuring the slope of a line, the denominator is first decided upon, a round number of units, say 2 or 10, being chosen, and the numerator corresponding to this change is read off in terms of the vertical units from the diagram.

In the case of the line representing $y = 5x - 9$ the slope is seen to be $\frac{25}{5} = 5$, *i. e.*, the slope is the coefficient of x in the original equation

The fixed point, a knowledge of which is necessary before the line can be located, is taken on the y axis through $x = 0$, *i. e.*, the point of intersection of the line with the vertical axis through $x = 0$ must be known. In the case shown in Fig. 82 the line intersects at the point for which $x = 0$, $y = -9$: also -9 is noted to be the value of the constant term in the equation from which the graph is plotted.

In general, if the equation to a straight line is written, $y = ax + b$; a is the slope of the line and b is the intercept on the vertical axis through the zero of the horizontal scale.

All equations of the first degree can be put into this standard form, and hence will all be represented by straight lines.

Example 6.—Consider the three equations—

$$4x + 5y = 8 \quad \dots\dots\dots\dots (1)$$
$$4x + 5y = 0 \quad \dots\dots\dots\dots (2)$$
$$4x + 5y = -12 \quad \dots\dots\dots\dots (3)$$

A similarity is at once noticed between the equations; a short investigation will show the full interpretation of that similarity when regarded from the graphical standpoint.

Whenever an equation is to be plotted it is always the best plan to find an expression for one variable in terms of the other; and it is usual to find y in terms of x in these simpler forms.

From (1) $5y = 8-4x$ $\quad\therefore\quad y = \frac{8}{5} - \frac{4x}{5} = 1{\cdot}6 - {\cdot}8x$ . . . (4)

From (2) $5y = -4x$, $\quad\therefore\quad y = -{\cdot}8x$ . . . . (5)

From (3) $5y = -12-4x$, $\quad\therefore\quad y = -2{\cdot}4 - {\cdot}8x$ . . . . (6)

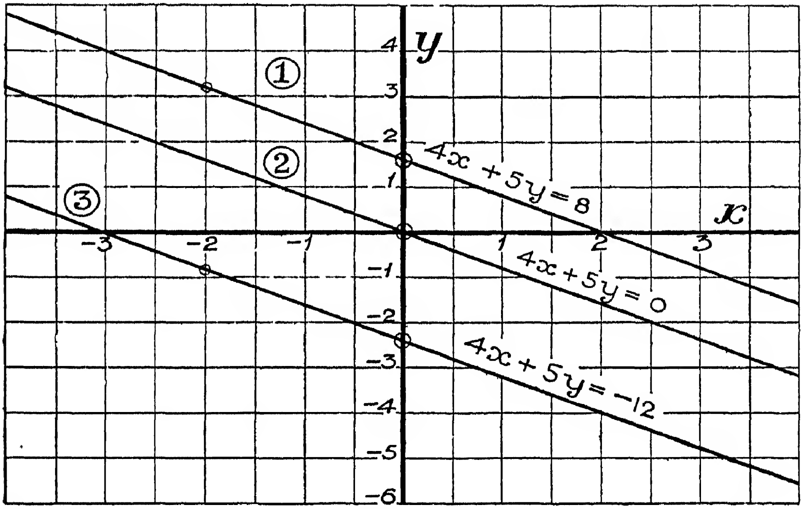


Fig. 83.—**Straight Lines and their Equations.**

Evidently all three equations, viz. (4), (5) and (6), are of the form $y = ax + b$, the value of a being constant throughout, viz. $-{\cdot}8$, whilst the value of b varies. From our previous work, then, we conclude that the three lines representing these equations have the same slope and are therefore parallel, being separated a distance vertically represented by the different values of b.

To plot, first calculate from the equations—

(1) $y = 1{\cdot}6 - {\cdot}8x$. (2) $y = -{\cdot}8x$. (3) $y = -2{\cdot}4 - {\cdot}8x$,

and tabulate the numerical values :—

(1)

x	$1{\cdot}6 - {\cdot}8x$	y
−4	1·6 + 3·2	4·8
−2	1·6 + 1·6	3·2
0	1·6 − 0	1·6
2	1·6 − 1·6	0
4	1·6 − 3·2	− 1·6

(2)

x	$-{\cdot}8x = y$
−4	3·2
−2	1·6
0	0
2	− 1·6
4	− 3·2

(3)

x	$-2{\cdot}4 - {\cdot}8x$	y
−4	−2·4 + 3·2	·8
−2	−2·4 + 1·6	− ·8
0	−2·4 − 0	− 2·4
2	−2·4 − 1·6	− 4 0
4	−2·4 − 3·2	− 5·6

These lines are parallel (see Fig. 83) and cross the y axis, (1) at 1·6, (2) at 0, and (3) at $-$ 2·4, or the values of b in the three cases are 1·6, 0 and $-$ 2·4 respectively.

Solution of Simultaneous Equations by a Graphic Method.—Knowing that a first-degree equation can be represented by a straight line, our attention must now be directed to some useful application of this property. One of the greatest advantages of graphs is that they can be utilised to solve equations of practically every description. As a first illustration we shall solve a pair of simultaneous equations by the graphic method.

Example 7.—To solve, by the graphic method, the equations—

$$5x+3y = 19 \quad \dots\dots\dots \quad (1)$$
$$9x-2y = 12 \quad \dots\dots\dots \quad (2)$$

Each of these equations can be represented by a straight line; and these lines will either be parallel or meet at a point, and at that point only. Such a point represents by its co-ordinates a value of x and a value of y; and since this point is common to the two lines, these values must be the solutions of the given equations.

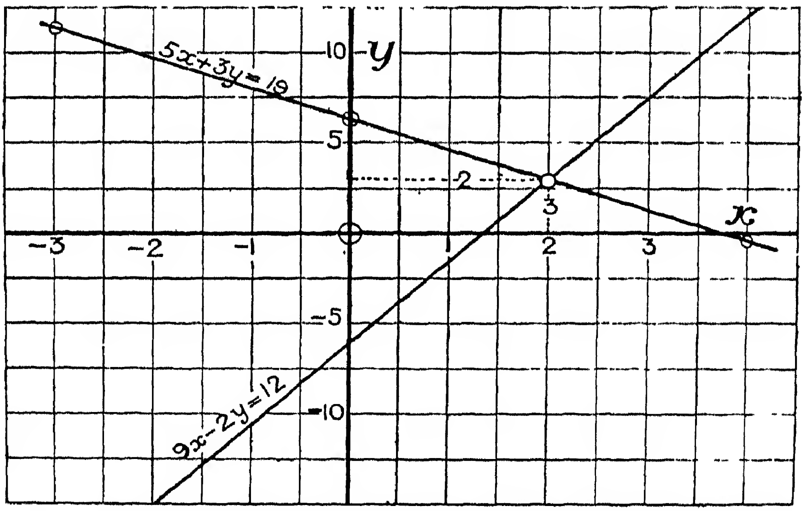


Fig. 84.—Solution of Simultaneous Equations.

[If the given equations were $5x+3y = 19$ and $5x+3y = 9$ it would be found on plotting that the lines were parallel; there could thus be no values of x and y satisfying the two equations at the same time, or, in other words, the equations are not consistent.]

For the example given, the lines are not parallel.

Two points are sufficient to determine a line, and therefore two values only of y need be calculated, but for certainty three are here taken, because if two only were taken, and an error made in one, the line would be entirely wrong.

Equation (1) $5x + 3y = 19$ from which $3y = 19 - 5x$

or $y = 6{\cdot}33 - 1{\cdot}67x$.

Table of values reads :—

x	$6{\cdot}33 - 1{\cdot}67x$	y
-3	$6{\cdot}33 + 5$	$11{\cdot}33$
0	$6{\cdot}33 - 0$	$6{\cdot}33$
4	$6{\cdot}33 - 6{\cdot}68$	$-{\cdot}35$

Equation (2) $9x - 2y = 12$

whence $-2y = 12 - 9x$ or $2y = 9x - 12$

$\therefore$ $y = 4{\cdot}5x - 6$.

Table of values reads :—

x	$4{\cdot}5x - 6$	y
-2	$-9-6$	-15
0	$0-6$	-6
4	$18-6$	12

These two lines must be plotted (see Fig. 84) to the same scales and on the same diagram and their point of intersection noted, viz. (2, 3).

$\therefore$ $\underline{x = 2,\ y = 3}$ are the solutions of the given equations.

[The scales chosen must be such that the point of intersection will be shown; to ensure that this shall be the case a rough mental picture of the diagram should be formed. This is not a difficult matter, as one soon becomes accustomed to reading a table from its graphical aspect. *E. g*, one can see at a glance in which direction the line is sloping, and a little further consideration decides the rate of its rising or falling.]

Exercises 21.—On plotting Co-ordinates, and plotting of Straight Lines representing Linear Equations.

1. On the same diagram plot the points $(2, -5)$; $(-3, 4)$; $(-9, -3)$, $(0, 11)$; and $(1{\cdot}2, 0)$. Indicate each point clearly.

2. Join up the four points $(-10, 10)$; $(5, 10)$; $(15, -2{\cdot}5)$; and $(-10, -2{\cdot}5)$ in the order given, and find the area in sq. units of the figure so formed.

3. On the same diagram plot the points $(1{\cdot}4, 2500)$; $(-{\cdot}75, 3740)$, $(-1{\cdot}82, -1140)$; $({\cdot}32, -4816)$. Indicate clearly the scales chosen.

4. Plot the straight line $3x - 8y = 19$ from $x = -4$ to $x = +5$. What is the slope of this line, and what is its intercept on the vertical axis through 0 on the horizontal?

5. Plot a straight line to show the change of x consequent on change of y between -10 and $+15$; the connection between y and x being ${\cdot}16y = 4{\cdot}28 - 4{\cdot}06x$.

6. The illumination I (foot candles) of a single arc lamp placed 22 ft. above the ground, at d feet from the foot of the lamp is given by

$$I = 1{\cdot}4 - {\cdot}01d.$$

Plot a graph to show the illumination for distances 0 to 12 ft. from the foot of the lamp.

7. Unwin's law states that the velocity of water in ft. per sec. in town supply pipes is $v = 1{\cdot}45d + 2$, where d is the diam. of pipe in ft. Plot a graph to give the diam. of pipe for any velocity from 0 to 13 ft. per sec.

8. The law connecting the ratio $\left(\frac{l}{d}\right)$, *i. e.*, $\frac{\text{length}}{\text{diam.}}$ of a journal with the speed (N, R.P.M.) is $\frac{l}{d} = {\cdot}003N + 1$.

Plot a graph to show values of this ratio for values of N from 20 to 180. If the diam. is $4{\cdot}5''$ what should the length be at 95 R.P.M.?

9. Plot a conversion chart to give the number of radians corresponding to angles between 0 and 360°. (1 radian = 57·3°.)

10. The law connecting the latent heat L with the absolute temperature τ, for steam is—

$$L = 1437 - {\cdot}7\tau.$$

Plot a graph to give the latent heat at any temperature between 460 and 1000 F.° absolute.

11. Plot a graph giving the resistance R of an incandescent lamp at any voltage V between 40 and 110. You are given that—

$$R = 2{\cdot}5V + 75.$$

What is the slope of the resulting graph?

Solve graphically the equations in Exs. 12 to 16:—

12. $5m - 6n = -6{\cdot}6$
$11n - 25 = 2m.$

13. $48x - 27y = 48$
$y - 51x = -51.$

14. $y + 1{\cdot}37 = 4x$
$9x - 17y = -49{\cdot}87.$

15. $7x + 3y = 10$
$35x - 6y = 1.$

16. $y = -1{\cdot}4x - {\cdot}3$
$2{\cdot}6x - y = 13.$

17. The co-ordinates of two points A and B are:—

A. Latitude (vertically) N 400 links; Departure W (horizontally) 700 links

B. Latitude S 160 links; Departure W 1500 links.

Plot the points A and B and find the acute angle which the line AB makes with the N and S line.

Determination of Laws.—The straight line as the representation of an equation finds its most direct and important application in the determination of laws embodying the results of experiments. An experiment has been made with some machine and a number of readings of the variable quantities taken; and it is desirable to express the connection between these quantities in a simple yet conclusive manner. If this is done the law of the machine is known for the range dealt with.

Example 8.—A test is carried out on a steam engine, and trials are made with the engine running at various loads. The amount of steam used per hour (W) and the Indicated Horse Power (I.H.P.) are calculated from the readings taken at each load, and the corresponding values are as follows :—

I (I.H.P.)	2	4	5	7	10	12
W (lbs. of steam per hour)	71	103	121	153	197	234

Find a simple relation connecting W and I.

It is reasonable to assume that to just start the engine a certain amount of steam would be required, which would in a sense be wasted, and that after once starting, the steam used would be practically

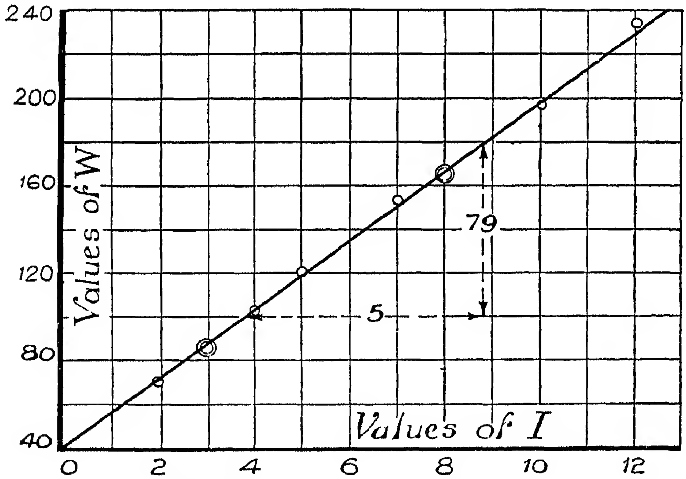


Fig 85 —Test on Steam Engine.

proportional to the power developed : accordingly we should expect a formula of the type $W = b + aI$ where a and b are constants to be determined. This we see is of the standard type $y = ax + b$, or putting it in a more general form (Vertical) $= a$ (Horizontal) $+ b$, where (Vertical) stands for the quantity plotted along the vertical ; therefore, a straight line should result when W is plotted against I.

On plotting (see Fig. 85) we see that a straight line fits the points very nearly, being above some and below others, *i. e.*, averaging the results.

The values of a and b may be found by either of two methods. The first is that used in the laboratory and is to be recommended when the slope of the line is more important than the intercept : it can be used on all occasions when the quantities given admit

of the vertical axis through the zero of the horizontal being drawn without diminishing the scale. This method is very quick, measurements on the paper being scaled off and a quotient easily found. The second method is the more general, but involves rather more calculation; both methods should, however, be studied.

First Method— $W = aI + b$

where a = the slope of the line and b = intercept on the vertical axis.

To find the slope, select some convenient starting-point, say, where the line passes through the corner of a square, and measure a round number of units along the horizontal, in this case (Fig. 85) 5 being taken.

(*Note*.—Distances are measured in terms of units, and not in inches.)

The vertical from the end of the 5 to meet the sloping line measures 79 units;

hence— $\text{slope} = \dfrac{\text{increase in W}}{\text{increase in I}} = \dfrac{79}{5} = 15{\cdot}8, \quad \therefore a = 15{\cdot}8.$

Intercept on axis of W through I = 0 is 40 units, $\therefore b = 40$.

Thus the equation is $\underline{W = 15{\cdot}8I + 40.}$

Second Method, or *Simultaneous Equation Method*—

Select two convenient points on the line, not too close together—

$$e.g.,\ \left.\begin{array}{l} W = 167{\cdot}5 \\ \text{when } I = 8 \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} W = 87{\cdot}5 \\ \text{when } I = 3 \end{array}\right\}$$

Substituting these corresponding values in the equation $W = aI + b$ two equations are formed, the solutions of which are the required values of a and b.

Thus— $167{\cdot}5 = 8a + b \quad \ldots \ldots \ldots \quad (1)$

$87{\cdot}5 = 3a + b \quad \ldots \ldots \ldots \quad (2)$

Subtracting— $80 = 5a$

whence $a = 16$.

Substituting in equation (2) $b = 87{\cdot}5 - 48 = 39{\cdot}5$

$\therefore$ as by first method (very closely) $\underline{W = 16I + 39{\cdot}5.}$

This particular line connecting the weight of steam per hour with the indicated horse-power is known as a Willans' line (named after Mr. Willans, who first put the results of steam-engine tests into this form).

To take a further example—

Example 9.—In a test on a crane the following values were found for the effort P_1 required to raise a weight W. Find the law of the crane.

W (lbs.) .	10	20	30	40	50	60	70	80	90	100
P_1 (lbs.) .	1	1·63	2·13	2·63	3·25	3·75	4·25	5	5·5	6

To find the equation in the form $P_1 = aW + b$ plot W along the horizontal (Fig. 86).

First Method— Slope $= \frac{2{\cdot}82}{50} = {\cdot}0564$, $\therefore\ a = {\cdot}0564$

Also the intercept on the axis through o of $W = {\cdot}41$, $\therefore\ b = {\cdot}41$

$$\therefore \quad \underline{P_1 = {\cdot}0564W + {\cdot}41.}$$

Second Method—

$$\text{when} \quad \left.\begin{array}{l} P_1 = {\cdot}7 \\ W = 5 \end{array}\right\} \quad \begin{array}{l} \text{and} \\ \text{when} \end{array} \quad \left.\begin{array}{l} P_1 = 3{\cdot}8 \\ W = 60 \end{array}\right\}$$

$$\therefore \quad 3{\cdot}8 = 60a + b \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\cdot 7 = 5a + b \quad \ldots\ldots\ldots\ldots \quad (2)$$

Subtracting— $3{\cdot}1 = 55a$

$\therefore$ $a = {\cdot}0564$

Substituting in (2) $b = {\cdot}7 - {\cdot}282 = {\cdot}418$

$$\therefore \quad \underline{P_1 = {\cdot}0564W + {\cdot}418.}$$

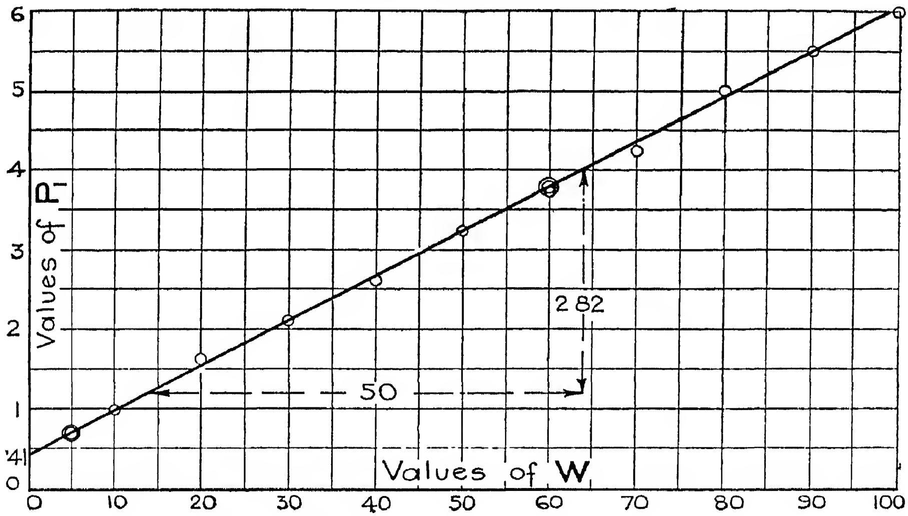


Fig. 86 —Test on a Crane.

This result suggests that ·41 lb. is required to just start the machine, *i.e.*, to overcome the initial friction, and that after that point for every pound lifted only ·0564 lb of effort is required.

If we are told, in addition, that the velocity ratio of the machine is 39, we can calculate the efficiency of the machine for any load.

$$\text{Velocity ratio} = \frac{\text{distance moved by effort}}{\text{distance moved by weight}}$$

and work done by effort = work done by weight; hence, theoretically, 1 lb. of effort should just lift 39 lbs of weight;

i.e., the connection between P and W (theoretically) is $P = \frac{1}{39}W$.

$$\text{Then the efficiency at any load} = \frac{\text{Theoretical effort}}{\text{Actual effort}} = \frac{P}{P_1}$$

$$= \frac{\frac{1}{39}W}{\cdot0564W + \cdot418} = \frac{\cdot0256W}{\cdot0564W + \cdot418}$$

$$= \frac{1}{2\cdot2 + \frac{16\cdot35}{W}}$$

$$e.\,g.,\text{ if } W = 50,\text{ efficiency} = \eta = \frac{1}{2\cdot2 + \frac{16\cdot35}{50}}$$

$$= \underline{\cdot396.}$$

Example 10.—The following are the results of a test on a 6-ton Hydraulic Jack (V.R. = 106).

Load (lbs.)	600	1020	1445	1885	2320	2740	3210	3625	4010
Effort (lbs.)	11	17	22	27·9	32·7	37·5	43·4	45·2	49

It is required to find an expression for the efficiency at any load, and also the maximum efficiency.

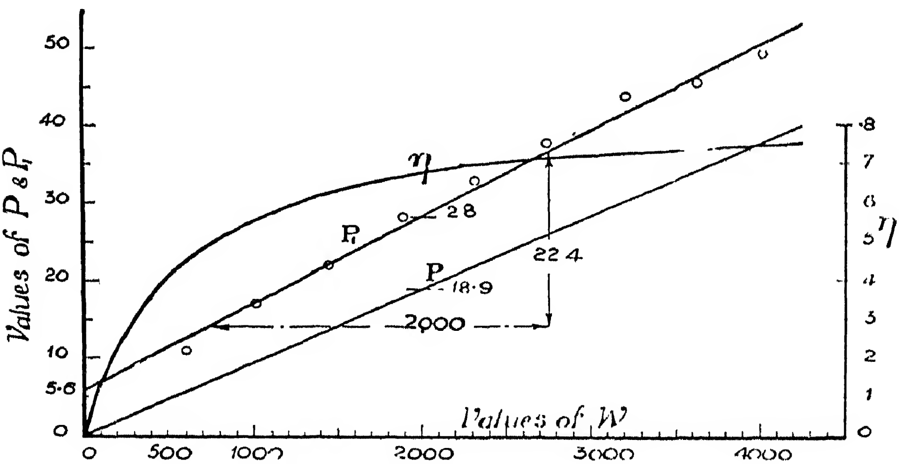


Fig. 87.—Test on Hydraulic Jack.

To a base of W (load) we plot the values of P_1 (practical effort) and average the results by a straight line, as in Fig. 87.

Theoretically, each pound of effort applied should lift 106 lbs. of load, hence a straight line can be drawn giving the theoretical effort (P) for all loads within the range dealt with.

Now, efficiency $\eta = \frac{P}{P_1}$; and therefore for any load find the quotient

$\frac{P}{P_1}$, which will be the efficiency at that load. A new scale must be chosen for efficiency, and the curve, a smooth one, because obtained from two straight lines, is plotted.

$$\left\{e.g., \text{ If } W = 2000, \ P = 18{\cdot}9, \ P_1 = 28, \ \eta = \frac{18{\cdot}9}{28} = {\cdot}675\right\}$$

To find the maximum efficiency, *i. e.*, the efficiency at 6 tons load.

$$P = \frac{1}{106}W = {\cdot}00944W$$

Also—
$$P_1 = 5{\cdot}6 + \frac{22{\cdot}4}{2000}W = 5{\cdot}6 + {\cdot}0112W \quad . \ . \ \text{(See Fig. 87)}$$

$$\eta = \frac{P}{P_1} = \frac{{\cdot}00944W}{5{\cdot}6 + {\cdot}0112W} = \frac{1}{\frac{5{\cdot}6}{{\cdot}0094W} + \frac{{\cdot}0112W}{{\cdot}00944W}}$$

$$\text{or efficiency at any load} = \underline{\frac{1}{\frac{593}{W} + 1{\cdot}19}}$$

Then for the efficiency at 6 tons load we must write 6×2240 for W, hence—

$$\text{maximum efficiency} = \frac{1}{\frac{593}{13440} + 1{\cdot}19} = \frac{1}{1{\cdot}23} = \underline{{\cdot}814.}$$

Exercises 22.—On the Determination of Laws.

1. Find the average value of $\frac{P}{W}$ (coefficient of traction) from the following figures (*i. e.*, find the slope of the resulting straight line).

W (lbs.) .	3	5·5	7·5	9·5	11·5	13·5	15·5	17·5
P (lbs.) .	1·25	2·25	2·75	3·75	4·25	5·25	6·25	7·25

This was for the case of wood on wood.

2. Recalculate but for cast iron on cast iron (dry).

W (lbs.)	33	53·3	63·2	72·9	93·2	113
P (lbs.) .	11·3	19	22	25	28	37·5

In Exs. 3 and 4 the slope of the line gives the value of the Young's Modulus E for the material. Find E in each case, stating the units. (Note that the stress is to be plotted vertically.)

3. For 1″ round, crucible cast steel.

Stress (lbs. per sq. in.)	2000	4000	6000	8000	10000	12000	14000	16000
Extension (inch per inch length) . . .	·00008	·00015	·00021	·00028	·00034	·00041	·00048	·00053

4. For 1″ round, hard-rolled phosphor-bronze.

Stress (lbs. per sq. in.)	2000	4000	6000	8000	10000	12000
Extension (inch per inch length)	·0001	·00022	·00034	·00044	·00055	·00067

5. Find the simple law connecting the Indicated Horse Power I with the Brake Horse Power B, given the following values of I and B :—

B	0	3·33	6·71	8·35	9·94
I	4·5	7·27	10·66	11·69	12·95

$\{I = aB + b\}$

6. The diameter under the thread for various diameters of bolts is given in the table for the Whitworth standard thread. Find the law connecting the smaller diameter, d_1, with the larger, d.

$\{d_1 = ad + b\}$

d	·0625	·09375	·125	·15625	·1875	·25	·375	·5	·625	·75
d_1	·0411	·067	·0929	·1162	·1641	·1859	·2949	·3932	·5085	·6219

7. Recalculate as for Ex. 6, but taking the figures for the British standard fine thread.

d	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{4}$
d_1	·199	·311	·420	·534	·643	·759	·872	1·108

Find the law connecting T and θ in the following cases (Exs. 8 and 9). $T = a\theta + b$. (T = twisting moment and θ = angle of twist.)

8.

T	0	30	60	90	120	150	180	210	240	270	300	330	360
θ	0	·4	·9	1·56	2·1	2·7	3·4	4	4·5	5·1	5·8	6·25	6·82

9.

T	0	1200	2400	3600	4800	6000	7200
θ	0	·34	·67	1·02	1·36	1·71	2·06

Express the results of the tests on incandescent lamps given in Exs. 10, 11 and 12 in the form $R = aV + b$. (R = resistance and V = voltage.)

10. Test on a metallic filament lamp.

V	75	78	80	82	84	86	88	90	92	94	96	98	100	102
R	144	147	148	149	151	153	155	157	158	159	160	161	162·5	164·5

11. Test on two metallic filament lamps in parallel.

V	54	60	65	70	75	80	85	90	95	105
A	·5	·55	·57	·59	·61	·63	·65	·67	·69	·72

(Values of resistance R must first be calculated from $R = \frac{V}{A}$, where A = ampere.)

12. Test on a metallic filament lamp.

V	86	80	70	60	50	40	30
R	277	267	259	231	208	174	150

The following two examples refer to tests on the variation of the resistance of a conductor with variation of temperature. Find the values of R_o (resistance at 0°) and a (temperature coefficient) in each case. [R_o is intercept on the vertical axis through 0 of the temperature scale, and $a = \frac{\text{Slope of line}}{R_o}$.]

13. Equation is of form $R_t = R_o(1 + at)$ where R_t = resistance at temperature t.

Temperature (t) .	10	25	35	50	80	90	100
Resistance (R_t) .	1·039	1·1	1·141	1·198	1·32	1·357	1·402

14.

t	17 1	25·4	30·3	36·2	41	49·4	61·3	67	74·3	80 1	93·8
R_t	1 214	1·259	1·285	1·317	1·341	1·369	1 44	1·473	1·505	1·532	1·622

15. The following results were obtained from the testing plant of the Pennsylvania Railroad Co. :—

x (B.Th U. across heating surface per min.)	207200	247500	295900	331000	367500	393500	443000	448500	481300
I (I.H.P.)	365 7	454 7	587·6	650	779·3	803·3	951 4	975·1	1036

Find the law connecting I and x in the form $I = ax + b$.

16. From the following figures find the value of g. ($g = 3{\cdot}29 \times$ slope of line : obtained by plotting t^2 horizontally and l vertically.)

l	34·5	30	28	25	21	16	12
t	1·87	1·76	1·67	1·6	1·49	1·26	1·11

[t = periodic time in seconds of a pendulum swing, l = length of simple pendulum in inches, and g = acceleration due to gravity, in feet per sec. per sec.]

17. Find the value of Young's Modulus E for the material of a beam, from the following :—

Load (W lbs.) .	0	36·5	56·5	96·5	136·5	176·5	216·5	256·5	296·5	316·5
Deflection (d in.)	0	·12	·198	·34	·51	·63	·79	·925	1·07	1·17

Also $d = \frac{Wl^3}{48EI}$, $l = 40''$, and $I = \cdot 0127$.

$\left(\textit{Hint.}\text{—Slope of line} = \frac{d}{W}.\right)$

Graphs representing Expressions of the Second Degree. —Consideration must now be paid to the graphs of such equations as $y = 5x^2 + 7x - 9$, or $x = ay^2 + by + c$. As mentioned before, the curves representing these equations will be smooth and of standard forms. The preliminary calculation must be performed in a manner similar to that already employed for the straight-line graphs. The only trouble likely to be experienced is with the signs: it must be remembered that -3 or $+3$ squared each gives 9, so that if $x = -3$ and $-x^2$ is required, the value is $-(9)$, *i. e.*, -9; also $-6x^2$ would be $-6 \times (-3)^2 = -6 \times 9 = -54$.

Since we are no longer dealing with straight lines, two points are not sufficient to determine the curve, so a number of values must be taken.

Example 11.—Plot, from $x = -5$ to $x = +4$, the graph representing the equation—

$$y = 5x^2 + 7x - 9.$$

Arranging the calculation in tabular form :—

x	x^2	$5x^2 + 7x - 9$	y
-5	25	$125 - 35 - 9$	81
-4	16	$80 - 28 - 9$	43
-3	9	$45 - 21 - 9$	15
-2	4	$20 - 14 - 9$	-3
-1	1	$5 - 7 - 9$	-11
0	0	$0 + 0 - 9$	-9
1	1	$5 + 7 - 9$	3
2	4	$20 + 14 - 9$	25
3	9	$45 + 21 - 9$	57
4	16	$80 + 28 - 9$	99

The scale for x must admit of a range of 9 units, whilst that for y requires a range of 110 units: and as the greater part of the curve is to be on the positive side of the x axis, this axis should be drawn fairly

low down on the paper and not in the centre (see Fig. 88). After plotting the points from the table of values, a smooth curve should be sketched in, passing through all the points; and if any one point is not well on the curve, the portion of the table in which the calculation for that point occurs must be referred to. The curve is a form of parabola, whose axis is vertical, and whose vertex is at the *bottom* of the curve: indeed, in all equations of the type $y = ax^2 + bx + c$ the curve will be of the form shown if a is positive; while if a is negative the axis will still be vertical, but the vertex will be at the *top* of the curve.

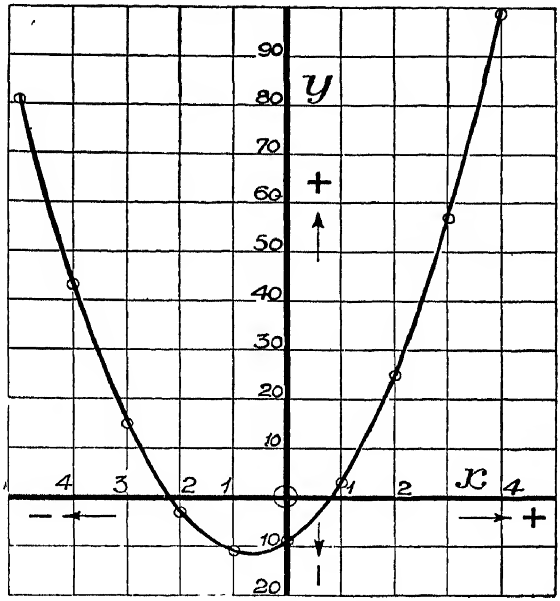


Fig. 88.—Curve of $y = 5x^2 + 7x - 9$.

As an illustration of the latter type—

Example 12.—Plot the curve $4y = -3x^2 - 8x + 2{\cdot}44$, from $x = -6$ to $x = +3$.

Division by 4 gives, $y = -{\cdot}75x^2 - 2x + {\cdot}61$.

Table of values :—

x	x^2	$-{\cdot}75x^2 - 2x + {\cdot}61$	y
-6	36	$-27 + 12 + {\cdot}61$	$-14{\cdot}39$
-5	25	$-18{\cdot}75 + 10 + {\cdot}61$	$-8{\cdot}14$
-4	16	$-12 + 8 + {\cdot}61$	$-3{\cdot}39$
-3	9	$-6{\cdot}75 + 6 + {\cdot}61$	$-{\cdot}14$
-2	4	$-3 + 4 + {\cdot}61$	$+1{\cdot}61$
-1	1	$-{\cdot}75 + 2 + {\cdot}61$	$+1{\cdot}86$
0	0	$0 + 0 + {\cdot}61$	$+{\cdot}61$
1	1	$-{\cdot}75 - 2 + {\cdot}61$	$-2{\cdot}14$
2	4	$-3 - 4 + {\cdot}61$	$-6{\cdot}39$
3	9	$-6{\cdot}75 - 6 + {\cdot}61$	$-12{\cdot}14$

Here the greater part of the curve is negative; hence the axis of

x must be higher than the centre of the paper. The plotting is shown in Fig. 89.

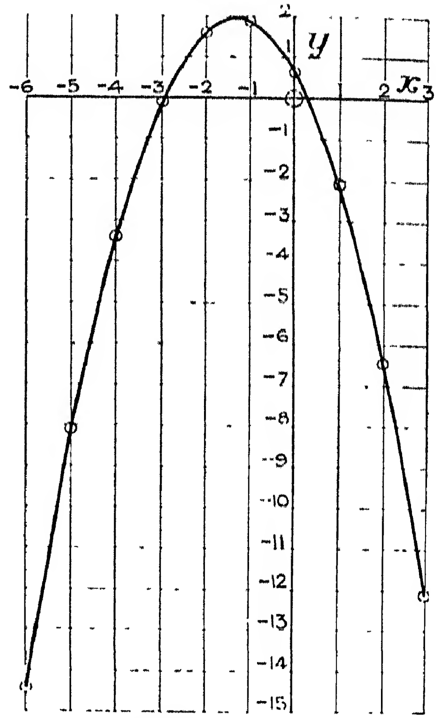


Fig. 89.—Curve of $4y = -3x^2 - 8x + 2\cdot44$.

Solution of Quadratic Equations.—The equations

$$5x^2+7x-9 = 0$$

and $\quad -\cdot75x^2-2x+\cdot61 = 0,$

or, in fact, any quadratic equation, can be solved by the aid of graphs. For the equations $y = 5x^2+7x-9$ and $5x^2+7x-9 = 0$ to be alike, y must equal 0. Now y is $= 0$ anywhere along the x axis: if, then, we wish to arrange that the y value or ordinate of the curve is to be 0, we must select the value or values of x that make it so; or, in other words, we must find those values of x at the points where the curve crosses the x axis. These values of x are the solutions or roots of the equation $5x^2+7x-9 = 0$. From the diagram (Fig. 88) we see that the curve crosses the x axis when $x = \cdot82$ and also when $x = -2\cdot22$: therefore $x = \cdot82$ or $-2\cdot22$ gives the two solutions of $5x^2+7x-9 = 0$.

In like manner the roots of $-\cdot75x^2-2x+\cdot61$ are $-2\cdot95$ and $\cdot28$. (See Fig. 89.)

Solution of Quadratic Equations on the Drawing Board.—Whilst on the question of the graphical solution of quadratic equations, mention may be made of a method that is simple and requires the use of set squares and compasses, but not squared paper

The general quadratic equation is $ax^2 + bx + c = 0$.

To solve this equation by the method of this paragraph: Set off a length OA (see a, Fig. 90) along a horizontal line, working from left to right, to represent a units to some scale. Through A draw AB perpendicular to OA; if b is positive a length to represent b must be measured, giving AB, so that the arrows continue in a right-hand direction. If c is positive draw BC perpendicular to AB, making BC to represent c units to the same scale as before, the

arrows still continuing to indicate right-hand movement about O. (If c were negative BC would be measured to the other side of AB.) Join OC. On OC as diameter describe a circle to meet AB in the points D and E. Then the roots of the equation are $\frac{DA}{OA}$ and $\frac{EA}{OA}$.

Proof of the construction.—Let F be the centre of the circle ODC (a, Fig. 90). Draw FG parallel to OA to cut AB in G and join C to H, the point at which the circle cuts OA.

Then, from the property of intersecting chords—

$$OA \times AH = EA \times AD$$

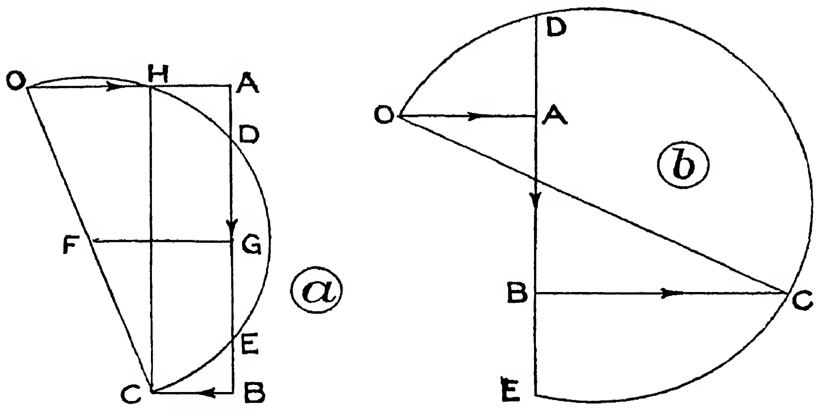


Fig. 90.—Solution of Quadratic Equations.

Dividing both sides by $(OA)^2$—

$$\frac{OA}{OA} \times \frac{AH}{OA} = \frac{EA}{OA} \times \frac{AD}{OA}$$

or

$$\frac{AH}{OA} = \frac{EA}{OA} \times \frac{AD}{OA} \quad \ldots\ldots\ldots \quad (1)$$

Now the angle OHC is a right angle since it is the angle in a semicircle and since angle OAB is a right angle also, CH and AB are parallel and AH = BC = c.

Also, since FG and OA are parallel and F bisects the line OC, then GA = GB.

Then—

$$EA+DA = ED+DA+DA = 2GD+2DA = 2GA = BA = -b$$

or

$$\frac{EA}{OA} + \frac{DA}{OA} = -\frac{b}{OA} = -\frac{b}{a} \quad \ldots\ldots\ldots \quad (2)$$

Let $\frac{DA}{OA} = \alpha$ and $\frac{EA}{OA} = \beta$.

Then from equation (1) $\frac{AH \text{ or } BC}{OA} = \alpha\beta$ or $\alpha\beta = \frac{c}{a}$

and from equation (2) $\alpha + \beta = -\frac{b}{a}$

The original equation $ax^2 + bx + c = 0$ might be written –

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

or $$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

which after factorisation becomes $(x - \alpha)(x - \beta) = 0$

whence $x = \alpha$ or β.

In other words, α and β or $\frac{DA}{OA}$ and $\frac{EA}{OA}$ are the roots of the original equation.

Example 13.—Solve the equation $5x^2 + 7x - 9 = 0$ by this method.

Starting from the point O (*b*, Fig. 90) set out OA to represent 5 units to some scale. Draw AB *downwards* from A, since 7, the coefficient of x, is positive, and make it 7 units long. From B draw BC 9 units long, to the *left* of the positive direction of AB (since the constant term is negative) Join OC and on it describe the circle cutting AB in D and also in E.

Then DA $= + 4{\cdot}04$ units, EA $= - 11{\cdot}1$ units and OA $= 5$ units

or the roots are— $\frac{DA}{OA}$, *i. e.*, $\frac{4{\cdot}04}{5}$ or $\underline{{\cdot}81}$

and $\frac{EA}{OA}$, *i. e.*, $\frac{-11{\cdot}1}{5}$ or $-\underline{2{\cdot}22}$.

Example 14.—Solve by the same means the equation—

$$-1{\cdot}5x^2 + 4x + 1{\cdot}22 = 0.$$

First, change the signs throughout to make the coefficient of x^2 positive, *i. e.*, the equation becomes—

$$1{\cdot}5x^2 - 4x - 1{\cdot}22 = 0.$$

Set out, in Fig. 91, OA $= 1{\cdot}5$ units, AB (upwards, for b is negative) $= 4$ units, and BC (to the right, to reverse the direction of movement about O, for c is negative) $= 1{\cdot}22$ units. The circle on OC as diameter cuts AB in D and E.

DA $= - {\cdot}42$ (for this would give left-hand rotation about O);

EA $= + 4{\cdot}45$, OA $= 1{\cdot}5$.

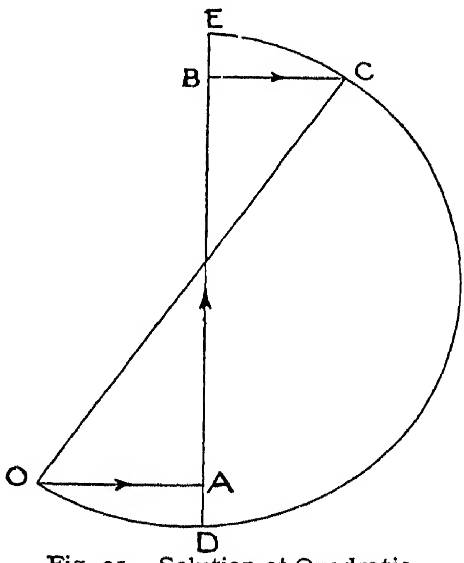


Fig. 91.—Solution of Quadratic Equation.

Then the roots are— $\frac{DA}{OA} = \frac{-\cdot42}{1\cdot5} = \underline{-\cdot28}$

and $\frac{EA}{OA} = \frac{4\cdot45}{1\cdot5} = \underline{+2\cdot96.}$

Graphs representing Equations of Higher Degree than the Second.—This work will best be understood by some examples.

Example 15.—Plot a curve to show the cubes of all numbers between 0 and 6. Use this curve to find the cube roots of 30 and 200.

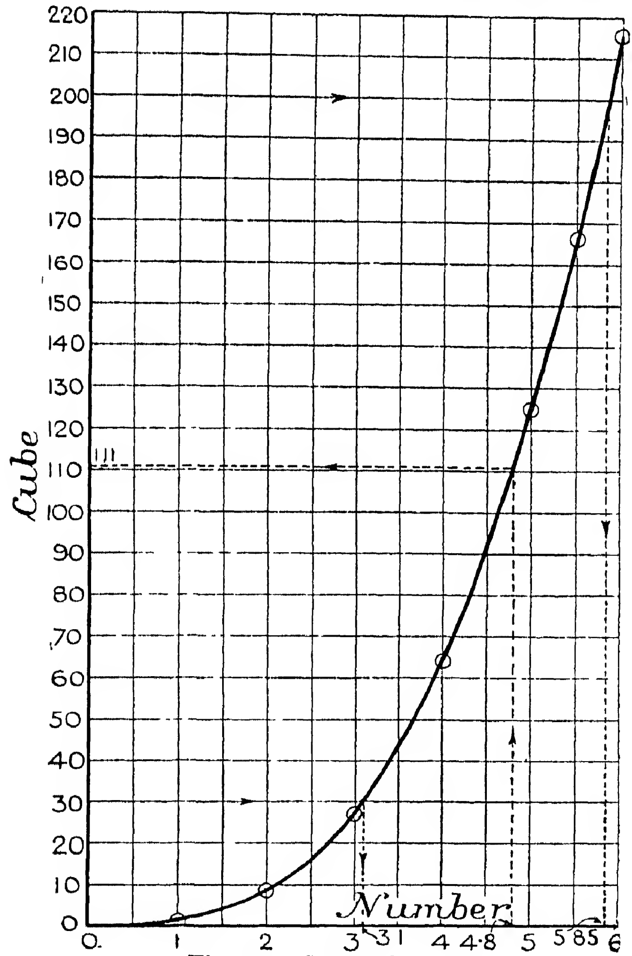


Fig. 92.—Curve of $y = x^3$.

If x represents the numbers and y the cubes then the equation of the curve will be $y = x^3$.

A few values of x may be taken, and the corresponding values of y calculated, the curve being plotted to pass through these points. All intermediate values can be interpolated from the curve.

The table of values reads :—

x	0	1	2	3	4	5	6
$y = x^3$	0	1	8	27	64	125	216

The points all lie on a smooth curve (see Fig. 92), which is known as a "cubic" parabola.

To read cubes, we must work from the horizontal scale to the curve and thence to the vertical scale; thus the cube of $4 \cdot 8 = 111$ while for the determination of cube roots the process is reversed; thus $\sqrt[3]{30} = 3 \cdot 1$ and $\sqrt[3]{200} = 5 \cdot 85$.

Example 16.—Represent the equation $y = x^3 - 8x^2 + 3x + 15$, by a graph (x to range from -4 to $+4$).

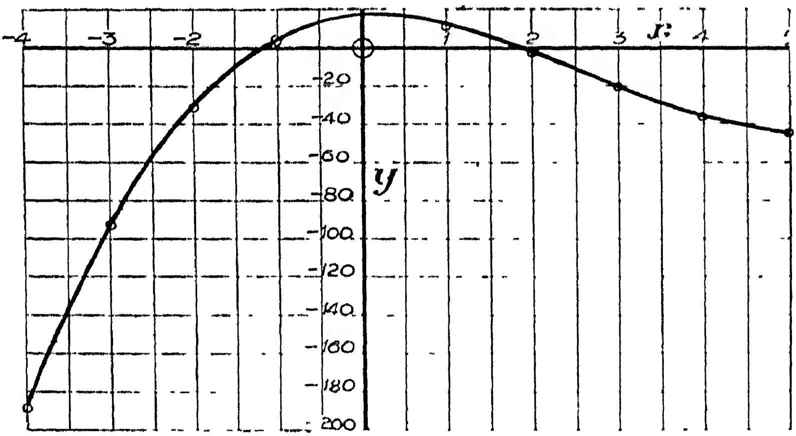


Fig. 93.—Curve of $y = x^3 - 8x^2 + 3x + 15$.

The table is arranged thus :—

x	x^2	$x^3 - 8x^2 + 3x + 15$	y
-4	16	$-64 - 128 - 12 + 15$	-189
-3	9	$-27 - 72 - 9 + 15$	-93
-2	4	$-8 - 32 - 6 + 15$	-31
-1	1	$-1 - 8 - 3 + 15$	$+3$
0	0	$0 - 0 + 0 + 15$	$+15$
1	1	$1 - 8 + 3 + 15$	$+11$
2	4	$8 - 32 + 6 + 15$	-3
3	9	$27 - 72 + 9 + 15$	-21
4	16	$64 - 128 + 12 + 15$	-37

The greater part of this curve is negative, hence the axis of x is taken well up to the top of the paper (Fig. 93).

A warning is again given concerning the evaluation of $-8x^2$; *e. g.*, when $x = -4$. First find x^2, *i. e.*, $(-4)^2$ or $+16$, then find $8x^2$, *i. e.*, $+128$, and finally $-8x^2 = -128$.

Example 17.—Solve, graphically, the equation—

$$2x^3 - 9x^2 - 2x + 24 = 0.$$

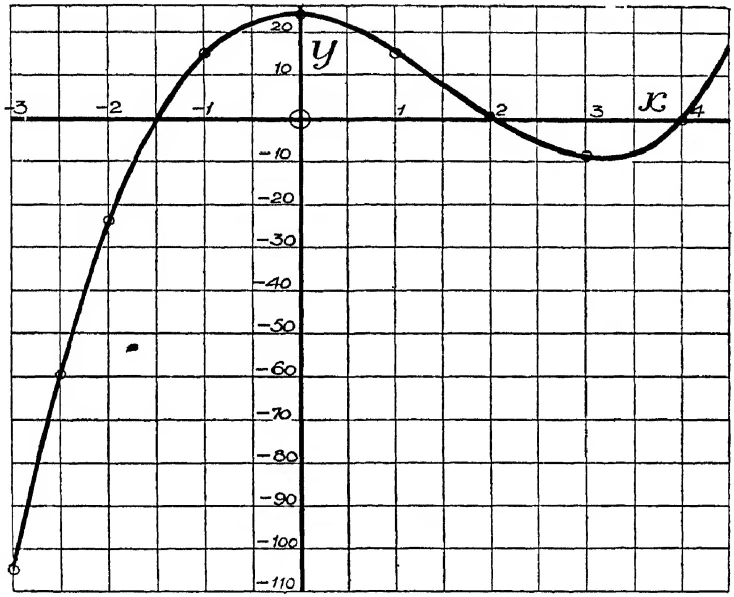


Fig. 94.—Curve of $y = 2x^3 - 9x^2 - 2x + 24$.

We shall first plot the curve $y = 2x^3 - 9x^2 - 2x + 24$ and then determine the values for x at the intersections of the curve with the x axis.

Let x range from -3 to $+5$, and arrange the table as indicated :—

x	x^2	x^3	$2x^3 - 9x^2 - 2x + 24$	y
-3	9	-27	$-54 - 81 + 6 + 24$	-105
-2	4	-8	$-16 - 36 + 4 + 24$	-24
-1	1	-1	$-2 - 9 + 2 + 24$	15
0	0	0	$0 - 0 - 0 + 24$	24
1	1	1	$2 - 9 - 2 + 24$	15
2	4	8	$16 - 36 - 4 + 24$	0
3	9	27	$54 - 81 - 6 + 24$	-9
4	16	64	$128 - 144 - 8 + 24$	0
5	25	125	$250 - 225 - 10 + 24$	39

On plotting the values of y against those of x the curve in Fig. 94 is obtained.

We observe that the curve is of a different character from the "square" parabola, in that it bends twice whereas the latter bends but once; there is thus one bend for a second-degree equation, two bends for a third-degree equation and so on. One can form some idea of the form of the curve from the equation by bearing in mind this fact.

The curve crosses the x axis at three points and three points only; and the three values of x satisfying the given equation are found from these points of intersection. Thus in Fig. 94—

$$\underline{x = -1{\cdot}5,\ 2,\ \text{and } 4.}$$

A cubic equation has three roots, although in some cases only one may be evident, the others being *imaginary*: if the curve were drawn to represent an equation, two of the roots of which were imaginary, it would cross the x axis at one point only, the bends being either both above or both below it.

Example 18.—A cantilever, 30 ft. long, carries a uniformly-distributed load of w tons per foot run. The deflection y at distance x from the fixed end is given by the formula—

$$y = \frac{w}{24\mathrm{EI}}(6l^2x^2 - 4lx^3 + x^4)$$

where I = moment of inertia of section of cantilever
E = Young's Modulus of material.
l = span.

If $w = 5$, I = 200, and E = 12500, show by a graph the deflected form of the cantilever.

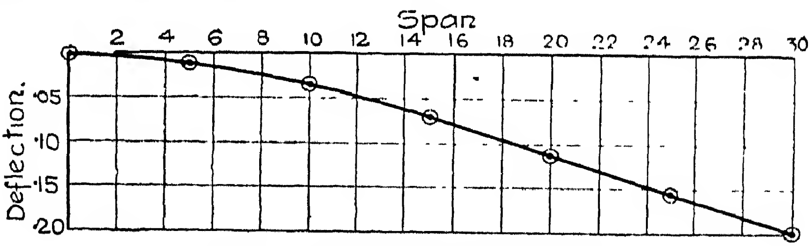


Fig. 95.—Deflection of Cantilever.

Substituting values—

$$\begin{aligned} y &= \frac{5}{24 \times 12500 \times 200}(5400x^2 - 120x^3 + x^4) \\ &= {\cdot}833 \times 10^{-7}(5400x^2 - 120x^3 + x^4) \\ &= {\cdot}833 \times 10^{-7} \times \mathrm{Y} \end{aligned}$$

(Y is substituted in place of the expression $5400x^2 - 120x^3 + x^4$.)

Since the powers of x combined with their respective coefficients give large numbers, it is found to be better to express all these large

numbers as simple numbers multiplied by a power of ten. Thus the product $5400x^2$ when $x = 5$, which has the value 135000, is written $1{\cdot}35 \times 10^5$, and similarly the other products are written in this abbreviated form. One has thus to deal with the addition and subtraction of small numbers, performing the multiplication or division by 10^{-5} at the end once instead of three times. To find values of y from those of Y we must divide by 10^7 and multiply by $\cdot 833$, and according to our scheme we find it convenient to note the values of $Y \times 10^{-5}$ (shown in the sixth column) and then multiply these by $\cdot 833$, dividing by 10^2. By arranging the work in columns one setting of the slide rule suffices for the multiplication by each particular constant, *i. e.*, in evaluating the values of $5400x^2$, 54 on the D scale would be set level with 1 on the C scale; and the figures in the second column would be taken on the C scale, while the figures on the D scale level with these would be the products of 5400 and x^2.

Tabulation :—

x	x^2	x^3	x^4	$5400x^2 - 120x^3 + x^4$	$Y \div 10^5$	y
0	0	0	0	$0 - 0 + 0$	0	0
5	25	125	625	$1\ 35 \times 10^5 - \cdot 15 \times 10^5 + \cdot 063 \times 10^5$	1·26	0105
10	100	1000	10^4	$5{\cdot}4 \times 10^5 - 1{\cdot}2 \times 10^5 + \cdot 1 \times 10^5$	4·3	·0358
15	225	3375	50600	$12{\cdot}14 \times 10^5 - 4\ 05 \times 10^5 + \cdot 506 \times 10^5$	8·6	·0716
20	400	8000	160000	$21{\cdot}6 \times 10^5 - 9{\cdot}6 \times 10^5 + 1{\cdot}6 \times 10^5$	13·6	·1133
25	625	15630	390000	$33{\cdot}7 \times 10^5 - 18\ 75 \times 10^5 + 3\ 9 \times 10^5$	18 85	·157
30	900	27000	810000	$48\ 6 \times 10^5 - 32{\cdot}4 \times 10^5 + 8\ 1 \times 10^5$	24·3	·2025

The deflected form is shown in Fig. 95, the scale for deflections being magnified in comparison with the linear scale.

Turning-points of Curves : Maximum and Minimum Values.—A quadratic curve has one bend, and a cubic has two : there must therefore be some one point on each of these bends which is either higher or lower than all other points in its immediate neighbourhood, for the curves are perfectly smooth and continuous. Such points are known as *turning-points* of the curve, and it is with these that we must now deal. If the curve is an ordinary parabola, let us say that representing the equation $y = 5x^2 + 7x - 9$ (see Fig. 88), there can only be one turning-point, and that is lower than all points on the curve round about it. Referring now to the ordinate at that particular point we note that it is less, algebraically (*i. e.*, taking account of sign), than any other ordinate near to it; it is therefore spoken of as a *minimum* value of the function. What is usually required is the value of the "independent variable" that makes the function a maximum or minimum : hence the highest or lowest point on the curve must be

found, by sliding a straight edge parallel to the x axis until it just touches the curve, the abscissa of this point being noted. Thus the function $5x^2 + 7x - 9$ has its minimum value when $x = -\cdot 7$.

The curve $y = -\cdot 75x^2 - 2x + \cdot 61$ would have no minimum value ("minimum" being understood to imply "less than any other value in the immediate vicinity"), but would have its ordinate a *maximum* when $x = -1\cdot 33$ (see Fig. 89). It is possible for a minimum value of an ordinate to be greater than a maximum.

Many instances occur in practice in which greatest or least values have to be found, or, more generally, values of some variables which cause some function to have maximum or minimum values. Questions of economy of material or time, best dimensions for certain conditions, etc., all arise, and may be classed under the heading of "maximum and minimum" problems. Before dealing with any of these, an ordinary theoretical example will be treated as a clear demonstration of the principles involved.

Example 19.—Find the value or values of x that make the function $x^3 + 2x^2 - 4x + 7$ a maximum or minimum. State clearly the nature of the turning-points.

First plot the curve $y = x^3 + 2x^2 - 4x + 7$. For this, the tabulation is as follows :—

x	x^2	$x^3 + 2x^2 - 4x + 7$	y
-4	16	$-64 + 32 + 16 + 7$	-9
-3	9	$-27 + 18 + 12 + 7$	10
-2	4	$-8 + 8 + 8 + 7$	15
-1	1	$-1 + 2 + 4 + 7$	12
0	0	$0 + 0 - 0 + 7$	7
1	1	$1 + 2 - 4 + 7$	6
2	4	$8 + 8 - 8 + 7$	15
3	9	$27 + 18 - 12 + 7$	40
4	16	$64 + 32 - 16 + 7$	87

A rough plotting is made (in Fig. 96) from the figures in this table; and for greater accuracy the portion between $x = -3$ and $x = -1$ and that between 0 and $1\cdot 5$ are drawn to a larger scale and more values of x are taken. One should always adopt such refinements as this; and especially does this apply when solving equations, viz. disregard the portion of the curve that is of no immediate use and deal with the useful portion in greater detail.

Apparently one turning-point is in the neighbourhood of -2 and another in the neighbourhood of 1, therefore take as additional

values for x, $-2 \cdot 5$, $-1 \cdot 5$, $\cdot 5$ and $1 \cdot 5$. Thus the subsidiary table reads :—

x	x^2	$x^3 + 2x^2 - 4x + 7$	y
$-2 \cdot 5$	$6 \cdot 25$	$-15 \cdot 63 + 12 \cdot 5 + 10 + 7$	$13 \cdot 87$
$-1 \cdot 5$	$2 \cdot 25$	$-3 \cdot 38 + 4 \cdot 5 + 6 + 7$	$14 \cdot 12$
$\cdot 5$	$\cdot 25$	$\cdot 13 + \cdot 5 - 2 + 7$	$5 \cdot 63$
$1 \cdot 5$	$2 \cdot 25$	$3 \cdot 38 + 4 \cdot 5 - 6 + 7$	$8 \cdot 88$

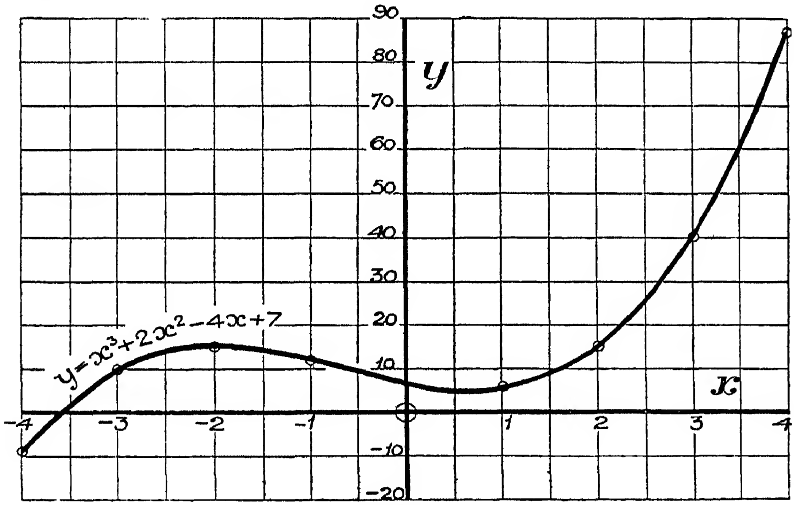


Fig 96.

Drawing only these portions of the curve (see (*a*) and (*b*), Fig. 97) we find that the trend is horizontal when $x = -2$ and also when $x = \cdot 67$. Therefore, $y = x^3 + 2x^2 - 4x + 7$ is a maximum when $\underline{x = -2}$, and a minimum when $\underline{x = \cdot 67}$.

Example 20.—We require to find for what external resistance R the power supplied from a battery of internal resistance r and electromotive force E is a maximum. We are told that $E = 8 \cdot 4$ and $r = \cdot 57$.

$$\text{The power} = (\text{Current})^2 \times \text{external resistance}$$

$$= \left(\frac{\text{E.M.F.}}{\text{total resistance}}\right)^2 \times \text{external resistance}$$

$$= \frac{RE^2}{(R + r)^2} = \frac{R \times (8 \cdot 4)^2}{(R + \cdot 57)^2}$$

Since $(8 \cdot 4)^2$ is a constant it can be disregarded throughout as it

does not affect the resistance for which the power is a maximum, but only the magnitude of the power.

Let $W = \frac{R}{(R + \cdot 57)^2}$; then we require a value of R that makes W a maximum, and R must be treated as the I.V., *i.e.*, R is plotted along the horizontal.

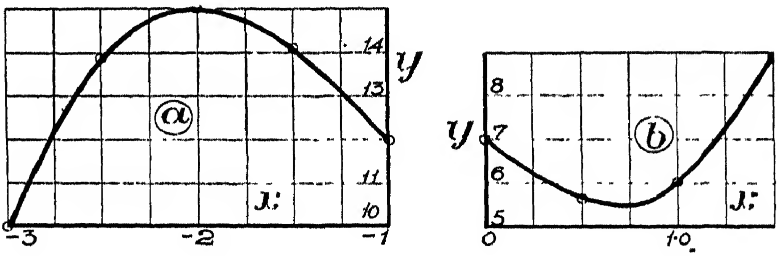


Fig. 97.

No negative values need be taken for R, but otherwise we have no idea as to its magnitude; a preliminary tabulation, and if necessary a preliminary graph, must consequently be first made

The table reads :—

R	(R + ·57)	$(R + \cdot 57)^2$	$\frac{R}{(R + \cdot 57)^2} = W$
0·0	·57	·325	0·000
·5	1·07	1·14	·438
1·0	1·57	2·46	·406
1·5	2·07	4·27	·352
2·0	2·57	6·6	·303

Apparently the curve rises fairly rapidly from R = 0 to R = ·5 and then falls again : hence we conclude that the maximum value of W will be obtained when R = ·5 or thereabouts. (If this reasoning cannot be followed from mere inspection of the table, a rough graph should be drawn to represent it.)

Accordingly, let us take values between R = ·2 and 1·0.

R	R + ·57	$(R + \cdot 57)^2$	$\frac{R}{(R + \cdot 57)^2} = W$
·2	·77	·592	·338
·4	·97	·941	·425
·5	1·07	1·145	·4367
·6	1·17	1·37	·4383
·8	1·37	1·88	·4263
1·0	1·57	2·46	·406

Plotting the portion between R = ·4 and ·8 (as in Fig. 98) we find that W has its maximum value when R = ·57, *i.e.*, the external resistance is equal to the internal resistance.

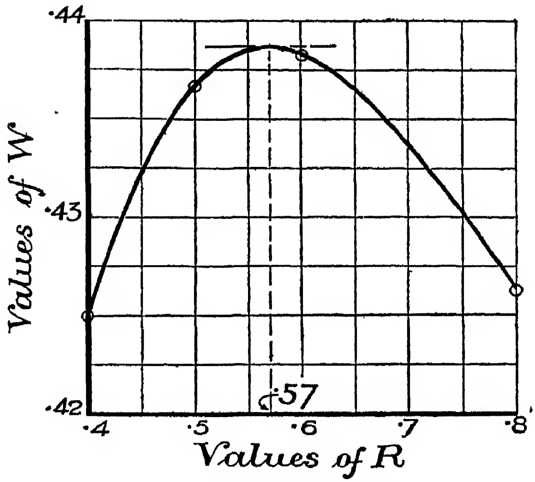


Fig. 98.—Curve of Power from an Electric Battery.

Example 21.—The horse power transmitted by a belt passing round a pulley and running at v feet per sec. is given by—

$$\text{H.P.} = \frac{v}{1100}\left(\text{T} - \frac{wv^2}{g}\right)$$

where T = maximum stress permissible in belt = 350 lbs./□″
w = mass of 1 foot length of belt = ·4 lb.
g = 32·2. {The belt is 4″ wide and ¼″ thick.}

Find the speed at which the greatest horse-power is transmitted under these conditions: find also the maximum horse-power transmitted.

Substituting the values of T, w and g, the equation becomes—

$$\text{H.P.} = \frac{1}{1100}\left(350v - \frac{\cdot 4v^3}{32\cdot 2}\right)$$

$$= \frac{1}{1100}(350v - \cdot 0124v^3).$$

The factor $\frac{1}{1100}$ may be disregarded in the curve plotting, as it is simply a constant, and does not affect the value of v without similarly affecting H.P.

Hence we plot the curve, $H_1 = 350v - \cdot 0124v^3$; and taking values of v from 0 to 160 we obtain the following table :—

v	v^3	$350v - \cdot 0124v^3$	H_1
0	0	0 — 0	0
20	8000	7000 — 99	6100
40	64000	14000 — 794	13206
60	216000	21000 — 2680	18320
80	512000	28000 — 6250	21750
100	10^6	35000 — 12400	22600
120	$1 \cdot 728 \times 10^6$	42000 — 21100	20900
140	$2 \cdot 352 \times 10^6$	49000 — 28700	20300
160	$4 \cdot 096 \times 10^6$	56000 — 50000	6000

H_1 is evidently a maximum somewhere in the neighbourhood of $v = 100$; accordingly, taking some intermediate values, the subsidiary table reads :—

90	729000	31500 — 9040	22460
95	855000	33200 — 10600	22600
105	$1 \cdot 16 \times 10^6$	36800 — 14400	22400
97	913000	33930 — 11310	22620

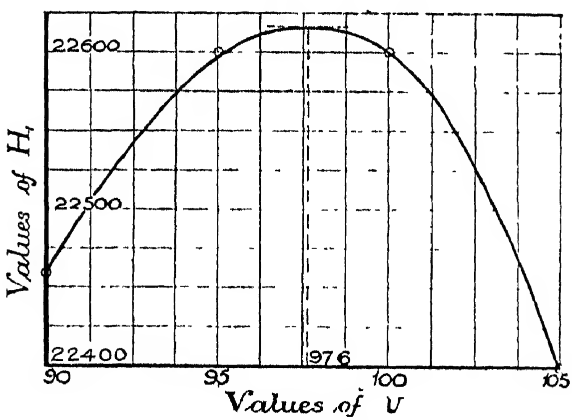


Fig. 99.—Curve of H.P. transmitted by Belt.

Plotting the portion of the graph from $v = 90$ to $v = 105$ (Fig. 99), we find that H_1 is a maximum when $v = 97 \cdot 6$. Maximum value of H_1 is 22615, *i.e.*, the maximum H.P. $= \frac{22615}{1100} = 20 \cdot 6$. Hence we conclude that the greatest H.P. is transmitted at a speed of $\underline{97 \cdot 6}$ ft. per sec. and that the greatest H.P. transmitted is $\underline{20 \cdot 6}$.

Exercises 23.—On the plotting of Graphs of Quadratic and Cubic Expressions: and on Maximum and Minimum Values.

1. Plot from $x = -5$ to $x = +3$ the curve $y = 3x^2 - 5x + 13$.

2. Plot from $x = -3$ to $x = +6$ the curve $y = 4{\cdot}15x - {\cdot}23x^2 + 1{\cdot}94$.

3. The centrifugal force on a pulley rim running at v ft. per sec. is found from $T = \frac{wv^2}{g}$. If $w = 3{\cdot}36$ and $g = 32{\cdot}2$, plot a curve to give values of T for values of v ranging from 70 to 200.

4. Plot a curve giving the H.P. transmitted by a belt running at velocity v from H.P. $= \frac{vt - {\cdot}014v^3}{550}$ when $t = 400$ and v is to range from 0 to 165.

5. Indicate by a graph the changes in B consequent on the variation of T from 10 to 50 when—

$$B = {\cdot}124 - \frac{{\cdot}684}{T}$$

6. If w = lbs. of water evaporated per lb. of fuel, and f = lbs. of fuel stoked per hour per sq. ft. of grate—

$$w = \frac{54}{f} + 8{\cdot}5.$$

Plot a curve to give values of w as f ranges from 12 to 40.

7. The weight per foot W of certain railroad bridges for electrical traffic can be calculated from $W = 50 + 5l$, where l = span in feet. Plot a graph to give the total weight of bridges, the span varying from 12 to 90 ft.

8. Johnson's parabolic formula for the buckling stress (lbs. per sq. in) of struts is (for W.I. columns having pin ends)—

$$p = 34000 - {\cdot}67\left(\frac{l}{k}\right)^2$$

Plot a curve to give values of p for values of $\left(\frac{l}{k}\right)$ from 0 to 150.

9. Plot as for Ex. 8, but for C.I. columns, for which the relation is expressed by the formula $p = 60000 - \frac{25}{4}\left(\frac{l}{k}\right)^2$; the range of the ratio $\frac{l}{k}$ being from 0 to 55.

10. For Yorke's notched weir or orifice for the measurement of the flow of water, the quantity flowing being proportional to the head,

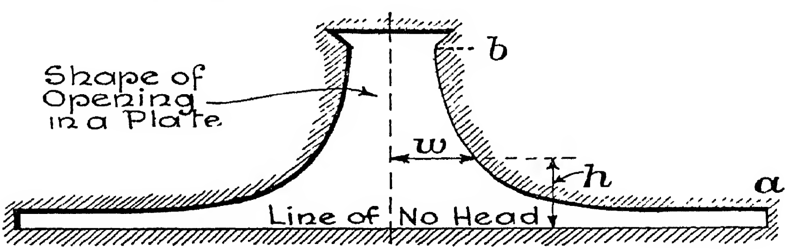


Fig. 100.—The Yorke Weir.

the half width w (see Fig. 100) at head h is given by $w = \frac{3\cdot43}{\sqrt{h}}$. Show the complete weir for a depth of 6″, taking the range of h from ·095″ to 6·095″.

11. The length of hob f to cut a worm wheel with teeth of 1″ circular pitch, N being the number of teeth, is found from—

$$f^2 = \cdot8742N - \cdot1373.$$

Plot a curve to show values of f for N ranging from 10 to 120.

12. The resistance R, in lbs. per ton for the case of electric traction, at a speed V miles per hour is given by $R = \frac{3(V + 12)}{V + 2} + \frac{V^2}{300}$

If V ranges from 0 to 40, show the variation of R by a graph.

13. The following equation occurs in connection with the reinforcement of rectangular beams $k = \sqrt{2rm + r^2m^2} - rm.$

Plot a curve to give values of k for values of r ranging from ·005 to ·02, taking the value of m as 15.

Solve, graphically, the equations in Exs. 14 to 17.

14. $x^2 - 5x - 6 = 0.$ **15.** $6x^2 - 56 = 5x.$

16. $\cdot14x^2 + \cdot87x - 1\cdot54 = 0.$

17. $(3 \times 10^5x^2) + (2\cdot8 \times 10^6x) + (31 \times 10^4) = 0.$

18. Find a value of x which makes M maximum or minimum, it being given that $M = 3\cdot42x - \cdot1x^2$.

19. The following values were given for the B.H.P. and I H P. for different values of the valve cut-off. Find the cut-off when the engine uses least steam, (*a*) per I.H.P. hour; (*b*) per B.H P. hour.

Cut-off	7½″	6″	4½″	3″
B.H.P.	111	115	115	110
I.H.P.	118	125	127	114
Steam per hour .	2160	2116	2080	2020

20. If 40 sq. ft. of metal are to be used in the construction of an open tank with square base, the dimensions being chosen in such a way that the capacity of the tank is to be a maximum for the metal used: Let x ft. be the length of the side of the base: then the volume is $10x - \frac{x^3}{4}$ cu. ft. By taking values of x from 0 to 7, find that value which gives the greatest volume of water. Hence find also the height of the tank $\left(\frac{\text{Volume}}{x^2}\right)$.

21. The table gives figures dealing with gas-engine tests.

Ratio of $\frac{\text{air}}{\text{gas}}$	11·7	10·43	9·13	7·74	5·38	4·40	3·60	3·14
Gas per I.H.P. hour	31·9	22	20·8	19	21·6	24·8	29·8	34·5

What are the best proportions of the mixture for least consumption of gas per I.H.P. hour?

22. In a non-condensing engine running at 400 revs. per min. the following results were obtained :—

Ratio of expansion r	4	4·4	4·8	5·2	5·6	6·0	8
lbs. of steam per I.H.P. hour	20·75	20·48	20·35	20·16	20	20·32	23·14

Find the most economic ratio of expansion.

23. The work done by a series electric motor in time t is given by

$$W = \frac{e(E - e)t}{R}$$

where e = back E.M.F., E = supply pressure, R = resistance of armature.

The electrical efficiency is $\frac{e}{E}$. Find the efficiency when the motor so runs that the greatest rate of doing useful work is reached. $\{R = \cdot 035, E = 110, t = 20.\}$

24. The total cost C, in pounds sterling, of a ship's voyage of 3000 nautical miles is given by

$$C = \frac{3000}{v}\left(3\cdot 2 + \frac{v^3}{2200}\right)$$

where v is the speed in knots. Find the speed at which the cost has its minimum value and state the cost at this speed.

25. To find the best angle of thread for a worm gear with steel worm and brass worm gear, calculations were made with the following results :—

Angle (degrees)	0	10	20	30	45	60	75
Efficiency . .	0	·466	·61	·671	·68	·605	·28

Find the best angle and the maximum efficiency.

26. If $W = 4C^2 + \frac{74}{C}$, find a value of C between 0 and 5 that makes W a maximum or minimum.

27. The efficiency η of a Pelton wheel is given by $\eta = \frac{4u(v - u)}{v^2}$. Find the value of u in terms of v which makes η a maximum. Find also the maximum efficiency.

28. If $\eta = \frac{2u(v - u)^2}{v^3}$ and $v = 25$, find the value of u for the maximum value of η; find also the maximum value of η.

29. The ratio of horse-power to weight of a petrol motor is $\frac{D - 1}{D^2}$ where D = diam. of cylinder in inches. Find the value of D which makes this ratio a maximum.

30. Sixteen electric cells are connected up, in $\frac{16}{x}$ rows of x cells per row. The current from them is $\dfrac{x}{\frac{x^2}{16}+4}$. Find the arrangement for maximum current.

31. Find a value of V between 0 and 10 that makes R a maximum, when

$$R = \frac{V^2}{54} - \frac{3(V-12)}{V+12}.$$

32. Plot from $x = -4$ to $x = +4$ the curve

$$y = 2x^3 - 5x^2 + 9x - 4.$$

33. Plot from $x = -2$ to $x = +6$ the curve

$$2y = \cdot56x^2 - 1\cdot07x - 1\cdot48x^3 + \cdot88.$$

Solve, graphically, the equations in Exs. 34 to 37.

34. $2x^3 - x^2 - 7x + 6 = 0.$ **35.** $20x^3 + 11x^2 + 27 = 138x.$

36. $x^3 + 5x^2 - \cdot08x - 8\cdot82 = 0.$ **37.** $50p^3 + 4 = 23p - 5p^2.$

38. Find the turning-points of the function $2x^3 + 3x^2 - 36x + 15$, stating their nature.

39. If x is the distance of the point of contraflexure from the end of a built-in girder whose length is l, find x in terms of l by the solution of the equation—

$$1 - \frac{6x}{l} + \frac{6x^2}{l^2} = 0.$$

40. To find d, the depth of flow through a channel under certain conditions of slope, etc., it was necessary to solve the equation

$$d^3 - 1\cdot305d - 1\cdot305 = 0.$$

Find the value of d to satisfy this equation.

41. From tests with model planes Thurston calculated the following figures :—

Inclination of plane to horizontal (degrees).	−2	−1	0	1	2	3	4	5	6	8	10	14
Weight supported per H.P.	16·8	31·1	51·5	90·5	157	203	230	256·5	259·2	233	196	128·5

Plot these values, to a base of angles, and find for what inclination the greatest weight is lifted per H.P. developed.

42. The table gives corresponding values of the ratio r $\left(\dfrac{\text{face pitch of airscrew}}{\text{diameter}}\right)$ and efficiency (η). By plotting, determine the value of r for maximum efficiency.

r	0	·1	·2	·3	·4	·5	·6	8	1 0	1·1	1·2	1·4	1·5	1·6	1 7
η	0	·0765	·16	·24	·32	·40	·48	64	795	·84	·875	·875	·824	·71	·35

CHAPTER V

FURTHER ALGEBRA

Variation.—If speed is constant during a journey, the time taken is proportional to the distance, *i. e.*, the bigger the distance the longer is the time taken, or, to extend this statement, twice the time would be required for twice the distance.

This is expressed by saying that the time varies as the distance, or more shortly $t \propto d$ where the sign $\propto$ stands for *varies as*. We cannot say that $t = d$, but the statement of the variation is well expressed by the equation $\frac{t_1}{t_2} = \frac{d_1}{d_2}$ or $\frac{t_1}{d_1} = \frac{t_2}{d_2}$, where t_1 and d_1 are the values of the time and distance in one instance, and t_2 and d_2 are corresponding values in some other.

If, in the second arrangement, k is the number to which each fraction is equal, it will be seen that—

$$\frac{t_1}{d_1} = k \qquad \therefore \quad t_1 = kd_1$$

$$\frac{t_2}{d_2} = k \qquad \therefore \quad t_2 = kd_2$$

or, in general, $t = kd$.

Hence the sign of variation may be replaced by the sign of equality together with a constant factor.

e. g., suppose the time for a journey of 300 miles is 15 hours,

$$\text{then} \qquad 15 = k \times 300$$

$$\text{or} \qquad k = \frac{1}{20}$$

i. e., the constant factor is $\frac{1}{20}$ so long as the units are miles and hours, and the speed is uniform.

Variation such as this is known as *direct* variation, since t varies *directly* as d. Suppose now that the length of journey is fixed, then the bigger the speed the less will be the time taken; halve

the speed and the journey takes double the time. Here the time varies *inversely* as the speed when the distance is constant;

$$\text{or} \qquad t \propto \frac{1}{v}$$

$$i.\,e., \qquad t = l \times \frac{1}{v} = \frac{l}{v}$$

where l is some constant.

If both speed and distance vary, the time will vary directly as the distance and inversely as the speed;

$$\text{or} \qquad t \propto d \quad \text{and also } t \propto \frac{1}{v}$$

$$i.\,e., \qquad t \propto \frac{d}{v}$$

$$\text{or} \qquad t = \frac{md}{v} \qquad \ldots \ldots \ldots \ldots \quad (1)$$

This variation is known as *joint variation.*

A proof of statement (1) is here given, as the reason for it is not self-evident.

Suppose the original values of time, distance, and speed are t_1, d_1, and v_1 respectively.

Change the distance to d_2, keeping the speed constant: the time will now be t, the value of which is determined from the equation—

$$\frac{t}{t_1} = \frac{d_2}{d_1} \qquad \ldots \ldots \ldots \ldots \quad (2)$$

Now make another change; keep the distance constant at d_2 but let the speed become v_2, then the time will change to t_2 and

$$\frac{t_2}{t} = \frac{v_1}{v_2} \qquad \ldots \ldots \ldots \ldots \quad (3)$$

Multiplying equations (2) and (3) together—

$$\frac{t}{t_1} \times \frac{t_2}{t} = \frac{d_2}{d_1} \times \frac{v_1}{v_2}$$

$$\text{or} \qquad \frac{t_2}{t_1} = \frac{d_2 v_1}{d_1 v_2}$$

$$\text{or} \qquad \frac{t_2 v_2}{d_2} = \frac{t_1 v_1}{d_1} = \text{constant} = m, \text{ say.}$$

$$\therefore \qquad t_2 = \frac{md_2}{v_2} \quad \text{or} \quad t_1 = \frac{md_1}{v_1}$$

$$\text{or, in general,} \qquad t = \frac{md}{v}$$

Questions on variation should be worked in the manner outlined in the following examples.

Example 1.—The loss of head of water flowing through a pipe is proportional to the length and inversely proportional to the diameter. If in a length of 10 ft. of $\frac{1}{2}''$ diam. pipe the head lost is 4·6 ft., what will it be for 52 ft. of $3\frac{1}{4}''$ diam. pipe?

Taking the first letters to represent the words—

$h \propto l$ when d is constant; and $h \propto \frac{1}{d}$ when l is constant.

Then, when both l and d vary—

$$h \propto \frac{l}{d} \quad \text{or} \quad h = \frac{kl}{d}, \quad \text{where } k \text{ is a constant.}$$

We must first find the value of k. In the first case—

$$4{\cdot}6 = \frac{k \times 10}{\frac{1}{2}}$$

$$\therefore \qquad k = \frac{4{\cdot}6 \times {\cdot}5}{10} = {\cdot}23, \quad \text{so that } h = \frac{{\cdot}23l}{d}$$

Substituting this value in the second case—

$$h = {\cdot}23 \times \frac{l}{d} = \frac{{\cdot}23 \times 52}{3{\cdot}25} = \underline{3{\cdot}68 \text{ ft.}}$$

Example 2.—The weight of shafting varies directly as its length and also as its cross section. If 1 yard of wrought-iron shafting of $1''$ diam. weighs 8 lbs., what is the weight of 50 ft. of W.I. shafting of $\frac{1}{2}''$ diam.?

If for weight, length and area, W, l and a respectively are written, then $W \propto l$ and also $W \propto a$; and when both l and a vary $W \propto la$.

Also we know that the area of a circle depends on the diameter squared; hence—

$$a \propto d^2$$

and $\qquad W \propto ld^2 \quad \text{or} \quad W = kld^2$

In the first case— $\qquad 8 = k \times 3 \times 1^2$

$$\therefore \qquad k = \frac{8}{3} \quad \text{and} \quad W = \frac{8}{3}ld^2$$

Substituting this value in the second case—

$$W = \frac{8}{3}ld^2 = \frac{8}{3} \times 50 \times (\tfrac{1}{2})^2$$
$$= \underline{33{\cdot}3 \text{ lbs}}$$

Example 3.—The diam. d of a shaft necessary to transmit a certain horse-power H is proportional to the cube root of the horse-power. If a shaft of $1{\cdot}5''$ diam. transmits 5 H.P., what H.P. will a $4''$ diam. shaft transmit?

Here— $\qquad d \propto \sqrt[3]{H} \quad \text{or} \quad H^{\frac{1}{3}}$

$$\therefore \qquad d = kH^{\frac{1}{3}}$$

Substituting the first set of values—

$$1{\cdot}5 = k \times 5^{\frac{1}{3}}$$

or $$k = \frac{1{\cdot}5}{5^{\frac{1}{3}}}$$

∴ $$d = \frac{1{\cdot}5}{5^{\frac{1}{3}}} \times \mathrm{H}^{\frac{1}{3}}$$

When $d = 4$—

$$4 = \frac{1{\cdot}5}{5^{\frac{1}{3}}} \times \mathrm{H}^{\frac{1}{3}}$$

Transposing— $$\mathrm{H}^{\frac{1}{3}} = \frac{4 \times 5^{\frac{1}{3}}}{1{\cdot}5}$$

Cubing— $$\mathrm{H} = \frac{4^3 \times 5}{1{\cdot}5^3} = \frac{64 \times 5}{3{\cdot}375} = \underline{94{\cdot}8.}$$

An application of this branch of the subject occurs in connection with the whirling of shafts. It is known that the deflection d of a shaft, as for a beam, is proportional to the cube of its length l, and also that the critical speed of rotation c is inversely proportional to the square root of the deflection.

In mathematical language—

$$d \propto l^3 \quad \ldots \ldots \ldots \ldots \quad (1)$$

and $$c \propto \frac{1}{\sqrt{d}} \quad \ldots \ldots \ldots \ldots \quad (2)$$

We desire to connect c with l.

From equation (1)— $$d = kl^3$$

∴ $$\sqrt{d} = \sqrt{k}l^{\frac{3}{2}}$$

Substituting in the modified form of equation (2), viz. $c = \dfrac{m}{\sqrt{d}}$

$$c = \frac{m}{\sqrt{k}l^{\frac{3}{2}}} = \frac{p}{l^{\frac{3}{2}}}$$

where p is some constant, *i. e.*, the critical speed is inversely proportional to the $\frac{3}{2}$ power of the length.

Thus if the equivalent lengths of the shaft under different modes of vibration (*i. e.*, for the higher critical speeds) are l, $\frac{l}{2}$, $\frac{l}{3}$, etc., the critical speeds are in the ratio 1, 2·82, 5·2, etc.; for comparing the first and third—

$$l_1 = 1 \qquad l_2 = \tfrac{1}{3}$$
$$c_1 = 1 \qquad c_2 = ?$$

but $$c_1 = \frac{p}{l_1^{\frac{3}{2}}}$$

i. e., $$p = c_1\, l_1^{\frac{3}{2}}$$

also $$p = c_2\, l_2^{\frac{3}{2}}$$

Thus— $c_2 l_2^{\frac{3}{2}} = c_1 l_1^{\frac{3}{2}}$

$$c_2 = c_1\left(\frac{l_1}{l_2}\right)^{\frac{3}{2}} = 1(3)^{\frac{3}{2}}$$

$$= \sqrt{27} = 5{\cdot}2$$

Hence— $\dfrac{c_2}{c_1} = \dfrac{5{\cdot}2}{1}$

Example 4.—The energy E stored in a flywheel varies as the fifth power of the diameter d and also as the square of the speed n.

Find the energy stored in a flywheel of 6 ft. diam., whilst it changes its speed from 160 to 164 revs. per min., if the energy stored at 100 R.P.M. is 25000 ft. lbs.

$$\text{E} \propto d^5n^2$$

$\therefore$ $\text{E} = kd^5n^2.$

When $n = 100$, $d = 6$, $\text{E} = 25000$,

so that $25000 = k \times 6^5 \times 100^2$

or $k = \dfrac{25000}{6^5 \times 10^4}$

Thus— $\text{E at } n = 164 = k \times 6^5 \times 164^2$

and $\text{E at } n = 160 = k \times 6^5 \times 160^2$

$\therefore$ $\text{Difference} = k \times 6^5(164^2 - 160^2)$

$$= \frac{25000 \times 6^5(4)(324)}{6^5 \times 10^4}$$

$$= \underline{3240 \text{ ft. lbs.}}$$

Example 5.—A direct-acting pump having a ram of 10″ diam. is supplied from an accumulator working under a pressure p of 750 lbs. per sq. in. When no load is on, the ram moves through a distance of 80 ft. in 1 min. at a uniform speed v. Estimate the value of the coefficient of hydraulic resistance or the coefficient of friction, viz. the friction force when the ram moves at a velocity of 1 ft. per sec.; the total friction force varying as the square of the speed.

Find also the time the ram would take to move through 80 ft. when under a load of 15 tons.

If the whole system is running light, the full pressure is used to overcome the friction, *i.e.*, $p \propto v^2$, since total friction force varies as (velocity)^2.

Thus— $p = kv^2$ where k is the coefficient of hydraulic resistance;

also $v = \dfrac{80}{60} = 1{\cdot}33$ ft. per sec., and $p = 750$

then $750 = k \times (1{\cdot}33)^2$

or $k = \dfrac{750}{(1{\cdot}33)^2} = 422$

i. e., the coefficient of hydraulic resistance is $\underline{422}$ if the units of pres. .re and velocity are lbs. per sq. in. and ft. per sec. respectively.

The intensity of pressure due to a load of 15 tons—

$$= \frac{15 \times 2240}{\frac{\pi}{4} \times 10^2} = 428 \text{ lbs. per sq. in.}$$

Then, to find the velocity in the second case—

Total pressure = pressure to overcome the friction + pressure to move the load,

i. e., $$p = kv_1^2 + p_1$$

where v_1 is the new velocity, and $p_1 = 428$.

Then— $$750 = kv_1^2 + 428 = 422v_1^2 + 428$$

or $$v_1^2 = \frac{750 - 428}{422} = \cdot 7632$$

and $$v_1 = \cdot 8736.$$

Hence the time required for 80 ft. of the motion—

$$= \frac{80}{\cdot 8736} \times \frac{1}{60} = \underline{1 \cdot 526 \text{ mins.}}$$

Example 6.—The linear dimensions of a ship are λ times those of a model. If the velocity of the ship = V, find the speed of the model at which the resistance is $\frac{1}{\lambda^3}$ times that of the ship, given that the fluid resistance varies as the area of surface S and also as the square of the velocity.

Let R = resistance of ship; then from hypothesis $R \propto S$, and also $R \propto V^2$.

Then— $$R = KSV^2$$

and r = resistance of model = Ksv^2.

Now $\frac{s}{S} = \frac{1}{\lambda^2}$ $\left\{\begin{array}{l}\text{for surfaces of similar solids are proportional to the} \\ \text{squares of corresponding linear dimensions}\end{array}\right\}$

and we are told that— $$\frac{r}{R} = \frac{1}{\lambda^3}$$

Hence— $$\frac{R}{r} = \frac{KSV^2}{Ksv^2}$$

i. e., $$\lambda^3 = \lambda^2 \frac{V^2}{v^2}$$

or $$\frac{v}{V} = \frac{1}{\sqrt{\lambda}}$$

i. e., $$\underline{v = \frac{V}{\sqrt{\lambda}}}$$

v and V (which is $v\sqrt{\lambda}$) are spoken of as "corresponding speeds."

Exercises 24.—On Variation.

1. The weight of a sphere is proportional to the cube of the radius. A sphere of radius $3 \cdot 4''$ weighs $47 \cdot 8$ lbs.; what will be the weight of a sphere of the same material, of which the radius is $4 \cdot 17''$?

2. The candle-power (C.P.) of a lamp is proportional to the square of its distance from a photometer. A lamp of 16 C.P. placed at 58 cms. from a screen produced the same effect as a second lamp placed 94 cms. from this screen. If this second lamp was absorbing 100 watts, find its efficiency, where η = watts per C.P.

3. The velocity of sound in air is proportional to the square root of the temperature τ (centigrade absolute, *i. e.*, t° C. + 273). If the velocity is 1132 ft. per sec. at temperature 18° C., find the law connecting v and τ; find also the velocity at 52° C.

4. The force of the earth's attraction varies inversely as the square of the distance of the body from the earth's centre. Assuming that the diameter of the earth is 8000 miles, find the weight a mass of 12 tons would have if it could be placed 200 miles above the earth's surface.

5. The total pressure on the horizontal end of a cylindrical drum immersed in a liquid is proportional to the depth of the end below the surface and to the square of the radius of the end.

If the pressure is 1200 lbs. when the depth is 14 ft. and the radius is 1 yard, find the pressure at a depth of 6 yards when the radius is 8 ft.

6. The loss of head due to pipe friction is directly proportional to the length, to the square of the velocity and inversely proportional to the diameter. If $2 \cdot 235$ ft. of head are lost in 50 ft. of $2''$ pipe, the velocity of flow being 4 ft. per sec., find the diameter of pipe along which 447 ft. of head are lost, the length of the pipe being 1 mile and the velocity of flow $8 \cdot 7$ ft./sec.

7. The electrical resistance of a piece of wire depends directly on its length and inversely on its diam. squared. The resistance of 85 cms. of wire of diam. $\cdot 045$ cm. was found to be $2 \cdot 14$ ohms. Find the diam. of the wire of which 128 cms. had a resistance of $8 \cdot 33$ ohms.

8. The power in an electric circuit depends on the square of the current and also on the resistance. The power is $15 \cdot 34$ kilowatts when 23 amps. are flowing through a resistance of 29 ohms. If a current of 9 amps. flows through a resistance of 17 ohms for 50 mins., what would be the charge at $2d.$ per unit?

(1 unit = 1 kilowatt-hour.)

9. The electrical resistance of a conductor varies directly as the length and inversely as the area of cross section. The resistance of 70 cms. of platinoid wire of diam. $\cdot 046$ cm. was found to be $1 \cdot 845$ ohms. Find the resistance of $1 \cdot 94$ metres of platinoid wire of diam. $\cdot 028$ cm.

10. The number of teeth T necessary for strength in a cast-iron wheel varies directly as the H.P. transmitted, inversely as the speed and inversely as the cube of the pitch p of the teeth.

If T = 10 when $p = 2''$ and ratio of H P. to speed (in R.P M.) = $\cdot 101$, find the H.P. transmitted when there are 30 teeth, the pitch of the teeth being $6''$, and the speed being 30 revs. per min.

11. The coefficient of friction between the bearing and shaft varies directly as the square root of the speed of the shaft and inversely as the pressure. The coefficient was $\cdot 0205$ when the speed was 10 and

the pressure was 30; find the pressure when the coefficient is ·0163 and the speed is 45.

12. The I.H.P. of a ship varies as the displacement D, as the cube of the speed v, and inversely as the length L. If I.H.P. = 2880 when D = 8000 tons, v = 12 knots, and L = 400 ft., find the speed for which I.H.P. = 30600, the displacement being 20000 tons and the length being 580 ft.

13. The pressure of a gas varies inversely as the volume and directly as the absolute temperature τ (see proof in Question 18). The pressure is 1 kgrm. per sq. in. when the volume is 6·90 and the absolute temperature is 468; find the absolute temperature when the pressure is 8·92 kgrms. per sq. in. and the volume is 1·39.

14. In some experiments on anti-rolling tank models, the number of oscillations per min. of a model of length 10·75 ft. was 27. If the number of oscillations per min. is inversely proportional to the square root of the ratio of the linear dimensions, find the number of oscillations of a similar ship 430 ft. long.

15. Assuming the same relations between volume, pressure and absolute temperature as in Question 13; if the pressure is 108 lbs. per sq. in. when the volume is 130·4 cu. ins. and the absolute temperature is 641, find the absolute temperature when the pressure is 41·3 lbs. per sq. in. and the volume is 283 cu. ins.

16. The time of vibration of a loaded beam is inversely proportional to the square root of the deflection caused by the loading. When the deflection was ·0424″ the time was ·228 sec.; find the deflection when the time was ·45 sec.

17. If the cost per foot of a beam of rectangular section of breadth b and depth h varies as the area of section, and the moment of resistance of the beam is proportional to the breadth and also to the square of the depth, find the connection between the cost per foot and the moment of resistance.

18. Boyle's law states that the pressure of a gas varies inversely as its volume, the temperature being constant; Charles's law states that the pressure is proportional to the absolute temperature, the volume being kept constant. Prove rigidly that $\frac{PV}{\tau}$ = constant.

Series.—A succession of numbers or letters the terms of which are formed according to some definite law is called a *series*.

Thus 6, 9, 12 . . . . . . is a series for which the law is that each term is greater by 3 than that immediately preceding it.

Again, $4a$, $16ab$, $64ab^2$ . . . . . is a series in which any term is obtained by multiplying the next before it by $4b$. In these particular series, taken as illustrations, the terms are said to be in *progression*, the former in *Arithmetical Progression*, written A.P., and the latter in *Geometrical Progression*, written G.P.

Other series with which the engineer has to deal are those known as the *Exponential* and the *Logarithmic*; and in the expansion or working out of certain binomial or multinomial functions or expressions a "series" results.

Arithmetical Progression.—Consider the series of numbers 2, 9, 16, 23 . . . etc.

The 2nd term is obtained from the 1st by adding 7.
,, 3rd ,, ,, ,, ,, ,, 2nd ,, ,, 7.
,, 4th ,, ,, ,, ,, ,, 3rd ,, ,, 7.

i. e., each term *differs* by the same amount from that immediately preceding it. The numbers in such a series are said to be in Arithmetical Progression; and since the terms increase, this is an increasing series.

Again, 1, -4, -9, -14 . . . . . . is an A.P., the common difference in this case being -5. This is a decreasing series.

In general, an A.P. can be denoted by—

$$a,\ (a+d),\ (a+2d)$$

where a is the 1st term and d is the common difference.

Now— the 2nd term $= a+d \ = a+(2-1)d$
and the 3rd term $= a+2d = a+(3-1)d$
So that the 20th term $= a+19d$

i. e., the general term, or **the n^{th} term $= a + (n-1)d$.**

Thus the 15th term is obtained by adding 14 differences to the 1st term, or 15th term $= a+14d$.

If three numbers are in A.P., the second is said to be the *arithmetic mean* between the other two; *e. g.*, 95, 85, 75 are three numbers in A.P., where 85 is the A.M. between 95 and 75 and $85 = \frac{95+75}{2}$: or the arithmetic mean of two numbers is one-half their sum.

To find the sum of n terms of an A.P., which is denoted by S_n—

$$S_n = a + (a+d) + (a+2d) + \ldots\ldots \{a+(n-1)d\}$$

Also, by writing the terms in the reverse order—

$$S_n = \{a+(n-1)d\} + \{a+(n-2)d\} + \{a+(n-3)d\} + \ldots\ldots a$$

Adding the two lines—

$$2S_n = \{2a+(n-1)d\} + \{2a+(n-1)d\} + \{2a+(n-1)d\} \ldots\ldots \text{ to } n \text{ terms}$$

or $$2S_n = n\{2a+(n-1)d\}$$

$$\therefore \quad \mathbf{S_n = \frac{n}{2}\{2a+(n-1)d\}}$$

If we call the last term l, then $l = a+(n-1)d$, and the formula for the sum can be written—

$$\mathbf{S_n = \frac{n}{2}\{a+l\} \quad \text{or} \quad n\left(\frac{a+l}{2}\right)}$$

i. e., the sum can most easily be found by multiplying the average term, *i. e.*, $\frac{a+l}{2}$, by the number of terms n.

Many problems on A.P. can be worked by means of a graph. If ordinates represent terms, and abscissæ the numbers of the terms, an A.P. will be represented by a sloping straight line for which the "slope" is the common difference d and the ordinate on the axis through 1 of the horizontal scale is the first term.

The sum will be the area under the line, with one-half the sum of the first and last terms added.

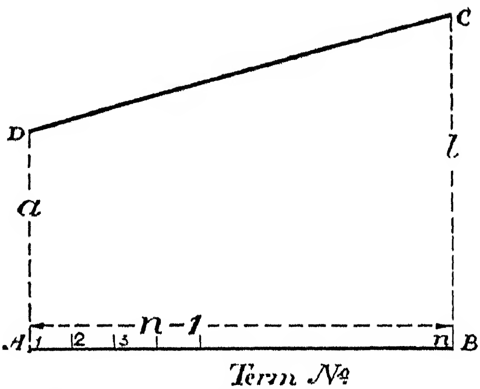


Fig. 101.—Arithmetical Progression.

For the area under the line, viz. ABCD (Fig. 101)—

$$\text{is} \qquad \tfrac{1}{2}(\text{AD} + \text{BC}) \times \text{AB} = \left(\frac{a+l}{2}\right) \times (n-1)$$

$$\text{but} \qquad S_n = n\left(\frac{a+l}{2}\right) = (n-1)\left(\frac{a+l}{2}\right) + \left(\frac{a+l}{2}\right)$$

$$= \text{area under line} + \tfrac{1}{2}\left(\begin{matrix}\text{first and}\\ \text{last terms}\end{matrix}\right)$$

Example 7.—Find the sum of 12 terms of the series, 4, 2, 0 . . .; find also its 10th term.

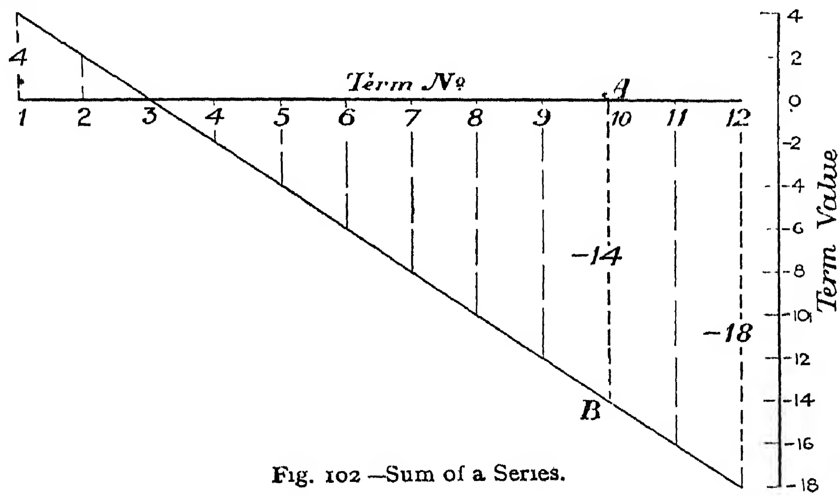


Fig. 102.—Sum of a Series.

In this case, $n = 12$, $a = 4$, $d = 4$ subtracted from $2 = -2$.

$$\therefore \qquad S_{12} = \frac{12}{2}\{(2 \times 4) + (12 - 1) \times -2\}$$
$$= 6\{8 - 22\} = \underline{-84}.$$

Also the 10th term $= a + 9d = 4 - 9 \times 2$
$$= \underline{-14}$$

or graphically, area under the line—

$$= \tfrac{1}{2}\{4 \times 2\} - \tfrac{1}{2}\{9 \times -18\}. \quad \text{(Fig. 102.)}$$
$$= 4 - 81 = -77$$

and $\qquad \frac{1}{2}(a + l) = \frac{4 - 18}{2} = -7$

$$\therefore \qquad S_{12} = -77 - 7 = \underline{-84}$$

and ordinate AB represents the 10th term and $= \underline{-14}$.

Example 8.—Insert 4 arithmetic means between 1·6 and 9·4, *i. e.*, insert 4 terms between 1·6 and 9·4 equally spaced so that together with the terms given they form an A.P.

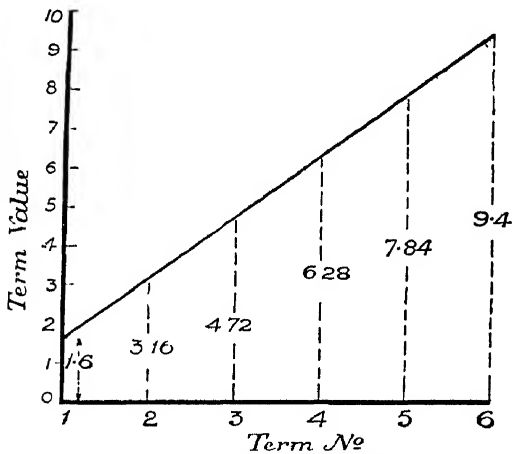


Fig. 103.—Arithmetic Means.

The total number of terms must be 6 (two end terms together with the 4 intermediate), so that—

1st term $= 1{\cdot}6$

and 6th term $= 9{\cdot}4$

but the 6th term $= a + 5d$ and $a = 1{\cdot}6$

$\therefore$ $1{\cdot}6 + 5d = 9{\cdot}4$

and $5d = 7{\cdot}8$ or $d = 1{\cdot}56$.

Hence the means are <u>3·16, 4·72, 6·28, and 7·84.</u>

The graphical construction would be quicker in this instance. Referring to Fig. 103, draw a vertical through 1 on the horizontal scale to represent 1·6, and a vertical through 6 to represent 9·4; join

the tops of the ordinates by a straight line and read off the ordinates through 2, 3, 4 and 5.

Example 9.—In calculating the deflection of a Warren girder due to the strain in the members of the lower flange, if U = the force in a member caused by a unit load at the centre of the girder, F = the force in the bar due to the external loads, a = area of section of member, d = length of one bay, h = height of the girder, and n = number of bays, then deflection $= \frac{F}{aE} \times$ sum of all the separate values of the product U × length of member.

If $d = 20$ ft., $h = 12'\text{-}6''$, $n = 8$, $\frac{F}{a} = 4$ tons per sq. in., and E = 12500 tons per sq. in., find the deflection.

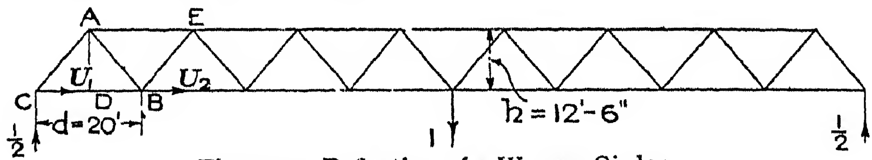


Fig. 104.—Deflection of a Warren Girder.

Dealing with the first bay (see Fig. 104) and taking moments round the point A—

$$\tfrac{1}{2} \times \text{CD} = \text{U}_1 \times \text{AD} \quad \text{or} \quad \tfrac{1}{2} \times \frac{d}{2} = \text{U}_1 \times h$$

$$i.\,e., \qquad \text{U}_1 = \frac{d}{4h}$$

For the second bay, by taking moments round E, $\text{U}_2 = \frac{3d}{4h}$; while for the third bay $\text{U}_3 = \frac{5d}{4h}$, and so on.

Hence the sum of the separate values of $\text{U} = \frac{d}{4h} + \frac{3d}{4h} + \ldots$ to n terms, *i. e.*, it is the sum of an A.P. of which the first term is $\frac{d}{4h}$ and the common difference is $\frac{d}{2h}$

Hence—
$$\text{S}_n = \frac{n}{2}\left\{\left(2 \times \frac{d}{4h}\right) + (n-1)\frac{d}{2h}\right\}$$
$$= \frac{n^2 d}{4h}$$

The sum of the products of U × length of member is this total × d, since all the members have the same length, viz. d.

Then the deflection—
$$= \frac{\text{F}}{a\text{E}} \times \frac{n^2 d^2}{4h}$$

and for this particular case, by substituting the numerical values,

$$\text{deflection} = \frac{4 \times 64 \times 400}{12500 \times 4 \times 12 \cdot 5} \text{ ft.}$$
$$= \underline{1 \cdot 97 \text{ ins.}}$$

Geometrical Progression.—The numbers 5, 7, 9·8 . . . . . are part of a series in which each term is obtained from the preceding one by the use of a common multiplier 1·4. Such a series is known as a Geometrical Progression, or a G.P.

4, -2, 1, $-\frac{1}{2}$ . . . . . . is a G.P., with 1st term 4, and the common multiplier or *ratio* is $-\frac{1}{2}$.

Generally a G.P. may be expressed by—

a, ar, ar^2 . . . . . . (r being the common ratio).

The 2nd term $= ar^1 = ar^{2-1}$

the 3rd term $= ar^2 = ar^{3-1}$

$\therefore$ **the n^{th} term $= ar^{n-1}$**

e. g., the 51st term $= ar^{50}$.

If three terms are in G.P., the middle one is said to be the *geometric mean* of the other two: it is equal to the square root of their product, for

if a, m and b be in G.P.

$$\frac{m}{a} = \frac{b}{m} \quad \text{and } m^2 = ab$$

or $m = \sqrt{ab}$

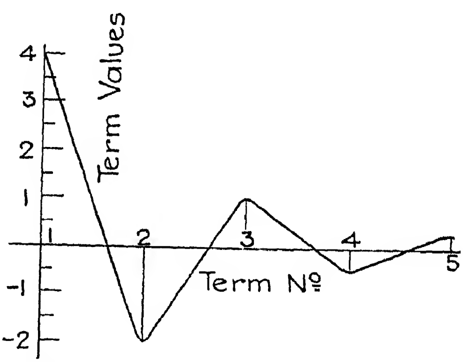


Fig. 105 - Geometrical Progression

[If the true weight of a body is required, but the weighing balance has unequal arms, weigh in each pan, and call the balancing weights W_1 and W_2 respectively: then the true weight W is the geometrical mean between W_1 and W_2, or $W = \sqrt{W_1W_2}$.]

To find the sum to n terms, written S_n—

$$S_n = a + ar + ar^2 + \ldots\ldots ar^{n-2} + ar^{n-1}$$

and $$rS_n = ar + ar^2 + \ldots\ldots ar^{n-2} + ar^{n-1} + ar^n$$

then $$S_n(1-r) = a - ar^n \quad \text{(Subtracting)}$$

and $$\mathbf{S_n = \frac{a(1-r^n)}{1-r} \quad or \quad \frac{a(r^n-1)}{r-1}}$$

the first form being used when the ratio is less than 1.

Referring once again to the series 4, -2, 1 . . . . . . the *numerical* value of the terms, plus or minus, soon becomes so small that the sum, say, of 60 terms is practically the same as that of 50, and the series is said to be rapidly converging. This fact is well illustrated by the graph of term values plotted to a base of

term numbers as in Fig. 105; the area between the curve and the horizontal axis being extremely small after even the fifth term of the series has been reached.

Hence the sum of the entire series, called the sum to infinity (of terms) and written S_∞, can be expressed definitely.

From the rule— $$S_n = \frac{a(1 - r^n)}{1 - r}$$

if r is very small and n is very great, r^n will be very small indeed compared with 1, and may be neglected.

$$\therefore \qquad \mathbf{S_\infty = \frac{a}{1 - r}}$$

Example 10.—Find the sum of 5 terms of the series 2, ·002, ·000002 . . . . . . and compare with the sum to infinity.

In this case $a = 2$ and $r = \cdot 001$.

Then— $$S_5 = \frac{a(1 - r^n)}{1 - r} = \frac{2\{1 - (\cdot 001)^5\}}{1 - \cdot 001}$$

$$= \frac{2}{\cdot 999}\{1 - (1 \times 10^{-15})\}$$

whereas $S_\infty = \frac{a}{1 - r} = \frac{2}{1 - \cdot 001} = \frac{2}{\cdot 999}$, and therefore the two are essentially the same.

Example 11.—The 5th term of a G.P. is 243 and the 2nd term is 9; find the law of the series, viz find the values of r and a.

$$\text{5th term} = ar^4 = 243 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$
$$\text{2nd term} = ar^1 = 9 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Dividing equation (1) by equation (2)—

$$\frac{ar^4}{ar} = \frac{243}{9} = 27$$

$$\therefore \qquad r^3 = 27$$

$$\text{and} \qquad r = 3.$$

Substituting in equation (2), $a \times 3 = 9$ and $a = 3$.

Hence the series is— $\underline{3, 9, 27, \text{etc.}}$

It is of interest to note that the logarithms of numbers in G.P. will themselves be in A.P.

Thus, if the numbers are— 28·4, 284, 2840 . . . . . .

(*i. e.*, in a G.P. having the common ratio = 10),

then their logs are— 1·4533, 2·4533, 3·4533 . . . . . .

(*i. e.*, are in A.P. with common difference 1).

Use may be made of this property when a number of geometric means are required to be inserted between two numbers.

Suppose that five geometric means are required between 2 and 89. Mark off on a strip of paper a length to represent the distance between 2 and 89 on the A or B scale of the slide rule. Divide this distance into $5 + 1$, *i. e.*, six equal divisions: place the paper alongside the scale with its ends level with 2 and 89 respectively: then the readings opposite the intermediate markings will be the required means to as great a degree of accuracy as is required in practice.

The means are $\underline{3{\cdot}76,\ 7{\cdot}1,\ 13{\cdot}3,\ 25{\cdot}1,\text{ and }47{\cdot}3.}$

To check this by calculation—

$$a = 2, \quad \text{and} \quad ar^6 = 89.$$

Hence, by division— $r^6 = \dfrac{89}{2} = 44{\cdot}5.$

Taking logs— $6 \log r = 1{\cdot}6484$

$\therefore \quad \log r = {\cdot}2747.$

Now— $\log ar = \log 2 + \log r = {\cdot}3010 + {\cdot}2747 = {\cdot}5757$

$\therefore \quad ar = 3{\cdot}763.$

Also log— $ar^2 = \log ar + \log r = {\cdot}5757 + {\cdot}2747 = {\cdot}8504$

$\therefore \quad ar^2 = 7{\cdot}084.$

Similarly, the other means are found to be 13·34, 25·11, and 47·26.

It has already been demonstrated that the plotting of the values of the terms in an A.P. to a base of "term numbers" gives a straight line. Consequently it will be seen that if the logs of the values of the terms in a G.P. are plotted to a base of "term numbers," a straight line will pass through the points so obtained, since the logs of numbers in G.P. are themselves in A.P. Consequently many problems on G P. can be solved by means of a straight-line plotting.

Example 12.—The values of the resistances of an electric motor starter should be in G.P. Thus if r_1 = resistance of armature and rheostat on the first step, and r_2, r_3, r_4, etc., are the corresponding values on the subsequent steps, then $\frac{r_1}{r_2} = \frac{r_2}{r_3} = \frac{r_3}{r_4}$, etc., and the value of this ratio is $\frac{C_s}{C}$, where C_s = starting current and C = full load working current.

Find the separate resistances of the 9 steps in a motor starting switch for a 220 volt motor, if the maximum (*i.e.* starting) current

must not exceed the full load working current of 80 amps. by more than 40%, and the armature resistance is ·133 ohm.

Here we are told that $\frac{C_s}{C} = 1{\cdot}4$ or the common ratio of the G.P. is $\frac{1}{1{\cdot}4}$; while the value of r_1 can be calculated by Ohm's law, viz.

$$r_1 = \frac{\text{voltage}}{\text{starting current}} = \frac{220}{1{\cdot}4 \times 80} = 1{\cdot}964 \text{ ohms.}$$

The problem now is to insert 7 geometric means between 1·964 and ·133; and this can be done in the following simple manner. Along a horizontal line indicate term numbers as in Fig. 106, and erect verticals through the points 1, 2 . . . . . . 9.

Set the index of the A scale of the slide rule level with the point 1, and mark the point P at 1·964 (at the right-hand end of the rule): similarly the point Q should be indicated, the distance 9Q representing ·133 (at the left-hand end of the rule). Join PQ.

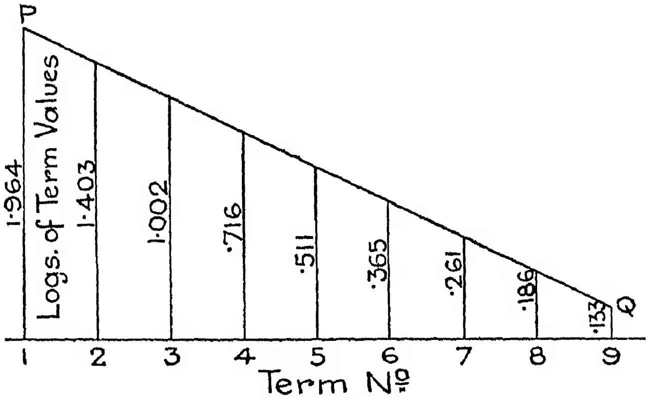


Fig. 106.—Resistances of an Electric Motor Starter.

Then the ordinates to this line through the points 2, 3 . . . . 8, read off according to the log scale (*i. e.*, by the use of the A scale of the slide rule, the index being placed at the horizontal in each case), give the required means, which are 1·403, 1·002, ·716, ·511, ·365, ·261, and ·186.

Compound Interest furnishes an example of geometrical progression.

If the original principle be P and the rate of interest be r—

Then the interest at the end of 1st year—

$$= Pr \text{ and the amount} = A_1 = P + Pr$$

interest at the end of 2nd year—

$$= rA_1$$
$$= r(P+Pr) \text{ and amount} = A_1 + I_2$$
$$= (P+Pr)(1+r)$$

i. e., $I_2 = Pr(1+r)$ and $A_2 = P(1+r)^2$
$I_3 = Pr(1+r)^2$ and $A_3 = P(1+r)^3$
$\therefore$ $I_n = Pr(1+r)^{n-1}$ and $A_n = P(1+r)^n$.

The consecutive interests are thus—

$$Pr,\quad Pr(1+r),\quad Pr(1+r)^2 \;\ldots\ldots$$

i. e., they are in a G.P. of 1st term Pr and common ratio $(1 + r)$.
Hence total interest for n years—

$$= S_n = \frac{Pr[(1+r)^n - 1]}{(1+r)-1} = \frac{Pr}{r}\{(1+r)^n - 1\}$$

or the amount at the end of n years $= P + \text{Interest}$

$$= P + P\{(1+r)^n - 1\}$$
$$= P(1+r)^n$$

Further Applications of G.P.—If an electric condenser be discharged through a ballistic galvanometer, and the lengths of the consecutive swings of the needle are measured, it will be found that they form a G.P.; the ratio, of course, being less than 1, because the amplitude of the swing decreases.

If $a_1 = 1$st swing and $a_2 = 2$nd swing,

then $\quad a_2 = ka_1$

$\quad a_3 = ka_2 = k^2a_1$

and $\quad a_n = k^{n-1}a_1$.

The logarithm of the ratio $\left(\frac{a_1}{a_2}\right)$, *i. e.*, $\log\left(\frac{1}{k}\right)$ according to our notation, is called the **logarithmic decrement** of the galvanometer.

Thus if the respective swings were, in divisions on a scale, 36, 31·4, 21·75, etc., the ratio $k = \frac{31{\cdot}4}{36}$ and the logarithmic decrement of the galvanometer $= \log\frac{36}{31{\cdot}4} = {\cdot}1594$.

To find the practical mechanical advantage $\left(\frac{W}{P}\right)$ for the pulley-block shown in Fig. 107. The pull P on one side of the pulley becomes cP after passing round the pulley (due to friction, and bending of the rope): after passing round the second pulley, the pull is now c^2P, and so on.

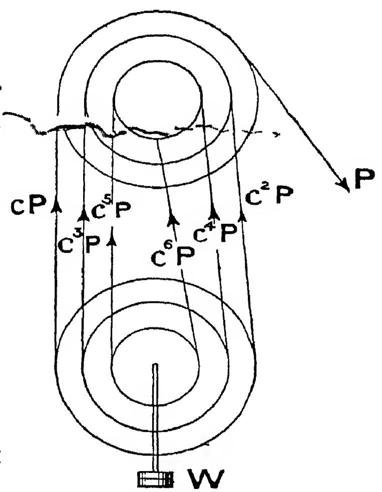


Fig. 107.—Pulley Block.

Hence— $W = cP(1+c+c^2+c^3+c^4+c^5)$
if there are 6 strings

$$= \frac{cP(1-c^6)}{1-c}$$

$$\therefore \qquad \frac{W}{P} = \frac{c}{1-c}(1-c^6)$$

for the case of 6 strings from the lower block.

This result may be put into a more general form by writing n in place of 6; n being the velocity-ratio of the blocks.

Thus— $$\frac{W}{P} = \frac{c}{1-c}(1-c^n)$$

In an actual experiment with a 1 : 1 block, the value of c was found to be ·837. Taking this value, the result given above may then be written—

$$\frac{W}{P} = \frac{\cdot 837}{1-\cdot 837}[1-(\cdot 837)^n] = 5\cdot 13[1-(\cdot 837)^n]$$

By the use of this formula the maximum efficiency of any pulley-block can be determined. Thus for a 4 : 1 block $n = 4$

and $$\frac{W}{P} = 5\cdot 13[1-(\cdot 837)^4] = 5\cdot 13(1-\cdot 4907) = 2\cdot 613.$$

Theoretically, $\frac{W}{P} = 4$, and hence the maximum efficiency—

$$= \frac{2\cdot 613}{4} = \cdot 653$$

Series may occur which, whilst not actually in arithmetical or geometrical progression, may be so arranged that the rules of the respective series may be applied.

Example 13.—Find the sum of n terms of the series, the rth term of which is—

(1) $3r + 1$; (2) 5×3^r.

(1) rth term $= 3r + 1$
Hence— the 1st term $= (3 \times 1) + 1$
the 2nd term $= (3 \times 2) + 1$

and $S_n = (3 \times 1) + (3 \times 2) + (3 \times 3) + \ldots\ldots + (1+1+1 \ldots\ldots \text{to } n \text{ terms})$
$= 3$ (sum of natural numbers to n terms) $+ n$

$$= 3 \times \frac{n}{2}\{2 + (n-1)1\} + n$$

$$= \frac{3n}{2}\{n+1\} + n$$

$$= \frac{3n^2}{2} + \frac{3n}{2} + n = \underline{\frac{3n^2}{2} + \frac{5n}{2}}$$

(2) rth term $= 5 \times 3^r$

Hence— the 1st term $= 5 \times 3^1$

2nd term $= 5 \times 3^2$

and the nth term $= 5 \times 3^n$

also $S_n = (5 \times 3) + (5 \times 3^2) + (5 \times 3^3) + \ldots\ldots$

$$= 5\left\{\frac{3(3^n - 1)}{3 - 1}\right\} = \frac{15}{2}(3^n - 1) \quad \text{or} \quad \underline{7{\cdot}5(3^n - 1)}.$$

Methods of allocating Allowance for Depreciation.—The principles of the previous paragraph may be applied to deal with the various systems of allowance for the depreciation of machinery, etc., which may be calculated by one of three methods.

First Method, involving arithmetical progression, and sometimes spoken of as the "straight-line method."

According to this scheme, the annual contribution to the depreciation fund is constant, and no interest is reckoned.

Let P = the original price of the machine, R = its residual value at the end of its life, n years, and let D = the annual contribution to the depreciation fund.

E. g., if a machine costs £500, has a scrap value of £80, and its life is 21 years, the annual contribution $= \dfrac{£500 - 80}{21} = £20$.

Then—

Value at end of 1st year = P—R—D

(*i. e.*, neglecting its value as scrap).

Value at end of 2nd year = P—R—2D

and Value at end of nth year = P—R—nD

whilst the contributions to the depreciation fund would total nD.

Its value as a working machine would be 0 at the end of the period, *i. e.*, P—R—nD = 0, or nD = P—R; hence its value as scrap would be taken into account in fixing D, for $D = \dfrac{P-R}{n}$, which is not so great as $\dfrac{P}{n}$

Taking the figures suggested above—

Value of the machine at end of 1st year = £500—80—20 = £400, *i. e.*, its value as a working machine, and the depreciation fund would then stand at £20.

At end of 2nd year, value = £500—80—40 = £380 and depreciation fund = £40.

Thus the value + depreciation fund always = £420 = "working" value of machine, which is as it should be.

Second Method.—According to this method of reckoning, the same amount is added to the depreciation fund yearly, but interest is reckoned thereon.

Let the rate of interest $= £r$ per annum per £1.

At end of 1st year, depreciation fund $= D$

" " 2nd " " " $= D + rD + D$ (since rD is the interest on the first contribution).

$= 2D + rD$

" " 3rd " " " $= (2D + rD) + r(2D + rD) + D$

$= 3D + 3rD + r^2D.$

We wish to find a general expression giving the magnitude of the depreciation fund at the end of any year; to do this, the expression last obtained must be slightly transposed.

$$3D + 3rD + r^2D = \frac{D}{r}(3r + 3r^2 + r^3) \left(\begin{matrix}\text{multiplying and}\\ \text{dividing by } r\end{matrix}\right)$$

$$= \frac{D}{r}(1 + 3r + 3r^2 + r^3 - 1)$$

$$= \frac{D}{r}\{(1 + r)^3 - 1\}$$

This is the value of the fund at the end of the 3rd year.

In like fashion, the value of the fund at the end of the 4th year—

$$= \frac{D}{r}\{(1 + r)^4 - 1\}$$

so that at the end of the nth year, the depreciation fund stands at—

$$= \frac{D}{r}\{(1 + r)^n - 1\}$$

This must be equal to the *working value* or $P - R$,

i. e., $$\frac{D}{r}\{(1 + r)^n - 1\} = P - R$$

$$\therefore \quad D = \frac{r(P - R)}{(1 + r)^n - 1}$$

E. g., if the original value $= £500$
the scrap value $= £80$
no. of years $= 21$
and rate of interest $= 3\%$, *i. e.*, $r = \cdot 03$

Then— $$D = £\frac{\cdot03(500 - 80)}{(1\cdot03)^{21} - 1}$$
$$= £\frac{\cdot03 \times 420}{\cdot857}$$
$$= \underline{£14\cdot7}$$

Explanation.

Let $x = (1\cdot03)^{21}$

$\log x = 21 \log 1\cdot03$

$= 21 \times \cdot0128$

$= \cdot2688$

$x = 1\cdot857$

There is one disadvantage in connection with the second method : the depreciation fund does not grow rapidly enough in the early years.

Keeping to the same figures as before,

depreciation fund at end of 1st year $= £14\cdot7$

" " " 2nd " $= (£2 \times 14\cdot7) + (£\cdot03 \times 14\cdot7)$
$= £29\cdot84$

" " " 3rd " $= (£3 \times 14\cdot7) + (£\cdot09 \times 14\cdot7)$
$+ (£\cdot0009 \times 14\cdot7)$
$= £45\cdot44.$

If the value of the machine decreases each year by £20, the depreciation fund would not be sufficiently large to ensure no loss in the event of the loss of machine in the first few years of its life : on the other hand, provided nothing untoward happens, only about three-quarters of the depreciation has to be allowed for yearly, *i. e.*, £14·7 as against £20.

Third Method.—The disadvantage of the second method may be eliminated by setting aside each year a constant percentage of the value of the preceding year.

Let this constant percentage be K: then at the end of the first year KP will be assigned to the depreciation fund.

At the end of the 2nd year the fund will stand at KP + per centage of value at end of 1st year—

$$= KP + (P - KP) \times K$$
$$= P(2K - K^2)$$
$$= P\{1 - (1 - 2K + K^2)\}$$
$$= P\{1 - (1 - K)^2\}$$

At the end of the 3rd year—

depreciation fund $= KP + K(P - KP) + K(P - KP)(1 - K)$
$$= P\{K + K - K^2 + K - 2K^2 + K^3\}$$
$$= P\{3K - 3K^2 + K^3\}$$
$$= P\{1 - (1 - 3K + 3K^2 - K^3)\}$$
$$= P\{1 - (1 - K)^3\}$$

Hence at the end of nth year—

$$\text{depreciation fund} = P\{1-(1-K)^n\}$$

$$\text{This must} = P-R$$

so that $\quad P-R = P-P(1-K)^n$

or $\quad P(1-K)^n = R$

$$(1-K)^n = \frac{R}{P}$$

Taking the nth root of each side—

$$(1-K) = \sqrt[n]{\frac{R}{P}}$$

or

$$K = 1-\sqrt[n]{\frac{R}{P}}$$

To compare with the results by the other method, take the figures as before, viz. $P = 500$, $R = 80$, and $n = 21$.

Then— $\quad K = 1-\sqrt[21]{\dfrac{80}{500}}$

$$= 1-\cdot 9164$$

$$= \cdot 0836$$

Then—

depreciation fund at end of 1st year—

$$= £41\cdot 8$$

Ditto end of 2nd year = £80

Ditto end of 3rd year = £116

i. e., the yearly allowance is greater at commencement.

Explanation.

Let—

$$x = \sqrt[21]{\frac{8}{50}}$$

$$\log x = \frac{1}{21}\{\log 8 - \log 50\}$$

$$= \frac{1}{21}\{\cdot 9031 - 1\cdot 699\}$$

$$= -\frac{\cdot 7959}{21}$$

$$= -\cdot 0379$$

$$= \bar{1}\cdot 9621$$

$$x = \cdot 9164$$

Exercises 25.—On Arithmetic and Geometric Progressions.

1. Find the 7th term and also the 29th term of the series 16, 18, 20 . . . .

2. Which term of the series − 81, −75, −69 . . . is equal to 33?

3. The 3rd term of an A.P. is 34 and the 17th term is − 8; find the sum of the first 30 terms.

4. Insert 8 arithmetic means between 2·8 and 10·9.

5. Three numbers are in A.P.; the product of the first and last is 216, and 4 times the second together with twice the first is 84. Find the numbers.

6. How many terms of the series 1·8, 1·4, 1 . . . must be taken so that the sum of them is − 67·2?

7. In boring a well 400 ft. deep the cost is 2*s.* 3*d.* for the first foot and an additional penny for each subsequent foot; what is the cost of boring the last foot and also of boring the entire well?

8. A manufacturer finds that his expenses, which in a certain year are £4000, are increasing at the rate of £28 per annum. He, however, sells 4 more machines each year than during the preceding, and after 16 years his total profit amounts to £14240. Find the selling price of each machine and the total number sold over this period if his profit the first year was £800.

9. A tank is being filled at the rate of 2 tons the first hour, 3 tons the second hour, 4 tons the third hour, and so on. It is completely filled with water in 10 hours. If the base measures 10 ft. by 15 ft. find the depth of the tank.

10. A body falls 16 ft. in the 1st second of its motion, 48 ft. in the 2nd, 80 ft. in the 3rd, and so on. How far does it fall during the 19th second and how long will it take to fall 4096 ft.?

11. A slow train starts at 12 o'clock and travels for the first hour at an average speed of 15 m.p.h., increasing its speed during the second hour to one of 17 m.p.h. for the hour, and during the third hour to 19 m.p.h., and so on. A fast train, starting at 1.30 from the same place travels in the same direction at a constant speed of 32 m.p.h. At what time does this train overtake the first?

12. Find the 5th term of the series 1, 1·2, 1·44 . . .

13. Find the sum to infinity of the series, 40, 10, 2·5 . . .

14. Insert 3 geometric means between $1\frac{1}{3}$ and $6\frac{3}{4}$.

15. Calculate the sum of 15 terms of the series 5, 6·5, 8·45 . . .

16. In levelling with the barometer it is found that as the heights increase in A.P., the readings decrease in G.P. At a height of 100 ft. the reading was 100; at a height of 300 ft. the reading was 80; at 500 ft. the reading was 64. What was the reading at a height of 2700 ft.?

17. Find the sum of the series 15, −12, 9·6 . . . to 7 terms and the sum to infinity of the series ·8, ·02, ·0005 . . .

18. When a belt passes round a pulley it is known that the tensions at equal angular intervals form a G.P. If the tension for a lap of 15° is 21·08 lbs. and that for 90° is 27·38 lbs., find the least tension in the belt, *i. e.*, at 0° (the angular intervals are each 15°).

19. The sum of the first 6 terms of a G.P. is 1020 and the common ratio is 2·4, find the series.

20. Find the 20th term of the series 3, 12, 33, 72, 135 . . . [the nth term is of the form $n(a + bn + cn^2)$].

21. A contractor agrees to do a piece of work in a certain time and puts 150 men on to the work. After the first day four men drop off daily and the work in consequence takes 8 days longer than was anticipated. Find the total number of days which the work actually takes.

22. Find the deflection of the Warren girder shown in Fig. 104 due to the strain in the members of the upper flange. [*Hint.*—By taking moments about the point B, the value of $U_{AE} = \frac{d}{2h}$, etc.]

23. A lathe has a constant countershaft speed, four steps on the cone and one double back-gear. There are thus 12 possible speeds for the spindle; the greatest being 150 revs. per sec. and the least being 3 revs. per sec. If the spindle speeds are in G.P., find the respective speeds.

Napierian Logarithms.—Suppose that £1 is lent out at 2% compound interest per annum.

Then the amount at the end of 1st year $= £(1 + \cdot 02)$; and this is the principal for 2nd year.

$$\therefore \qquad I_2 = (1 + \cdot 02) \times \cdot 02$$

$$\text{and} \qquad A_2 = (1 + \cdot 02)^2$$

If, however, the interest is to be reckoned and added on each month the amount at the end of the first year will be greater, for the interest $= \frac{\cdot 02}{12}$ (*i. e.*, per month),

and, amount at end of 1st month $= \left(1 + \frac{\cdot 02}{12}\right)$

„ „ 2nd „ $= \left(1 + \frac{\cdot 02}{12}\right)^2$

„ „ 1st year $= \left(1 + \frac{\cdot 02}{12}\right)^{12}$ . . . . (1)

Assume now that the interest is added day by day, *i. e.*, practically continuously, then at the end of 1st year—

$$\text{Amount} = \left(1 + \frac{\cdot 02}{365}\right)^{365} \quad \ldots\ldots\ldots \quad (2)$$

If the interest is calculated and added on each second, that being as near continuity as we need approach—

Amount at end of year—

$$= \left(1 + \frac{\cdot 02}{31536000}\right)^{31536000} \quad \ldots\ldots\ldots \quad (3)$$

By means of laborious calculation the actual values of these amounts could be found, and it would be observed that the amount in (2) was greater than that in (1), and the amount in (3) was greater than that in (2); the difference in the values being very slight, and not perceptible unless a great number of decimal places were taken.

It would appear at first sight that by increasing the *number* of additions of interest to the earlier amounts, the final amount could be made indefinitely large: this, however, is not the case, for the amount approaches a figure beyond which it does not rise, but to which it approximates more nearly the larger the value of the exponent (*i. e.*, 12, 365, etc.). This final amount is £2·718 for a principal of £1; in other words, when the interest, added continuously, is proportional to the previous amount, the final amount will reach a limiting value, being £2·718. The symbol

"e" is given to the figure 2·718 . . . from Euler, the discoverer of the series.

Later work will show that e can be expressed as a sum of a series, viz.—

$$e = 1 + 1 + \frac{1}{1\times2} + \frac{1}{1\times2\times3} + \frac{1}{1\times2\times3\times4} + \ldots\ldots$$

and from the foregoing reasoning it will be seen that it is a *natural* number: it occurs as a vital factor in the statement of many natural phenomena.

E. g., a chain hanging freely due to its own weight lies in a curve whose equation may be written—

$$y = \frac{c}{2}\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\}$$

or more simply— $\quad Y = \frac{1}{2}\{e^{X} + e^{-X}\}$

i. e., it hangs in its natural curve (known as the "*catenary*"), and this curve, depending for its form entirely on e, can only have this one form if e is a constant, and, further, a particular constant.

Again, if an electric condenser discharges through a large resistance, the rate at which the voltage (*i. e.*, the difference in potential between the coatings of the condenser) is diminishing is proportional to the voltage. The equation which gives the voltage at any time t, is $v = ae^{-\frac{t}{KR}}$ where K, R and a are constants settled from the given conditions. Then e is a constant, but one determined quite apart from any particular set of conditions.

Actually the most natural way to calculate logarithms is to work from e as base, such logs being called natural, Napierian, or hyperbolic logs; the common logs, *i. e.*, those to base 10, which are far more convenient for ordinary use, being obtained from the Napierian logs. In higher branches of mathematics all the logs are those to base e, for if natural laws are being followed, then any logs that may be necessary must, of course, be natural logs.

It is, therefore, desirable to understand how to change from logs of one base to logs of another. The rule can be expressed in this form—

$$\log_e N = \log_{10} N \times \log_e 10 \quad \ldots\ldots\ldots \quad (1)$$

To remember this, omit the logs and write the law in the fractional form—

$$\frac{N}{e} = \frac{N}{10} \times \frac{10}{e}$$

which is equally correct, as proved by cancelling.

Again— $\log_{10}N = \log_e N \times \log_{10} e$

or $$\frac{N}{10} = \frac{N}{e} \times \frac{e}{10}$$

Proof of statement (1)—

Let— $\log_e N = x$, $\log_{10} N = y$ and $\log_e 10 = z$
then $N = e^x$, $N = 10^y$ and $10 = e^z$
and $e^x = 10^y = (e^z)^y = e^{yz}$
or $x = y \times z$
i. e., $\log_e N = \log_{10} N \times \log_e 10$

Taking e as 2·718 (its actual value, like that of π, is not commensurate) $\log_{10} e = \log_{10} 2{\cdot}718 = {\cdot}4343$ and $\log_e 10$ is the reciprocal of this, viz. 2·3026 or 2·303 approximately.

Hence, from the rules given above—

$$\log_e N = 2{\cdot}303 \log_{10} N$$
$$\log_{10} N = {\cdot}4343 \log_e N.$$

To avoid confusion with these multipliers it should be borne in mind that e is a smaller base than 10, and therefore it must be raised to a higher power to equal the same number.

Hence the log to base e of any number must be greater than its log to base 10.

If tables of Napierian logs are to hand, the foregoing rules become unnecessary; but a few hints as to the use of such tables will not be out of place, for reading from tables of Napierian logs is somewhat more involved than that from tables of common logs.

Examples are here added to demonstrate the determination of natural logs by the two processes.

Example 14.—Using the tables of natural logs (Table IV at the end of the book), find $\log_e 48{\cdot}72$, $\log_e \frac{507}{461}$ and $\log_e {\cdot}00234$.

$$\log_e 48{\cdot}72 = \log_e (4{\cdot}872 \times 10) = \log_e 4{\cdot}872 + \log_e 10$$
$$= 1{\cdot}5835 + 2{\cdot}3026$$
$$= \underline{3{\cdot}8861}$$

$$\log_e \frac{507}{461} = \log_e \frac{5{\cdot}07}{4{\cdot}61} = \log_e 5{\cdot}07 - \log_e 4{\cdot}61$$
$$= 1{\cdot}6233 - 1{\cdot}5282$$
$$= \underline{{\cdot}0951}$$

$$\log_e \cdot 00234 = \log_e \frac{2\cdot 34}{1000} = \log_e 2\cdot 34 - \log_e 1000$$
$$= \log_e 2\cdot 34 - 3 \log_e 10$$
$$= \cdot 8502 - 3 \times 2\cdot 3026$$
$$= \cdot 8502 - 6\cdot 9078$$
$$= \underline{\bar{7}\cdot 9424}$$

It will be observed that for each power of ten in the number, 2·3026 has to be added or subtracted as the case may be.

Example 15.—Without tables of natural logs, find values of—

$$\log_e 9\cdot 63,\ \log_e \frac{1717}{435},\ \text{and } \log_e \cdot 2357.$$

$$\log_e 9\cdot 63 = \log_{10} 9\cdot 63 \times 2\cdot 303$$
$$= \cdot 9836 \times 2\cdot 303$$
$$= \underline{2\cdot 266}$$

$$\log_e \frac{1717}{453} = \log_{10} \frac{1717}{453} \times 2\cdot 303$$
$$= 2\cdot 303 \{\log_{10} 1717 - \log_{10} 453\}$$
$$= 2\cdot 303\{3\cdot 2347 - 2\cdot 6561\}$$
$$= 2\cdot 303 \times \cdot 5786$$
$$= \underline{1\cdot 332}$$

$$\log_e \cdot 2357 = \log_{10} \cdot 2357 \times 2\cdot 303$$
$$= \bar{1}\cdot 3724 \times 2\cdot 303$$

Separating the two distinct parts—

$$= (\bar{1} \times 2\cdot 303) + (\cdot 3724 \times 2\cdot 303)$$
$$= -\ 2\cdot 303 + \cdot 8576$$
$$= \underline{\bar{2}\cdot 5546}$$

{*the subtraction being performed so that the mantissa is kept positive*}.

Application of Logarithms to harder Computations.—In the first chapter the method of applying logs for purposes of evaluation of simple expressions was shown. Such values were found as $(21\cdot 25)^5$, $\sqrt[4]{\cdot 03}$, etc., *i. e.*, numbers raised to positive powers only. The rules there used are applicable to all cases, whatever the powers may be. A negative power may be made into a positive power by changing the whole expression from top to bottom of the fraction or *vice versa* $\left(\text{for } a^{-n} = \frac{1}{a^n}\right)$ so that the evaluation is obtained on the lines already detailed; or it may be obtained directly as here indicated.

N.B.—**Great care must be observed in connection with the signs : whenever distinct parts (*e.g.*, a positive and a negative) occur in a logarithm, these should be treated separately.**

Example 16.—Evaluate $(\cdot005134)^{\cdot134}$

Let the expression $= x$

then $x = (\cdot005134)^{\cdot134}$ and by taking logs,

$$\log x = \cdot134 \times \log \cdot005134$$
$$= \cdot134 \times \bar{3}\cdot7104$$
$$= (\cdot134 \times \bar{3}) + (\cdot134 \times \cdot7104)$$
$$= -\cdot402 + \cdot0952 = \bar{1}\cdot6932 = \log \cdot4934$$
$$\therefore \quad x = \underline{\cdot4934}$$

Notice that $\cdot402$ is subtracted from $\cdot0952$ although the former is the greater; this being done so that the mantissa of the log shall be positive.

Example 17.—Find the value of $(\cdot1473)^{-2\cdot1}$

Let— $x = (\cdot1473)^{-2\cdot1}$

Then— $\log x = -2\cdot1 \times \log \cdot1473 = -2\cdot1 \times \bar{1}\cdot1682$
$$= (-2\cdot1 \times \bar{1}) + (-2\cdot1 \times \cdot1682)$$
$$= +2\cdot1 - \cdot3532 = 1\cdot7468$$
$$= \log 55\cdot82$$
$$\therefore \quad x = \underline{55\cdot82}$$

Example 18.—Evaluate $\{\log_e 3\cdot187\}^{-\cdot024}$

Let— $x = \{\log_e 3\cdot187\}^{-\cdot024}$
$= y^{-\cdot024}$ where $y = \log_e 3\cdot187$.

The value of y must first be found.

From the tables $\log_e 3\cdot187 = 1\cdot1591$

Hence— $y = 1\cdot1591$ and $x = (1\cdot1591)^{-\cdot024}$

Now— $\log x = -\cdot024 \times \log 1\cdot159$
$$= -\cdot024 \times \cdot0641$$
$$= -\cdot001538 = \bar{1}\cdot998462 \text{ or } \bar{1}\cdot9985$$
$$= \log \cdot9965$$
$$\therefore \quad x = \underline{\cdot9965}$$

Example 19.—Evaluate $\dfrac{(42\cdot17)^{\frac{1}{3}} \times (\cdot0145)^{-2}}{(8\cdot91)^2 \times (58\cdot27)^{-\cdot116}}$

Let $x =$ this fraction.

Then—

$$\log x = \{\tfrac{1}{3}\log 42\cdot17 - 2\log \cdot0145\} - \{2\log 8\cdot91 - \cdot116\log 58\cdot27\}$$
$$= \{\tfrac{1}{3}\log 42\cdot17 + \cdot116\log 58\cdot27\} - \{2\log 8\cdot91 + 2\log \cdot0145\}$$

$$= (\cdot 5417 + \cdot 2046) - (1\cdot 8998 + \bar{4}\cdot 3228)$$

$$= \cdot 7463 - \bar{2}\cdot 2226$$

$$= 2\cdot 5237 = \log 333\cdot 9$$

$$\therefore \quad \underline{x = 333\cdot 9}$$

Explanation.

$$\log 42\cdot 17 = 1\cdot 6250$$
$$\tfrac{1}{3} \times \log 42\cdot 17 = \cdot 5417$$
$$\log 58\cdot 27 = 1\cdot 7654$$
$$\cdot 116 \times \log 58\cdot 27 = \cdot 2046$$
$$\log 8\cdot 91 = \cdot 9499$$
$$2 \times \log 8\cdot 91 = 1\cdot 8998$$
$$\log \cdot 0145 = \bar{2}\cdot 1614$$
$$2 \times \log \cdot 0145 = \bar{4}\cdot 3228$$

When substituting figures for the letters in formulæ and thence evaluating the formulæ, the importance of the preceding rules will be recognised. Empirical formulæ and also the direct results of rigid proofs are of no value at all if one cannot use them efficiently.

It is necessary for this purpose that one or two rules, in addition to, or in extension of, those already given should be rigidly observed, viz.—

Work one step at a time : keep all terms quite distinct until their separate values have been found : and remember that statements including + and − cannot be directly changed into log forms.

$$e.\,g.,\ x = 45 + (29)^{1\cdot 2}$$

would not read, when logs were taken throughout,

$$\log x = \log 45 + 1\cdot 2 \log 29 \quad \text{which is wrong.}$$

To evaluate this equation, $(29)^{1\cdot 2}$ would be found separately and its value afterwards added to 45.

In cases in which a number of separate terms have to be evaluated it is advisable to keep the separate workings for these to one side of the paper and quite distinct from the body of the sum.

Example 20.—A gas is expanding according to the law $pv^n = C$. Find the value of the constant C when $p = 85$, $v = 2\cdot 93$ and $n = 1\cdot 3$.

Substituting values— $C = 85 \times (2\cdot 93)^{1\cdot 3}$

In the log form— $\log C = \log 85 + 1\cdot 3 \log 2\cdot 93$

$$= 1\cdot 9294 + (1\cdot 3 \times \cdot 4669)$$
$$= 1\cdot 9294 + \cdot 607 = 2\cdot 5364$$
$$\therefore \quad C = \underline{343\cdot 9}$$

Example 21.—The insulation resistance of a length l inches of fibre-covered wire, of outside radius r_2, and inside radius r_1; the specific resistance of the insulator being S, is given by the formula—

$$R = \cdot 366 \times \frac{S}{l} \times \log_e \frac{r_2}{r_1}$$

Find the resistance of the insulation of 50 ft. of wire, of outside diam. ·25 cm. and inside diam. ·12 cm., when S = 3000 megohms.

$$R = \cdot 366 \times \frac{3000}{50 \times 12} \log_e \frac{\cdot 125}{\cdot 06}$$

$$= \cdot 366 \times 5 \times \cdot 734$$

$$= \underline{1\cdot 343 \text{ megohms.}}$$

Explanation.

$\left\{\frac{r_2}{r_1}\right.$ is a ratio, and $\therefore$ r_2 and r_1 may be in any units so long as both are in the same $\left.\right\}$

$$\log_e \frac{\cdot 125}{\cdot 06}$$
$$= \log_e \frac{1\cdot 25}{\cdot 6}$$
$$= \log_e 1\cdot 25 - \log_e \cdot 6$$
$$= \cdot 2231 - \bar{1}\cdot 4892$$
$$= \cdot 7339$$

Example 22.—For the flow of water over a rectangular notch, the quantity—

$$Q = aLH^{1\cdot 5} + bH^{2\cdot 5}$$

Find Q when $a = \cdot 27$, $L = 11\cdot 5$, $b = 28$, and $H = \cdot 517$.

Making substitutions for the separate parts—

Let— $Q = x + y.$

Then the values of x and y must be first found quite separately and then added. It is preferable in this example to treat the determination of the values of x and y as the main portion, *i. e.*, to work these in the centre of the page.

$$x = aLH^{1\cdot 5}$$
$$= \cdot 27 \times 11\cdot 5 \times (\cdot 517)^{1\cdot 5}$$

Then— $\log x = \log \cdot 27 + \log 11\cdot 5 + 1\cdot 5 \log \cdot 517$
$= \bar{1}\cdot 4314 + 1\cdot 0607 + (1\cdot 5 \times \bar{1}\cdot 7135)$
$= \bar{1}\cdot 4314 + 1\cdot 0607 - 1\cdot 5 + 1\cdot 0703$
$= \cdot 0624$
$x = 1\cdot 154$

Also— $y = bH^{2\cdot 5} = 28 \times (\cdot 517)^{2\cdot 5}$

Then— $\log y = \log 28 + 2\cdot 5 \log \cdot 517$
$= 1\cdot 4472 + (2\cdot 5 \times \bar{1}\cdot 7135)$
$= 1\cdot 4472 - 2\cdot 5 + 1\cdot 7838$
$= \cdot 7310$
$y = 5\cdot 383$

$\therefore$ $Q = x + y$
$= 1\cdot 154 + 5\cdot 383 = \underline{6\cdot 537}$

Alternative Method of Setting Out

No	Log
$\cdot 517$	$\bar{1}\cdot 7135$
	$1\ 5$
	$\cdot 35675$
	$\cdot 7135$
	$1\cdot 0703\ +$
	$1\ 5\ \ -$
	$\bar{1}\cdot 5703$
$11\cdot 5$	$1\ 0607$
$\cdot 27$	$\bar{1}\ 4314$
$1\cdot 154$	$\cdot 0624$

Example 23.—The dryness fraction q of a sample of steam, expanding adiabatically, viz. without loss or gain of heat, can be found from—

$$\frac{qL}{\tau} = \frac{q_1 L_1}{\tau_1} + \log_e \frac{\tau_1}{\tau}$$

where τ_1, q_1 and L_1 are the original conditions of absolute temperature, dryness and latent heat respectively; and τ, q and L are the final conditions of absolute temperature, dryness and latent heat.

One lb. of dry steam at 115·1 lbs. per sq. in. absolute pressure, expands adiabatically to a pressure of 20·8 lbs. per sq. in. absolute : find its final dryness.

From the steam tables, τ_1 (corresponding to pressure 115·1 lbs. per sq. in.) = 799° F. absolute temperature, and τ (for pressure = 20·8 lbs. per sq. in.) = 691° F. absolute temperature.

Also the respective latent heats are $L_1 = 879$ and $L = 954$.

Then if $q_1 = 1$, since the steam is originally dry—

$$q = \frac{\tau}{L}\left\{\frac{q_1 L_1}{\tau_1} + \log_e \frac{\tau_1}{\tau}\right\}$$

$$= \frac{691}{954}\left\{\frac{1 \times 879}{799} + \log_e \frac{799}{691}\right\}$$

$$= \cdot725\{1\cdot1 + \cdot145\}$$

$$= \cdot725 \times 1\cdot245$$

$$= \underline{\cdot903.}$$

Explanation.

$$\log_e \frac{799}{691} = \log_{10}\frac{799}{691} \times 2\cdot303$$

$$= 2\cdot303(2\cdot9025 - 2\cdot8395)$$

$$= 2\cdot303 \times \cdot0630$$

$$= \cdot145$$

Example 24.—For an air-lift pump for slimes (a mixture of water and very fine portions of crushed ore, of specific gravity = 1·1013) the formula for the horse-power per cu. ft. of free air can be reduced to—

$$\text{H.P.} = \cdot015042\left\{P_2\left(\frac{P_1}{P_2}\right)^{\cdot71} - P_1\right\}$$

Find H.P. when $P_1 = 12\cdot5$, $P_2 = 15$.

Substituting values—

$$\text{H.P.} = \cdot015042\left\{15\left(\frac{12\cdot5}{15}\right)^{\cdot71} - 12\cdot5\right\}$$

$$= \cdot015042\,\{15 \times \cdot8786 - 12\cdot5\}$$

$$= \cdot015042\,(13\cdot18 - 12\cdot5)$$

$$= \cdot01023$$

Explanation.

$$\text{Let } x = \left(\frac{12\cdot5}{15}\right)^{\cdot71}$$

$$\log x = \cdot71\,(\log 12\cdot5 - \log 15)$$

$$= \cdot71(1\cdot0969 - 1\cdot1761)$$

$$= \cdot71 \times -\cdot0792$$

$$= -\cdot0562$$

$$= \bar{1}\cdot9438$$

$$\therefore\ x = \cdot8786.$$

Logarithmic Equations.—Whenever it is required to solve equations containing awkward powers it is nearly always the best plan, and in many cases the only one, to use logarithms. Little explanation should be necessary after the previous work, and a few examples will suffice.

Example 25.—Find v from the equation, $pv^n = C$, when $C = 146$, $n = 1{\cdot}37$, and $p = 22$.

Substituting values— $22 \times v^{1\cdot37} = 146$

Then—

$$\log 22 + 1{\cdot}37 \log v = \log 146$$
$$1{\cdot}37 \log v = \log 146 - \log 22$$
$$\log v = \frac{\log 146 - \log 22}{1{\cdot}37}$$
$$= \frac{2{\cdot}1644 - 1{\cdot}3424}{1{\cdot}37} = \frac{{\cdot}822}{1{\cdot}37} = {\cdot}6$$
$$\therefore \quad v = \underline{3{\cdot}981}$$

Example 26.—If $h = \dfrac{{\cdot}0004v^{1\cdot87}l}{d^{1\cdot4}}$, giving the head h lost in length l of pipe of diam. d, the velocity of flow of the water being v, find d when $h = {\cdot}87$, $v = 4{\cdot}7$ and $l = 12$.

Transposing for d—

$$d^{1\cdot4} = \frac{{\cdot}0004v^{1\cdot87}l}{h}$$

Taking logs of both sides—

$$1{\cdot}4 \log d = \log {\cdot}0004 + 1{\cdot}87 \log v + \log l - \log h$$
$$= \log {\cdot}0004 + 1{\cdot}87 \log 4{\cdot}7 + \log 12 - \log {\cdot}87$$
$$= \begin{cases} \bar{4}{\cdot}6021 \\ 1{\cdot}2568 - \bar{1}{\cdot}9395 \\ \underline{1{\cdot}0792} \\ \bar{2}{\cdot}9381 \end{cases}$$
$$= \bar{2}{\cdot}9986$$

Explanation.

$\log 4{\cdot}7 = {\cdot}6721$

$1{\cdot}87 \times \log 4{\cdot}7 = 1{\cdot}2568$

Then—

$$\log d = \frac{\bar{2}{\cdot}9986}{1{\cdot}4} = \frac{-2 + {\cdot}9986}{1{\cdot}4} = \frac{-1{\cdot}0014}{1{\cdot}4}$$
$$= -{\cdot}7153 = \bar{1}{\cdot}2847$$
$$\therefore \quad d = \underline{{\cdot}1926}$$

Example 27.—It is required to express the clearance in the cylinder of a gas engine as a fraction of the stroke. We are told that the temperature at the end of compression is 1061° F. abs. and at the end of expansion is 661° F. abs.; and that expansion is according to the law $pv^{1\cdot3} = C$. Also $\dfrac{pv}{\tau} = K$. (This example is important and should be carefully studied.)

Let p_c, τ_c and v_c be the pressure, absolute temperature and volume respectively at the end of compression; and let p_e, τ_e and v_e be the corresponding quantities at the end of expansion.

Then we can say— $\quad p_c v_c^{1\cdot 3} = C = p_e v_e^{1\cdot 3}$

$$\text{i. e.,} \quad \left(\frac{p_c}{p_e}\right) = \frac{v_e^{1\cdot 3}}{v_c^{1\cdot 3}} = \left(\frac{v_e}{v_c}\right)^{1\cdot 3} \quad \ldots\ldots\ldots\ldots \quad (1)$$

and also—
$$\frac{p_c v_c}{\tau_c} = K = \frac{p_e v_e}{\tau_e}$$

or
$$\left(\frac{p_c}{p_e}\right) = \left(\frac{v_e}{v_c}\right) \times \left(\frac{\tau_c}{\tau_e}\right) \quad \ldots\ldots\ldots\ldots \quad (2)$$

Hence from equations (1) and (2)—
$$\left(\frac{v_e}{v_c}\right)^{1\cdot 3} = \left(\frac{v_e}{v_c}\right)^{1} \times \left(\frac{\tau_c}{\tau_e}\right)$$

or, dividing through by $\frac{v_e}{v_c}$

$$\left(\frac{v_e}{v_c}\right)^{\cdot 3} = \frac{\tau_c}{\tau_e} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

What is required is $\frac{v_c}{v_e - v_c}$; and this can be found if $\frac{v_e}{v_c}$ is known.

For simplicity let $x = \frac{v_e}{v_c}$

Then from (3)— $\quad x^{\cdot 3} = \frac{\tau_c}{\tau_e} = \frac{1061}{661}$

In the log form $\cdot 3 \log x = \log 1061 - \log 661$

$$\text{or} \quad \log x = \frac{\log 1061 - \log 661}{\cdot 3} = \frac{3\cdot 0257 - 2\cdot 8202}{\cdot 3}$$
$$= \frac{\cdot 2055}{\cdot 3} = \cdot 685$$

$\therefore \quad x = 4\cdot 842$ which is thus the value of $\frac{v_e}{v_c}$

Hence— $\quad v_e = 4\cdot 842 v_c$

and $\quad v_e - v_c = 4\cdot 842 v_c - 1 v_c$
$$= 3\cdot 842 v_c$$

$$\therefore \quad \frac{v_c}{v_e - v_c} = \frac{v_c}{3\cdot 842 v_c} = \frac{1}{3\cdot 842} = \underline{\cdot 2604}$$

Exercises 26.—On Evaluation of Difficult Formulæ and on Logarithmic Equations.

1. Find the natural logs of $21\cdot 42$; $3\cdot 18$, $\cdot 164$.

2. Find the values of $\log_e \cdot 007254$; $\log_e 72\cdot 54$; $\log_e \frac{1871}{461}$

3. Tabulate the values of $\log_e \frac{\tau}{461}$, when $\tau = 461$, 500, 560, 613, 800, and 1000 respectively.

4. Evaluate $[\log_e 4\cdot 718]^{\cdot 6}$

Q

5. Find the value of $pv \log_e r$ when $p = 120$, $v = 4\cdot71$ and $r = 5\cdot13$.
Evaluate Exs. 6 to 14.

6. $(24\cdot91)^{-\cdot72}$ **7.** $(\cdot1183)^{4\cdot6}$ **8.** $\dfrac{2}{(\cdot0054)^{\cdot16}}$

9. $(3\cdot418)^2 \times (\cdot4006)^{-3\cdot4}$ **10.** $\dfrac{98^{\frac{1}{3}} \times (3\cdot051)^{-\frac{1}{11}}}{(\cdot0046)^{-\cdot05}}$

11. $(\cdot04105)^{-2\cdot3}$ **12.** $(\cdot3724)^{-2\cdot43}$

13. $\dfrac{(\log_e 1\cdot62)^3 \times (\log_{10}325\cdot6)^{-\cdot247}}{(8093)^{\cdot01542}}$

14. $1\cdot163 \times (\cdot0005)^{7\cdot76} \div \sqrt{(\log_{10}21\cdot67)^{-1}}$

15. The heat (B.Th.U) generated per hour in a bearing $= d\,l\,v^{1\cdot38}$ where d = diameter of bearing in inches, l = length of bearing in inches, v = surface velocity of shaft in feet per sec. Find the number of B.Th.U. generated per hour by a shaft of 5″ diam., rotating in a bearing 2 ft long with surface velocity of 50 ft. per sec.

16. Find the value of a velocity v from—

$$v = \frac{c\sqrt{2gh}}{\sqrt{1+\left(\dfrac{1}{K}-1\right)^2}}$$

when $c = \cdot97$, $K = \cdot63$, $g = 32\cdot2$ and $h = 49\cdot5$.

17. The collapsing pressure P lbs. per sq. in. for furnace tubes with longitudinal lap-joints may be calculated from Fairbairn's formula—

$$P = 7\cdot363 \times 10^6 \frac{t^{2\cdot1}}{l^{\cdot9}d^{1\cdot16}}$$

where t = thickness in inches, l = length in inches and d = diameter in inches. Find P when $t = \cdot043''$, $l = 38''$, and $d = 4''$.

18. Similarly for tubes with longitudinal and cross joints.
Calculate P if $t = \cdot12''$, $l = 60''$, and $d = 5\frac{1}{2}''$ from—

$$P = 15547000\frac{t^{2\cdot35}}{l^{\cdot9}d^{1\cdot16}}$$

19. The theoretical mean effective pressure (m e.p.) in a cylinder is calculated from—

$$p_m = \frac{P(1 + \log_e r)}{r} - P_b$$

where P = boiler pressure, P_b = back pressure, and r = ratio of expansion. The actual m.e.p. $= p_m \times$ diagram factor.

Find the actual m.e.p. in the case when $P = 95$, $P_b = 15$, cut off is at $\cdot3$ of stroke $\left(i.e., r = \dfrac{1}{\cdot3}\right)$ and diagram factor $= \cdot8$.

20. The H.P. required to compress adiabatically a given volume of free air, to a pressure of R atmospheres, is given by—

H.P. $= \cdot015P(R^{\cdot29} - 1)$ when the compression is accomplished in one stage and H.P. $= \cdot03P(R^{\cdot145} - 1)$ when the compression is accomplished in two stages.

Find H.P. in each case if $P = 14\cdot7$ and $R = 4\cdot6$.

21. Find H, a hardness number, from—

$$H = \frac{16PD^{n-2}}{\pi(2d)^n}$$

Given that $D = 24$, $d = 5$, $P = 58$, $n = 2\cdot35$.

22. Mallard and Le Chatelier give the following rule for the determination of the specific heat at constant volume (K_v) of CO_2 (carbon dioxide)—

$$44K_v = 4{\cdot}33\left(\frac{t}{100}\right)^{\cdot 367} \text{ where } t = C^\circ.$$

Find K_v when $t = 326$.

23. Find H.P. from $\text{H.P.} = {\cdot}01504\left\{P_2\left(\frac{P_1}{P_2}\right)^{\cdot 71} - P_1\right\}$

when $P_1 = 12{\cdot}5$, and $P_2 = 22$; the letters having the same meanings as in worked Example 24.

24. Find the efficiency η of a gas engine from—

$$\eta = 1 - \left(\frac{1}{r}\right)^{n-1} \quad \text{when } n = 1{\cdot}4 \text{ and } r = 5$$

25. The H.P. lost in friction when a disc of diameter D ft. revolves at N revs. per min. in an atmosphere of steam of pressure p lbs. per sq. in. abs., is given by—

$$\text{H.P.} = 10^{-13}pD^5N^3$$

Find the H.P. lost when the diameter is 5 ft., N = 500, and $p = 1$.

26. If $p = P\left(\frac{2}{1+n}\right)^{\frac{n}{n-1}}$ and $n = 1{\cdot}41$ find $\frac{p}{P}$

27. Calculate the entropy of water ϕ_w, and that of steam ϕ_s at absolute temperature τ from—

$$\phi_w = \log_e \frac{\tau}{461}$$

and

$$\phi_s = \log_e \frac{\tau}{461} + \frac{1437}{\tau} - {\cdot}7$$

The value of τ is 682.

28. In the case of curved beams, as for a crane hook—

$$P = SRD\,\frac{\pi}{2}\left\{\frac{1}{4}\left(\frac{D}{D+2R}\right)^2 + \frac{1}{8}\left(\frac{D}{D+2R}\right)^4 + \frac{5}{64}\left(\frac{D}{D+2R}\right)^6\right\}$$

where R = radius of inside of crane hook in ins. = 1·5, D = diam. of cross section in ins. = 2·1, P = safe load of hook in lbs., and S = maximum allowable tensile stress = 17000 lbs. per sq. in. Find the value of P.

29. A sample of steam of dryness ·83 at 380° F. expands adiabatically to 58° F.; calculate its dryness at the latter temperature from—

$$\frac{qL}{\tau} = \frac{q_1L_1}{\tau_1} + \log_e \frac{\tau_1}{\tau}$$

τ_1 is the initial temperature and $L = 1115 - {\cdot}7t \left\{\begin{array}{l} t = \text{F.}^\circ \\ \tau = \text{F.}^\circ \text{ absol.} \\ \quad i.e.,\ t + 461 \end{array}\right\}$

30. Steam 20% wet at 90 lbs. per sq. in. absolute pressure expands adiabatically to 25 lbs. per sq. in. absolute. Find its wetness at the second pressure. Note that:—

$p = 90$, $t = 320°$ F.; $p = 25$, $t = 240°$ F.; $L = 1115 - {\cdot}7t$

[Note also the difference between examples 29 and 30 as to the given data.]

31. The efficiency η of a perfectly-jacketed engine is given by—

$$\eta = \frac{a \log_e \frac{\tau_1}{\tau_2} + b(\tau_1 - \tau_2)}{a \log_e \frac{\tau_1}{\tau_2} + a + b\tau_1}$$

where $a = 1437$, $b = -\cdot 7$; τ_1 and τ_2 being the extreme temperatures (F.° abs.).

Find the efficiency of a jacketed engine working between 66° F. and 363° F.

32. Calculate the efficiency of an engine working on the Rankine cycle between 60° F. and 363° F., using the formula—

$$\eta = \frac{(\tau_1 - \tau_2)\left(1 + \frac{L_1}{\tau_1}\right) - \tau_2 \log_e \frac{\tau_1}{\tau_2}}{L_1 + \tau_1 - \tau_2}$$

τ_1 and τ_2 are absolute temperatures and $L = 1437 - \cdot 7\tau$.

33. Calculate the flow Q over a triangular notch from the formula—

$$Q = \frac{8}{15} \tan\frac{\theta}{2} \sqrt{2g} \cdot H^{\frac{5}{2}}$$

where $g = 32 \cdot 2$, $H = \cdot 28$, $\tan \frac{\theta}{2} = \cdot 577$.

34. Find the number of heat units H_j supplied for the jacket to an engine working between 60° F. and 363° F. from the formula—

$$H_j = 1437 \log_e \frac{\tau_1}{\tau_2} - (\tau_1 - \tau_2)$$

where τ_2 and τ_1 are absolute temperatures, initial and final respectively.

35. Francis' formula for the discharge of water over a rectangular notch is—

$$Q \text{ (cu. ft. per sec.)} = 3 \cdot 33 \, (L - \cdot 1nH)H^{\frac{3}{2}}$$

If the breadth $L = 5 \cdot 4$, the head $H = \cdot 4$, and $n = 2$, find Q.

36. If $i = \frac{\cdot 00037v^{2 \cdot 1}}{m^{1 \cdot 5}}$, find i when $m = 2 \cdot 16$, $v = 1 \cdot 65$.

37. The volume of 1 lb. of steam may be calculated from Callendar's equation—

$$V - w = \frac{R\tau}{p} - c\left(\frac{273}{\tau}\right)^{\frac{10}{3}}$$

where $w = \cdot 017$, $c = 1 \cdot 2$, $R = 154$

V = vol. in cu. ft., p = pressure in lbs. per sq. foot, τ = temperature in centigrade degrees absolute (*i. e.*, $t°$ C. + 273).

Find V when $p = 10$ lbs. per sq. in. and $t = 89 \cdot 6°$ C.

38. Recalculate, when $P = 7200$ lbs. per sq. foot and $t = 138 \cdot 2°$ C.

39. Similarly, when $p = 100$ lbs. per sq. in. and temperature is 437° C. absolute.

40. In calculating the tensions of ropes on grooved pulleys we have the formula—

$$\frac{T}{t} = e^{\mu r \theta}$$

where θ is the angle of lap in radians, μ is the coefficient of friction, r is a coefficient depending on the angle of the groove, and T and t are the greatest and least tensions respectively. Calculate the value of T if the angle of lap is 66°, $\mu = \cdot 22$, $t = 45$ and $r = 1 \cdot 84$.

41. The efficiency of an ideal or perfect engine (working on the Diesel principle) is given by—

$$\eta = 1 - \frac{1}{r^{n-1}}\left\{\frac{d^{n-1}}{nd-1}\right\}$$

where $d = \dfrac{\text{volume at cut-off}}{\text{volume of clearance}}$, $r = \dfrac{\text{maximum volume}}{\text{volume of clearance}}$

Find the efficiency when $d = 1{\cdot}56$, $r = 14{\cdot}3$ and $n = 1{\cdot}4$.

42. Find the tensions T and t in a belt transmitting 20 H.P.; the belt lapping 120° round the pulley, which is of 3 ft. diam. and runs at 180 R.P.M. The coefficient of friction μ between the belt and pulley is ·3.

Given that $\dfrac{T}{t} = e^{\mu\theta}$ and θ = angle of lap in radians; and

H.P. $= \dfrac{\pi ND(T-t)}{33000}$, N = revs. per min. and D ft. = diam. of pulley.

43. The pressure of a gas is 165 lbs. per sq. in. when its volume is 2·257 cu. ft. and the pressure is ·98 lb. per sq. in. when the volume is 286 cu. ft. If the law connecting pressure and volume has the form pv^n = constant, find the values of n and this constant.

44. Find y from $4^{2y} = 58{\cdot}7$.

45. Solve for x in the equation $x^{1{\cdot}95} = 14x^{{\cdot}62}$

46. When $e^{5c} = 41{\cdot}28^{2{\cdot}9}$, find the value of c.

47. If $\frac{5}{8}\ (x^2)^{4{\cdot}3} = 9x$: solve this equation for x.

48. Given that $f_1^{\frac{3}{2}}\rho_1^{-\frac{1}{2}} = f_2^{\frac{3}{2}}\rho_2^{-\frac{1}{2}}$, and also that $\rho_1 = {\cdot}283$, $f_1 = 28$, and $f_2 = 19{\cdot}5$: find ρ_2.

49. In the law connecting pressures and temperatures of a perfect gas, find p_2 from the equation—

$$\frac{\tau_2}{\tau_1} = \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}$$

having given that $n = 1{\cdot}37$, $p_1 = 2160$, $\tau_2 = 1460$ and $\tau_1 = 2190$.

50. For a gas engine, $Pv^{1\cdot 33} = p(v+s)^{1\cdot 33}$
where P = compression pressure, p = suction pressure, v = clearance volume and s = total volume swept out by the piston.
If P = 8·91 p and s = ·138, find v.

51. If $v = aH^n$, and H = 3 when $v = 387$· and H = 80 when v = 2000, find the values of a and n.

52. If the pressure be removed from an inductive electric circuit, the current dies away according to the law—

$$C = \frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

where C is the current at any time t secs. after removal of the voltage, R and L are the resistance and self-inductance of the circuit respectively, and V is the voltage. If R = 350, L = 5·5 and V = 40000, find the time that elapses before the current has the value 80 amperes.

53. £120 was lent out at r% per annum compound interest, the interest being added yearly; and in 5 years the amount became £150. Find the rate per cent. $\left[\text{Amount} = \text{Principal}\left(1 + \frac{r}{100}\right)^n\right]$

54. If $P_eV_e^{1{\cdot}33} = P_dV_d^{1{\cdot}33}$; $\left(\dfrac{V_d}{V_e}\right) = {\cdot}206$; and $P_d = 44000$; find P_e.

55. The insulation resistance R of a piece of submarine cable is being measured; it has been charged, and the voltage is diminishing according to the law—

$$v = be^{-\frac{t}{KR}}$$

where b is some constant, and t = time in secs. and $K = \cdot 8 \times 10^{-6}$. If $v = 30$, and at 15 secs. after it is noted to be 26·43, find the value of R.

56. Calculate the efficiency of a Diesel Engine from the formula—

$$\eta = 1 - \frac{1}{n}\left\{\frac{\left(\frac{r_c}{r_e}\right)^n - 1}{r_c^{n-1}\left(\frac{r_c}{r_e} - 1\right)}\right\}$$

where $n = 1\cdot41$, r_c = compression ratio = 13·8 and r_e = expansion ratio = 7·4.

57. Determine the ratio of the maximum tension to the minimum tension in a belt lapping an angle θ radians round a pulley, the coefficient of friction being μ, from

$$\frac{T_{max.}}{T_{min.}} = \frac{2e^{\mu\theta} + 1}{3}$$

The coefficient of friction is ·18 and the angle of lap is 154°.

58. The work done in the expansion of a gas from volume v_2 to volume v_1 is given by—

$$W = \frac{4000(v_1^{1-n} - v_2^{1-n})}{1-n}$$

Find this work when $v_1 = 10$, $v_2 = 1$, and $n = 1\cdot13$.

59. If $T = te^{\mu\theta}$ (the letters having the same meanings as in Example 40): $\theta = 2\cdot88$ radians, $\mu = \cdot15$ and $t = 40$, find the value of T.

60. Similarly if $\theta = 165°$, and $\frac{T}{t} = 1\cdot78$, find μ.

61. In the expansion of a gas it is given that $pv^n = c$, and that $p = 107\cdot3$ when $v = 3$: and $p = 40\cdot5$ when $v = 6$: find the law connecting p and v in this case.

62. In a "repeated load" test on a rotating beam of $\frac{9}{16}$" rolled Bessemer steel, the connection between the stress F in lbs. per sq. in. and the number of revolutions N to fracture was found to be—

$$F = \frac{214300}{N^{\cdot147}}$$

Find the value of N when F = 40700.

63. In a similar test on a specimen of ½" bright drawn mild steel—

$$F = \frac{73300}{N^{\cdot04}}$$

Determine the value of F which makes N = 48300.

64. The total magnetic force at a point in a magnetic field—

$$= \frac{2\pi nCr^2}{(r^2 + x^2)^{\frac{3}{2}}}$$

Find this force when $C = \cdot4$, $n = 10$, $r = 4$ and $x = 5\cdot9$.

65. From the results on a test on the measurement of the flow of water over a rectangular notch, complete the following table; it

being given that coeff. of discharge $= \frac{\text{actual discharge}}{\text{theoretical discharge}}$, and theoretical discharge $= 40{\cdot}15\, bh^{\frac{3}{2}}$ (lbs. per min.).

b	h	Actual Discharge (lbs. per minute).	Theoretical Discharge.	C_d
1·75	·829	35		
1·75	1·41	79		
1·75	1·81	112·6		

66. Also calculate as in the preceding Example, but for a submerged rectangular orifice, for which the theoretical discharge

$$= 40{\cdot}15b(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}).$$

h_2	h_1	b	Actual Discharge.	Theoretical Discharge.	C_d
2·325	1·075	1·25	88·8		
3·34	2·09	1·25	109·6		
4·415	3·165	1·25	133		
6·11	4·86	1·25	156·6		

67. The skin resistance per sq. ft. of a ship model is proportional to some power of the speed. If the resistance is ·0821 at velocity 5, and ·612 at velocity 14, find the law connecting resistance and velocity.

68. The loss of head due to pipe friction is proportional to some power of the velocity. If loss of head was 14·13 when velocity was 10·23, and loss was 6·31 when velocity was 6·76, find the law connecting loss of head h, and velocity v.

69. Relating to the flow of water through pipes it is required to find a value of d (the diameter of the pipe) to satisfy the two equations—

$$i = \frac{\cdot 00045v^{1\cdot 95}}{d^{1\cdot 25}} \quad \text{and} \quad \frac{\pi}{4}d^2v = 14$$

If i (hydraulic gradient) $= \frac{1}{2640}$, find this value.

70. When a disc revolves in air the H.P. lost in air friction varies as the 5·5 power of the diameter of the disc and the 3·5 power of the revolutions. If H.P. lost is ·1 when diameter is 4 and disc makes 500 R.P.M. find diameter when 10 H.P. is lost, the disc revolving at 580 R.P.M.

71. When a disc revolves in a fluid it is found that the friction F per sq. foot of surface is proportional to some power of the velocity V. For a brass surface—

F per sq ft. .	·22	1·26
V ft./sec. . .	10	25

Find a formula connecting F and V.

CHAPTER VI

PLANE TRIGONOMETRY

Trigonometric Ratios.—If the ordinary $30° : 60°$ set-square be examined it will be found that for all sizes the ratios of corresponding sides are equal. If one of the angles is selected and the sides named according to their position with regard to that angle, the ratios of pairs of sides may be termed the *trigonometrical ratios* of the angle considered. The word *trigonometry* implies measurement of angles; the measurement of the angles being made in terms of lengths of lines.

For example, let the sides of the set-square be as shown in Fig. 108: then the angle 30° can be defined as that angle in a right-angled triangle for which the side opposite to it is 2″, whilst the hypotenuse is 4″, *i. e.*, the ratio of the $\frac{\text{opposite side}}{\text{hypotenuse}} = \frac{2}{4} = \cdot 5$.

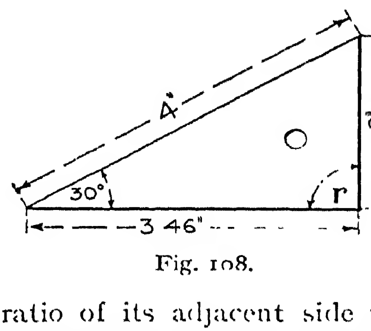


Fig. 108.

Again, the side 3·46″ long is that "lying next" or *adjacent* to the angle 30°, so that the angle 30° could thus be alternatively defined by the ratio of its adjacent side to the hypotenuse, or by the ratio of the adjacent side to the opposite side.

To these ratios special names are given.

The ratio $\frac{\text{opposite side}}{\text{hypotenuse}}$ is called the "sine" of the angle considered.

The ratio $\frac{\text{adjacent side}}{\text{hypotenuse}}$ is called the "cosine" of the angle considered.

The ratio $\frac{\text{opposite side}}{\text{adjacent side}}$ is called the "tangent" of the angle considered.

These three are the most important: if they are inverted

three other ratios are obtained, viz. **the cosecant or** $\left(\frac{1}{\text{sine}}\right)$, **secant or** $\left(\frac{1}{\text{cosine}}\right)$ **and cotangent or** $\left(\frac{1}{\text{tangent}}\right)$.

As a general rule these ratios, which, as defined, only apply to right-angled triangles, are written in the abbreviated form : sin, cos, tan, cosec, sec and cot.

In the triangle ABC, Fig. 109

$$\sin A = \frac{\text{opposite to A}}{\text{hypotenuse}} = \frac{a}{c}$$

whilst

$$\sin B = \frac{\text{opposite to B}}{\text{hypotenuse}} = \frac{b}{c}$$

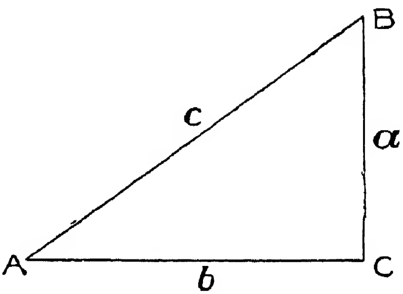


Fig. 109.

$$\cos A = \frac{\text{adjacent to A}}{\text{hypotenuse}} = \frac{b}{c}, \quad \cos B = \frac{a}{c}$$

$$\tan A = \frac{\text{opposite to A}}{\text{adjacent to A}} = \frac{a}{b}, \quad \tan B = \frac{b}{a}$$

$$\text{cosec } A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{a}, \quad \text{cosec } B = \frac{c}{b}$$

$$\sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{b}, \quad \sec B = \frac{c}{a}$$

$$\cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}, \quad \cot B = \frac{a}{b}$$

The angles A and B together add up to 90°; each being called the *complement* of the other, and it may be noticed that any ratio of one of the angles is equal to the co-ratio of its complement.

Hence the syllable " co " in cosine, cosec and cotan, indicates the complement of the sine, sec and tan respectively.

Thus sine A = co-sine of its complement B.
tan B = co-tan of its complement A.

For any angle the ratios could be found by careful drawing to scale and measurement of sides; this is not very accurate, however, and is certainly very tedious, and therefore tables are provided, in which the ratios of all angles from 0° to 90° are expressed. The changes in the values of the sine and cosine as the angle increases from 0° to 90° are illustrated by Fig. 110, in which the quadrant is that of a circle of unit radius—

i. e., OA = OC = OD = 1″.

Now $\sin \angle BOA = \frac{BA}{OA} = \frac{BA}{1''} = BA$, and in like manner $\sin \angle EOC = EC$ and $\sin \angle FOD = FD$. Also $\cos \angle BOA = OB$, $\cos \angle EOC = OE$ and $\cos \angle FOD = OF$. Thus the sine of the angle depends on the horizontal distance from the line ON of the end of the revolving line, while the cosine depends on the vertical distance from OP.

When the angle is very small, A is very near to ON and consequently the sine is small; and as OA approaches ON more closely, the value of the sine decreases until, when the angle is 0° the sine is 0, because the revolving line lies along ON. When the angle is 90°, the revolving line lies along OP and the horizontal distance of its end from ON has its greatest value, viz. 1. Thus the value of the sine increases from 0 to 1 as the angle increases from 0° to 90°.

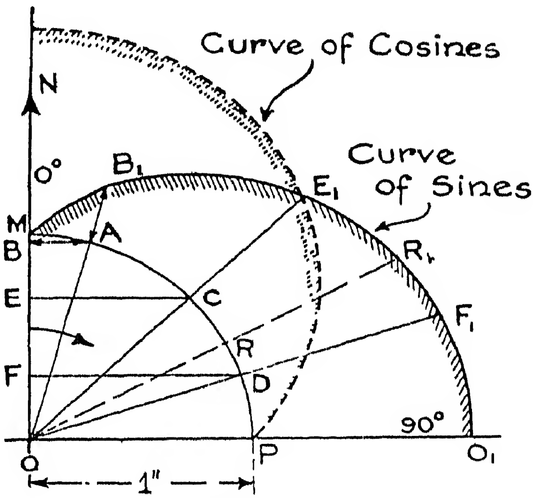


Fig. 110.

Along OA, produced, set off $AB_1 = AB = \sin \angle BOA$; and in like manner obtain the points E_1, F_1 and O_1. Draw a smooth curve through the points M, B_1, E_1, F_1 and O_1: then this is a curve of sine values, since the intercepts between the quadrant perimeter and this curve give the values of the sine, thus $\sin \angle MOR = RR_1$.

Similarly the curve of cosine values can be drawn, and it is seen that it is of the same form as the curve of sine values, but it is reversed in direction.

To read Table I at the end of the book. In this table one page suffices for the various ratios, these being stated for each degree only from 0° to 90°. This table is compact and has educational advantages, for it demonstrates clearly that as the angle increases the sine increases whilst the cosine decreases; and that a ratio of an angle is equal to the co-ratio of its complement, and so on.

Down the first column and up the last are the angles expressed

in degrees, whilst in the adjacent columns the corresponding values in circular measure (radians) are given. Thus $31^\circ = \cdot 5411$ radian, and $73^\circ = 1{\cdot}2741$ radians.

The values of the sines appear in the 4th column from the beginning and the 4th from the end, as do also the cosine values; but for cosines the tables must be read in the reverse direction.

No difficulty should be experienced in this connection if it be remembered that one must **always work away from the title of the column.** Thus for cosines read down the 7th column and up the 4th column.

Values of tangents and cotangents appear in the 5th and 6th columns; again working away from the title—

E. g.,	sine $17^\circ = \cdot 2924$,	sine $61^\circ = \cdot 8746$
	cos $23^\circ = \cdot 9205$,	cos $49^\circ = \cdot 6561$
	tan $42^\circ = \cdot 9004$,	tan $88^\circ = 28{\cdot}6363$
	cot $5^\circ = 11{\cdot}4301$,	cot $59^\circ = \cdot 6009$.

To read Table V at the end of the book, which should be used when greater subdivision of angles is required. Suppose that sin $43^\circ 22'$ is required: if Table I is followed, sin 43° must be found, viz. $\cdot 6820$, and sin 44°, viz. $\cdot 6947$, and $\frac{22}{60}$ of their difference must be added to $\cdot 6820$.

Thus— $$\sin 43^\circ 22' = \cdot 6820 + \frac{22}{60}(\cdot 6947 - \cdot 6820)$$
$$= \cdot 6867$$

This process is rather tedious: accordingly, referring to Table V, look down the 1st column until 43° is reached, then along the line until under $18'$, the figure is $\cdot 6858$; $4'$ have now to be accounted for; for this, use the difference columns, in which under $4'$, 8 is found—

$$\therefore \text{ sine } 43^\circ 22' = \cdot 6858 + \cdot 0008 = \cdot 6866.$$

The tangent tables, Table VII, would be applied in the same manner, but here the value of the ratio gets very large when in the neighbourhood of 90° so that the difference columns cannot be given with accuracy. When the angle $= 45^\circ$, the tangent $= 1$ and the tangent continues to increase as the angle increases, therefore it happens occasionally that the integral part of the value has to be altered in the middle of a line. To signify this a bar ($^-$) is written over the first figure: *e. g.*, tan $63^\circ = 1{\cdot}9626$, whilst tan $63^\circ 30'$ is written $\bar{0}057$, and this means $2{\cdot}0057$, the bar indicating that the integer at the commencement of the line must be increased by 1.

When using the cosine table, viz. Table VI, it must be remembered that an increase of the angle coincides with a *decrease* of the cosine, so that differences must be *subtracted: e. g.*, if the value of cos $52° 55'$ is required.

$$\cos 52° 54' = \cdot 6032; \text{ diff. for } 1' = 2$$
$$\therefore \quad \cos 52° 55' = \cdot 6032 - \cdot 0002 = \cdot 6030.$$

Values of cosecants and secants can be found by inverting the values of sines and cosines respectively.

Example 1.—The angle of advance θ of an eccentric in a steam engine mechanism can be found from $\sin\theta = \dfrac{\text{lap} + \text{lead}}{\frac{1}{2}\text{ travel}}$. Find θ when the lap is $\cdot 72''$, the lead is $\cdot 12''$ and the travel is $3\cdot 6''$.

Substituting the numerical values, $\sin\theta = \dfrac{\cdot 72 + \cdot 12}{1\cdot 8} = \dfrac{\cdot 84}{1\cdot 8}$
$= \cdot 4667.$

We have now to find the angle whose sine is $\cdot 4667$.

Turning to the table of natural sines we find $\cdot 4664$ (the sine of $27° 48'$) to be the nearest figure *under* $\cdot 4667$; this leaves $\cdot 0003$ to be accounted for. In the difference columns in the same line we see that a difference of 3 in the sine corresponds to a difference of 1 min. in the angle; hence $1'$ must be added to $27° 48'$ to give the angle whose sine is $\cdot 4667$. Hence $\theta = \underline{27° 49'}$.

Example 2.—If $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, $a = 4\cdot 2$, $b = 7\cdot 8$ and $c = 6$; find A.

Substituting the numerical values—

$$\cos A = \frac{7\cdot 8^2 + 6^2 - 4\cdot 2^2}{2 \times 7\cdot 8 \times 6} = \frac{60\cdot 84 + 36 - 17\cdot 64}{93\cdot 6}$$
$$= \frac{79\cdot 2}{93\cdot 6} = \cdot 8461.$$

From the table of natural cosines we find that the angle having the ratio the nearest *above* $\cdot 8461$ is $32° 12'$; for this the cosine is $\cdot 8462$, and therefore the difference of $\cdot 0001$ has to be allowed for. In the difference columns we see that a difference of $\cdot 0002$ corresponds to $1'$; and thus $\cdot 0001$ corresponds to $30''$ Hence $A = \underline{32° 12' 30''}$.

Exercises 27.—On the Use of the Tables of Trigonometric Ratios.

1. Read from the tables the values of: sin $61°$; tan $19°$; cos $87°$; tan $\cdot 2269$ radian

2. Find the values of sin $77\frac{1}{3}°$; cos $15° 24'$; tan $58° 13'$; cos $1\cdot 283$ radians.

3. Evaluate $\dfrac{2\cos 53°}{\tan 17\frac{1}{2}°}$

4. In a magnetic field, if H = horizontal component and T = the total force due to the earth, then $H = T \cos d$. Find T when $H = \cdot 18$ and $d = 63°$.

5. The tangent of the angle of lag of an electric current $= \dfrac{\text{reactance}}{\text{resistance}}$ and reactance $= 2\pi \times$ frequency $\times$ inductance. If frequency is 40, inductance $\cdot 0021$ and resistance $1 \cdot 7$, find the angle of lag.

6. The mean rate of working in watts = amperes $\times$ volts $\times$ cos (angle of lag). Find the mean rate when $A = 2 \cdot 43$, $V = 110$ and lag $= 19\frac{1}{4}°$. What is the mean rate of working if the current lags 90° behind the voltage?

7. The pitch of a roof $= \dfrac{\text{rise}}{\text{span}} = \frac{1}{2} \tan A$, where A is the angle of the roof. Find the angle of the roof for which the span is 36 ft. and the rise is 12 ft.

8. If an axle of radius r runs on a pair of antifriction wheels of radius R, and θ is the angle between the lines joining the respective centres, then—

$$\frac{F_1}{F} = \frac{r}{R \cos \frac{\theta}{2}}$$

where F = force required to overcome the friction on a plane axle and F_1 = force required to overcome the friction when using the antifriction wheels. Find F if $\theta = 47\frac{1}{2}°$, $r = 3''$, $R = 10''$ and $F_1 = 47$.

9. If D = pitch diameter of spiral toothed gear, N = number of teeth in gear, P = normal diametral pitch, and a = tooth angle of gear, then—

$$D = \frac{N}{P \cos a}$$

If $D = 5 \cdot 108$, $N = 24$ and $P = 5$, find a.

10. In calculating principal or maximum stresses, if $\tan 2\theta = \dfrac{2s}{f}$, $s = 2852$ and $f = 3819$, find θ.

11. The number of teeth in the cutter for spiral gears—

$$= \frac{\text{no. of teeth in the gear}}{\cos^3 (\text{angle of spiral})}$$

Find the number of teeth in the cutter when the angle of the spiral is 50° and there are 48 teeth in the gear. (*N.B.*—$\cos^3 A$ means the cube of the cosine of A; but $\cos A^3$ is the cos of A^3.)

12. In connection with the design of water turbines the equation $\dfrac{u}{w - V} = \tan \theta$ occurs, where w = tangential velocity of the water at inlet, u = radial velocity of water at inlet, V = velocity of the blade at inlet, and θ is the inclination of the blade at inlet. If $u = 8 \cdot 95$ ft. per sec., $V = 47 \cdot 7$ ft. per sec. and $\theta = 60\frac{1}{2}°$, find w.

13. In the formula giving the value of the horizontal pressure p on a retaining wall of height h, the earth surface being level, w is the weight of 1 cu. ft. of earth and ϕ is the angle of repose of the earth, *i. e.*, the greatest angle at which the loose earth would remain at rest.

$$\text{Then} \qquad p = \frac{1 - \sin \phi}{1 + \sin \phi} \times wh$$

Find the value of p when $w = 130$, $\phi = 23\frac{1}{2}°$ and $h = 24$ ft.

14. Calculate the value of M, the moment of friction of a collar bearing, from—

$$M = \frac{2\mu W(R_1^3 - R_2^3)}{3 \sin a(R_1^2 - R_2^2)}$$

when $R_1 = 4 \cdot 5$, $R_2 = 3 \cdot 75$, $W = 2000$, $\mu = \cdot 17$ and $a = 12°$.

15. The total pressure P on the rudder of a ship is given by—

$$P = 4 \cdot 6 KAV^2 \frac{\sin a}{\cdot 39 + \cdot 61 \sin a}$$

where V = speed of ship in knots, A = area of rudder, K = ·7, and a = angle of rudder with fore and aft plane. Calculate P, given that V = 16, A = 8 and $a = 15\frac{1}{4}°$.

16. The force P_1 applied horizontally to move a weight W up a rough plane inclined at an angle a to the horizontal, is given by—

$$P_1 = \frac{W(\mu + \tan a)}{1 - \mu \tan a}$$

Find P_1 if W = 3000, $a = 8\frac{1}{2}°$, and μ, the coefficient of friction, = ·12.

17. The total extension d of a helical spring is given by—

$$d = \frac{Wa^2l}{JG}(1 - \cdot 2 \sin^2 a)$$

If a = radius of coil = 4″, G = 12 × 10⁶ lbs. per □″, J = ·15, l = length = 29″, W = 12 lbs. and $a = 14°$, find the total extension.

18. The range of a projectile is given by $\frac{V^2 \sin 2A}{g}$, where V = velocity of projection, A = elevation of gun and $g = 32 \cdot 2$. Find the range, if the projectile is fired at an elevation of 29° 15′ with a velocity of 1520 ft. per sec.

19. p_n = intensity of the normal pressure of wind on a surface inclined at θ to the direction of wind, and p = intensity of pressure on the surface perpendicular to its direction—

$$p_n = p \cdot \frac{2 \sin \theta}{1 + \sin^2 \theta}$$

If $p = 35$ and $\theta = 22\frac{1}{2}°$, find p_n.

20. The maximum power-factor of a motor $= \cos \phi = \frac{\text{H.P.} + 4}{\text{H.P.} + 5}$

If H.P. is 4·78 find ϕ, the angle of lag of the current.

21. If P = effort on crosshead of a steam engine, T = crank-pin effort, θ = crank angle, $n = \frac{\text{connecting rod}}{\text{crank}}$; and if P = 450 lbs., $n = 11$ and $\theta = 1 \cdot 5$ radians; find T, from $T = P\left\{\sin \theta + \frac{\sin 2\theta}{\sqrt{n^2 - \sin^2 \theta}}\right\}$

{*Hint.*—Sin 171·9° = sin 8·1°}

22. Calculate the value of $y = Re^{-Kt} \sin(wt + \theta)$ when R = 3·5, K = ·4, t = ·02, w = 5, θ = ·16; the angle being expressed in radians.

23. The electrical induction B in an air gap is given by—

$$B = \frac{C \sin \frac{\theta}{2}\left(1 + \frac{\lambda}{2}\right) R \times 10^9}{An \times 10^7}$$

Find B when A = 3·515, n = 20, λ = ·0867, C = 42·05, R = 10382 and $\tan 2\theta = \cdot 1052$.

24. Find a value of θ to satisfy the equation—

$$\tan\theta = \frac{4d(l-2x)}{l^2}$$

where $d = 5$, $l = 30$ and $x = 4{\cdot}5$. This equation refers to stiffened suspension bridges, where θ is the angle of inclination of the cable to the horizontal at a horizontal distance x from one end of the bridge, l is the span of the bridge and d is the sag.

Application of Trigonometric Ratios.—We will first deal with a very simple case.

Example 3.—The angle of elevation of the top of a chimney at a point on the ground 120 ft. from the foot of the chimney is 25°. Find the height of the chimney.

Before proceeding to the actual working of the example, the term **angle of elevation** must be explained. The zero of the theodolite (an angle measuring instrument) would be observed when the telescope was directed along the horizontal: the telescope would then be moved in a vertical plane until the top of the chimney was seen and the angle then noted. **This angle is called the angle of elevation and is the angle between the horizontal and the line joining the eye to the object.**

If the instrument be placed on the chimney top, the same angle would be read, but it would now be called the **angle of depression** because the object (the earth) is *below* the level of the eye.

In the example before us, let h ft. = height of chimney (Fig. 111)

Then— $\frac{h}{120} = \frac{\text{opp}}{\text{adj.}}$ (for 25°)

and therefore $= \tan 25°$.

Now, from the tables—

$\tan 25° = {\cdot}4663$

$\therefore \frac{h}{120} = {\cdot}4663$

and $h = 120 \times {\cdot}4663 = 55{\cdot}96$, say $\underline{56 \text{ ft.}}$

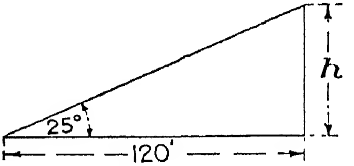


Fig. 111.

Example 4.—Two coils are connected in series over a 220 volt alternating-current main, and the drop across each coil is 126 volts. If the diagram illustrating the relation between the voltage drops is as in Fig. 112, find the difference in phase between the voltages in the two coils, *i. e.*, find the angle a.

Since the sides AB and BC are equal, the perpendicular from B on to AC bisects AC, or DC = 110; and also $\angle DBC = \frac{a}{2}$

Then— $\sin\frac{a}{2} = \frac{110}{126} = \cdot 8730$

$= \sin 60^\circ 49'$

and $\frac{a}{2} = 60^\circ 49'$

whence $a = \underline{121^\circ 38'.}$

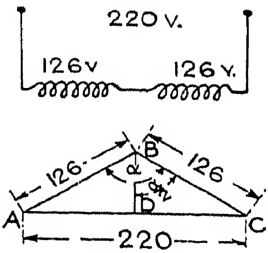


Fig. 112.

Example 5.—In a test on the Halpin thermal storage system, as fitted to a Babcock and Wilcox boiler, the volume of water taken from the storage tank to the boiler is to be determined by the difference in water level between start and finish. The tank being a cylinder of 57·81″ diam. and 251″ length, with its axis horizontal, see Fig. 113, the water level is 52·96″ from the tank bottom at the start and 14·86″ at the finish. Find the volume of water abstracted in cu. ft.

We have to find the area of ABCD, viz. the difference between the area AEBCD (at the start) and the area AEB (at the finish), and then multiply by the length of the tank.

To find the area of the segment AEB—

$$OF = OE - EF = 28\cdot 91 - 14\cdot 86 = 14\cdot 05$$

$$\cos a = \frac{OF}{OA} = \frac{14\cdot 05}{28\cdot 91} = \cdot 4859 = \cos 60^\circ 56'$$

and $\quad a = 60^\circ 56' \quad \text{or} \quad \frac{60\cdot 93}{57\cdot 3} = 1\cdot 063$ radians.

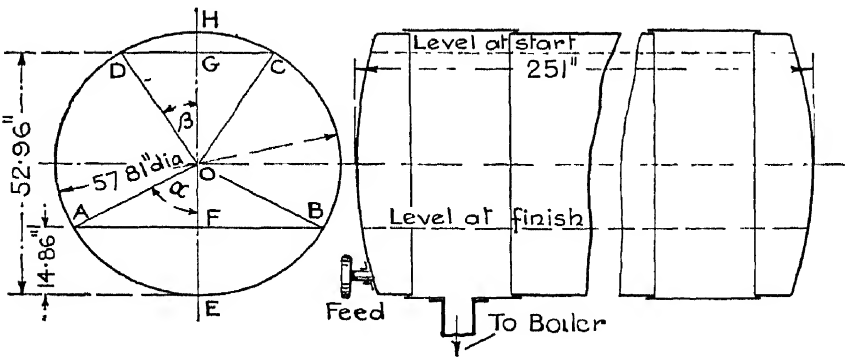


Fig. 113.—Halpin Thermal Storage System.

We can now use the rule previously given for the area of a segment, viz. area $= \frac{r^2}{2}(\theta - \sin\theta)$ where θ is the central angle in radians, for $r = 28\cdot 91$, $\theta = \angle AOB = 2a = 2\cdot 126$, and $\sin\theta = \sin 121^\circ 52'$

$= \sin(180^\circ - 121^\circ 52') = \sin 58^\circ 8' = \cdot 8493.$

[*Note.*—The proof of the rule $\sin A = \sin (180 - A)$ is given later in the book.]

Thus—
$$\text{area of AEB} = \frac{(28{\cdot}91)^2}{2}(2{\cdot}126 - {\cdot}8493)$$
$$= 533{\cdot}8 \text{ sq ins.}$$

To find the area of the segment DHC—
$$OG = EG - OE = 52{\cdot}96 - 28{\cdot}91 = 24{\cdot}05$$
$$\cos \beta = \frac{OG}{OD} = \frac{24{\cdot}05}{28{\cdot}91} = {\cdot}8318 = \cos 33^\circ\ 43'$$
$$\beta = 33^\circ\ 43' \text{ or } {\cdot}588 \text{ radian}$$
and
$$\sin 2\beta = \sin 67^\circ\ 26' = {\cdot}9234.$$

Hence—
$$\text{area of DHC} = \frac{(28{\cdot}91)^2}{2}(1{\cdot}176 - {\cdot}923)$$
$$= 105{\cdot}9 \text{ sq. ins.}$$
$$\text{Area of the whole circle} = \frac{\pi}{4} \times 57{\cdot}81^2 = 2625 \text{ sq. ins.}$$
$$\therefore \quad \text{area of ABCD} = 2625 - 533{\cdot}8 - 105{\cdot}9 = 1985 \text{ sq. ins.}$$
$$\text{and volume} = \frac{1985 \times 251}{1728} \text{ cu. ft.} = \underline{288{\cdot}4 \text{ cu. ft.}}$$

Example 6.—A seam dips at an angle of 62° to the horizontal for a distance of 900 ft. measured along the seam and then continues dipping at an angle of 40° to the horizontal. A shaft is started to cut the seam at a distance of 1200 ft. horizontally from the outcrop; at what depth will it cut the seam?

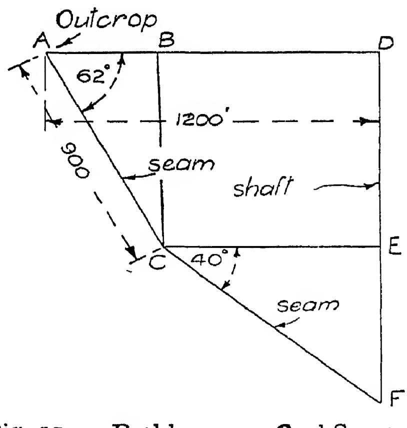


Fig. 114.—Problem on a Coal Seam

This example introduces the solution of two right-angled triangles: the lengths AD and AC (Fig. 114) are given and we require to find DF.

In the triangle ABC, $\dfrac{AB}{900} = \sin 28^\circ = {\cdot}4695$

$\therefore \quad AB = 900 \times {\cdot}4695 = 422{\cdot}6'$

Also— $\dfrac{BC}{900} = \sin 62^\circ = {\cdot}8829$

$\therefore \quad BC = 900 \times {\cdot}8829 = 794{\cdot}4'$

Hence— $CE = BD = 1200 - AB = 777{\cdot}4'$

In the triangle CEF, $\dfrac{EF}{CE} = \tan 40^\circ = {\cdot}8391$

$\therefore \quad EF = 777{\cdot}4 \times {\cdot}8391 = 652{\cdot}4$

$\therefore \quad DF = DE + EF = BC + EF = 794{\cdot}4 + 652{\cdot}4 = \underline{1446{\cdot}8 \text{ ft.}}$

Trigonometric Ratios from the Slide Rule.—The sine and tangent scales of the slide rule may be usefully employed in trigonometry questions; the multiplication of the side of the triangle by the trigonometric ratio being performed without the actual value of the ratio being read off.

To read values of trigonometric ratios: Reverse the slide so that the S scale is adjacent to the A scale and the T scale to the D scale. The sines of angles on the S scale will then be read off directly on the A scale. If the number is on the left-hand end of the rule, then ·0 must be prefixed to the reading, but if on the right-hand end of the rule, then a decimal point only.

e. g., to find $\sin 4^\circ$: place the cursor over 4° on S scale, and on A scale read off 698; this being on the left-hand end of the rule $\sin 4^\circ = \cdot 0698$.

Again, $\sin 67^\circ = \cdot 921$ for 921 is read off on the A scale above 67 on the S scale and is on the right-hand end of the rule.

As the angle approaches 90° the sine does not increase very rapidly and therefore the markings for the angles on the S scale in this neighbourhood are very close together. From 70° the usual markings are for 72°, 74°, 76°, 78°, 80°, 85° and 90°, the longer mark being at 80°.

To use the S scale for a scale of cosines, first subtract the angle from 90°, *i. e.*, find its complement, and then find the sine of this.

e. g., $\cos 37^\circ = \sin 53^\circ = \cdot 799$.

To combine multiplication with the reading of ratios, use the S scale just as the ordinary slide or B scale, multiplying, as it were, by the angles instead of by mere numbers.

e.g., suppose the value of the product $18{\cdot}5 \times \sin 72^\circ$ is required. The right hand of the S scale is set level with 185 on the A scale, the cursor is placed over 72 on the S scale, and the product 17·6 is read off on the A scale.

The tangents of angles from 0° to 45° will be read in a similar fashion, the T and D scales being used. $\tan 45^\circ = 1$, and after this the tangent increases rapidly, being infinitely large at 90°. For an angle greater than 45°: subtract the angle from 90° and divide unity by the tangent of the resulting angle.

e. g., suppose $\tan 58^\circ$ is required.

Actually— $$\tan 58^\circ = \frac{1}{\tan 32^\circ}$$

Hence: set 32° on the T scale level with 1 on the D scale; then

the reading on the D scale opposite 45°, *i. e.*, the end of the T scale, is the required value and is 1·6.

A further example.—Find the value of $\frac{87}{\tan 64°}$

$$\frac{87}{\tan 64°} = 87 \times \tan 26° = 42 \cdot 4.$$

[The setting being: 45° on the T scale against 87 on the D scale; the cursor over 26 on the T scale; then 42·4 on the D scale.]

Example 7.—A boat towed along a canal is 12 ft. from the near bank and the length of rope is 64 ft. The horse pulls with a force of 500 lbs.: find the effective pull on the boat, and that tending to pull the boat to the side of the canal.

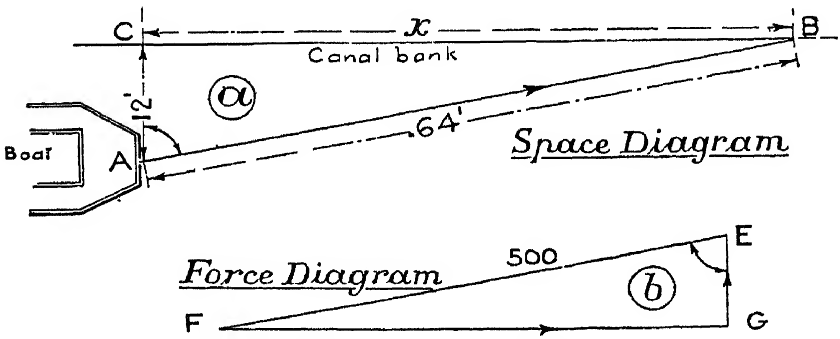


Fig 115.—Forces on Boat towed along a Canal.

The "space" diagram is first set out and from this x is calculated, viz. $x = \sqrt{64^2 - 12^2} = 62 \cdot 8$ (*a*, Fig. 115).

If a triangle be drawn (see *b*, Fig. 115) with sides parallel to those of the triangle ABC so that EF represents 500 lbs. to some scale, then EG and GF represent the pulls required to the same scales.

Or by calculation—

$$\frac{GE}{500} = \cos E = \cos A = \frac{12}{64}, \quad i.e., \ GE = 500 \cos E$$

$$\therefore \quad GE = \frac{12 \times 500}{64} = 93 \cdot 8$$

i.e., the pull towards the bank = 93·8 lbs.

$$\text{Also} \quad \frac{GF}{500} = \sin E = \sin A = \frac{62 \cdot 8}{64}, \quad i.e., \ GF = 500 \sin E$$

$$\therefore \quad GF = \frac{500 \times 62 \cdot 8}{64} = 491$$

i.e., the effective pull in the direction of the boat's motion = 491 lbs.

In general the components of a force R in two directions at right angles to one another (see Fig. 116) are R cos α, and R sin α

where α is the angle between R and the first of the components. As a further example of resolution into components, if T (Fig. 117) is the total magnetic force on a unit pole at some place and d is the angle of dip, H the horizontal component of the force is $= \mathrm{T}\cos d$, and V the vertical component $= \mathrm{T}\sin d$.

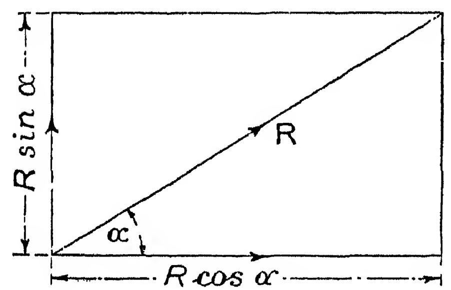


Fig. 116.

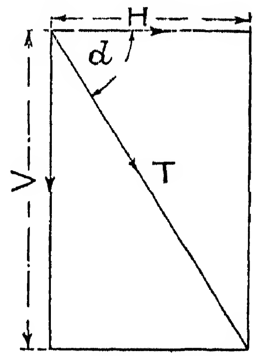


Fig. 117.

Components of Forces.

Calculation of Co-ordinates in Land Surveying.—When plotting the notes of a traverse survey, in which the sides of a polygon and the "included" or internal angles are measured in the field, it is necessary to first transform the dimensions of the lines and angles so as to give the co-ordinates of the corners as measured from the north and south line (or meridian) on the one hand, and from some chosen east and west line on the other hand. The survey is then plotted from the co-ordinates, with the object of introducing an accuracy of drawing which is impossible if the field-book dimensions are directly set out. In the latter case the angular error is cumulative, and, further, the plotting of angles at all times is more productive of error than the plotting of lines (*e. g.*, co-ordinates).

Quadrant bearings.—The co-ordinate axes being chosen as just stated, viz. North-South and East-West, every line of the traverse is referred to the meridian in terms of the smallest angle between it and the meridian, with the further statement of the "quadrant" (N.E., S.E., S.W., or N.W.) in which it is placed. Such angles are termed **quadrant or reduced bearings.**

Thus in Fig. 118—

The reduced bearing of the line A is 27° N.E., that of the line B is 36° S.E., that of the line C is 66° S.W., and that of the line D is 11° N.W.

Whole-circle bearings.—There is a second method of denoting the bearing of a line from the meridian and that is to simply take the angle that the line makes with the north but always in a right-handed direction. This is better than the quadrant method as requiring but one simple numerical statement.

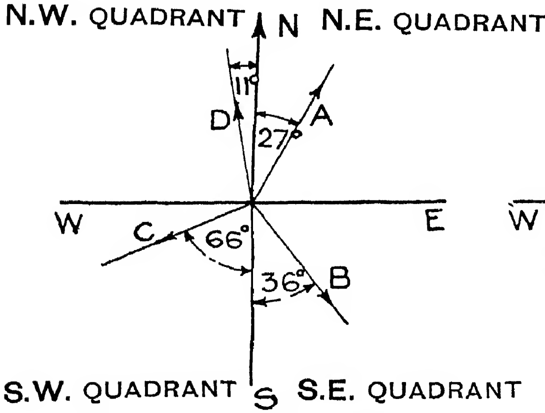


Fig. 118.—Reduced Bearings.

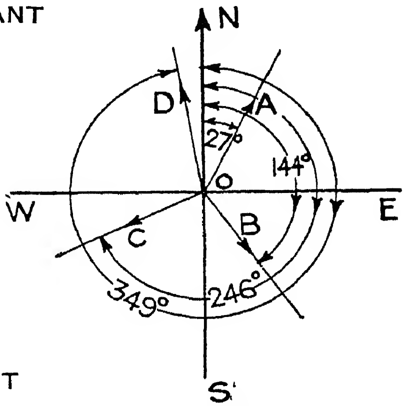


Fig. 119.—Whole-circle Bearings.

For example, in Fig. 119, the whole-circle bearings of the lines A, B, C and D are respectively 27°, 144°, 246° and 349°, all measured from the north line ON.

Example 8.—Measurements on a triangular plot of land ABC, Fig. 120, resulted in the following: AB = 7073 links, BC = 7736 links, CA = 5462 links, A = 75°, B = 43° and C = 62°. The reduced bearing (R.B.) of AB is 9° N.E. and the point A is taken as the origin for the co-ordinates. Find the reduced bearings of BC and CA, the co-ordinates of the points B and C, and also the area of ABC.

Right-hand order should be adhered to throughout, as indicated by the letters ABC.

To find the R.B. of BC. [It should be grasped that the bearing of C to B is not the same as the bearing B to C.] Mark on the diagram all the known angles, and then by combination with 90° or 180° all the required bearings can be found. Thus R.B. of BC = 43°−9° = 34° S.E., since 34° is the acute angle made by BC with the N. and S. line: the quadrant must also be stated, to definitely fix the direction of movement.

Similarly, the R.B. of CA = 180°−62°−34° = 84° S.W.

To calculate the co-ordinates of B—

$$\frac{BD}{AB} = \sin 9° \text{ and therefore } BD = AB \sin 9°$$

also $$AD = AB \cos 9°.$$

Thus the departure of B (*i. e.*, its distance E. or W. from A)
$= AB \times \sin$ (R.B. of AB)
and the latitude of B (*i. e.*, its distance N. or S. from A)
$= AB \times \cos$ (R.B. of AB)

Then— $BD = 7073 \times \sin 9°$
In the log form— $\log BD = \log 7073 + \log \sin 9°$
$= 3{\cdot}8496 + \bar{1}{\cdot}1943 = 3{\cdot}0439$
$\therefore$ BD = 1106 links, which is the departure of B east of A

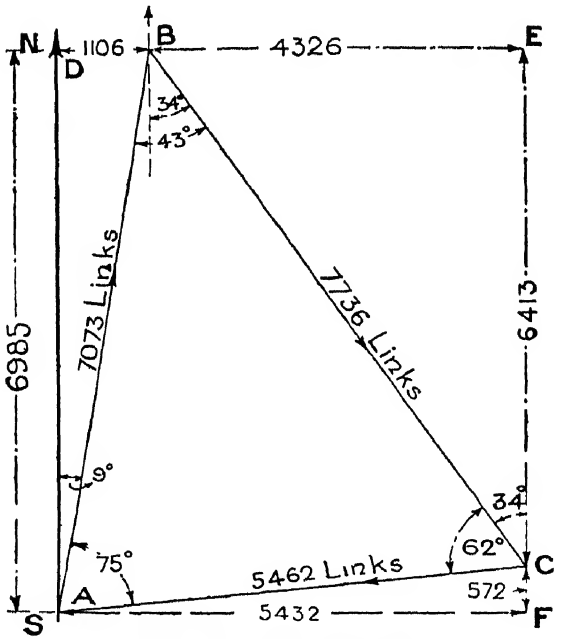


Fig. 120.—Plot of Land.

Again— $AD = AB \cos 9° = 7073 \times \cos 9°$
In the log form— $\log AD = \log 7073 + \log \cos 9°$
$= 3{\cdot}8496 + \bar{1}{\cdot}9946 = 3{\cdot}8442$
$\therefore$ AD = 6985 links, which is the latitude B north of A

Hence the co-ordinates of the point B are 1106, 6985.

For the point C— $BE = 7736 \sin 34°$
In the log form— $\log BE = \log 7736 + \log \sin 34°$
$= 3{\cdot}8885 + \bar{1}{\cdot}7476 = 3{\cdot}6361$
$\therefore$ BE = 4326 links, which is the departure of C east of B.

Again— $CE = 7736 \cos 34^\circ$

In the log form— $\log CE = \log 7736 + \log \cos 34^\circ = 3{\cdot}8885 + \bar{1}{\cdot}9186$
$= 3{\cdot}8071$

$\therefore$ $CE = 6413$ links, which is the difference of latitude between B and C.

Thus the co-ordinates of C are $(1106 + 4326)$ and $(6985 - 6413)$ or $\underline{(5432, 572)}$.

The figure may now be accurately plotted by means of the co-ordinates.

To calculate the area—

$$\triangle ABC = ADEF - ABD - BEC - ACF$$
$$= (5432 \times 6985) - (\tfrac{1}{2} \times 6985 \times 1106)$$
$$- (\tfrac{1}{2} \times 6413 \times 4326) - (\tfrac{1}{2} \times 572 \times 5432)$$
$$= (37{\cdot}95 \times 10^6) - (3{\cdot}863 \times 10^6) - (13{\cdot}88 \times 10^6) - (1{\cdot}553 \times 10^6)$$
$$= 18654000 \text{ sq. links}$$

Dividing by 100^2, $= 1865{\cdot}4$ sq. chns.

Dividing by 10, $= \underline{186{\cdot}54 \text{ acres}}$.

For greater precision tables of log sines and log cosines (viz. Tables VIII and IX at the end of the book) have been utilised in the working of this example. For general work the accuracy of the slide rule is sufficient, but in all cases these tables, which are used in the same way as the tables of natural sines and natural cosines, are convenient.

As shown earlier in the chapter the value of the sine or cosine of an angle varies between 0 and 1, and accordingly the values of the logs of these ratios vary between $-\infty$ (*i. e.*, the smallest quantity possible) and 0, since $\log 0 = -\infty$ (refer Chapter I) and $\log 1 = 0$. Except for small angles, therefore, the log sine will be of the nature of $\bar{1}\cdot \ldots\ldots$ or $\bar{2}\cdot \ldots\ldots$ whilst the value of the log cosine will be $\bar{1}\cdot \ldots\ldots$ or $\bar{2}\cdot \ldots\ldots$ unless the angle is large.

e. g., $\sin 27^\circ = {\cdot}4540$ and $\log \sin 27^\circ = \log {\cdot}4540 = \bar{1}{\cdot}6571$
$\sin 0^\circ 33' = {\cdot}0096$ and $\log \sin 0^\circ 33' = \log {\cdot}0096 = \bar{3}{\cdot}9823$
$\cos 87^\circ = {\cdot}0523$ and $\log \cos 87^\circ = \log {\cdot}0523 = \bar{2}{\cdot}7185$

Example 9.—From the following co-ordinates compute the true length, the bearing, and the angle with the horizontal of the line AB.

Station.	Feet.	Feet	Feet above Sea Level
A	Northing 4501·2	Westing 56·1	Reduced level 249·2
B	Southing 20·1	Easting 4788·1	Reduced level 329·2

By plotting the points A and B from the co-ordinates given, their actual positions are represented. Complete the triangle ABC by drawing AC vertically and BC horizontally (see Fig. 121).

Then— $\quad AC = 4501{\cdot}2 + 20{\cdot}1 = 4521{\cdot}3$

and $\quad BC = 4788{\cdot}1 + 56{\cdot}1 = 4844{\cdot}2$

To express the results with the same accuracy as that with which the figures have been measured we must use five figure log tables.

To find the angle CAB—

$$\tan CAB = \frac{4844{\cdot}2}{4521{\cdot}3}$$

i.e., $\log \tan CAB = \log 4844{\cdot}2 - \log 4521{\cdot}3$

$= 3{\cdot}68523 - 3{\cdot}65527$

$= {\cdot}01996$

$= \log \tan 46° 19'$.

$\therefore \quad CAB = 46° 19'$

The whole circle bearing of AB is thus $180° - 46° 19' = \underline{133° 41'}$.

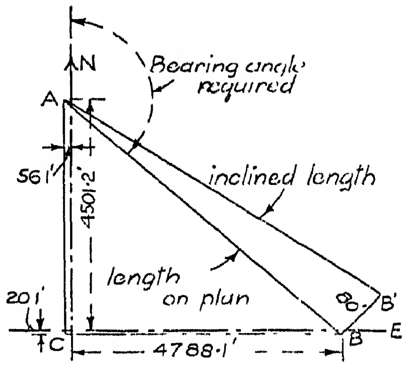


Fig. 121.

To find the length (on the plan) of AB—

$$\frac{CB}{AB} = \sin 46° 19'$$

$$AB = \frac{CB}{\sin 46° 19'}$$

$$= \frac{4844{\cdot}2}{\sin 46° 19'}$$

In the log form— $\quad \log AB = \log 4844{\cdot}2 - \log \sin 46° 19'$

$= 3{\cdot}68523 - \bar{1}{\cdot}85924 = 3{\cdot}82599$

$= \log 6698{\cdot}7$

$\therefore \quad AB = 6698{\cdot}7$

This length found is that of AB on the plan; the true length will be slightly greater than this, since it is the hypotenuse of the triangle of which the base is 6698·7; and the height is 80.

In the triangle ABB^1: $\quad \tan B^1AB = \dfrac{80}{6698{\cdot}7}$

and $\quad \log \tan B^1AB = \log 80 - \log 6698{\cdot}7 = 1{\cdot}90309 - 3{\cdot}82599$

$= \bar{2}{\cdot}07710$

$= \log \tan 41' 4''$

$\therefore$ the inclination of AB to the horizontal is $41' 4''$

and the true length of AB (or AB^1) $= \sqrt{(6698{\cdot}7)^2 + 80^2}$

$= \underline{6699{\cdot}3}$

Exercises 28.—On the Solution of Right-Angled Triangles, and the Calculation of Co-ordinates.

In the following Examples 1 to 7, ABC is a triangle right-angled at C. (In each case the figure should be drawn to scale.)

1. $c = 45''$, $A = 15°$, find a and b.
2. $a = 12''$, $B = 36°$, find b and c.
3. $c = 65''$, $A = 48°$, find a and b.
4. $b = 34''$, $B = 27°$, find a and c.
5. $c = 27{\cdot}37''$, $A = 54°$, find a and b.
6. $b = 72{\cdot}5''$, $A = 38\frac{1}{2}°$, find a and c.
7. $c = 23{\cdot}4''$, $B = 27\frac{1}{4}°$, find a and b.

8. A bomb dropped from an aeroplane strikes a building which is known to be one mile away from an observing station, at which the elevation of the aeroplane is seen to be 29°. Find the "range," *i. e.*, the distance of the aeroplane from the observer, and also its height.

9. A mountain railway at its steepest rise has a gradient of 1 in 7. What is the inclination to the horizontal of this gradient? [Note that the gradient is always the $\frac{\text{perpendicular}}{\text{hypotenuse}}$.]

10. From the top of a house, 37 ft. high, a bench mark (Government height above sea-level) is sighted, and the angle of depression is 48°. Find the horizontal distance from the house of the B.M., which is placed at a point 3 ft. above the ground.

11. The crank and connecting rod of a reciprocating engine are at right angles to one another. If the value of the ratio

$$\frac{\text{connecting rod length}}{\text{length of crank}}$$

is 4·7, find the angle which the crank makes with the line of stroke.

12. The rise of a roof is 11 ft. and the span is 84 ft.: find the angle of the roof.

13. The tangent of the angle of a screw is given by the pitch divided by the circumference of the screw. If the diameter is 5″ and the pitch angle is 7° 15′, find the pitch.

14. If the screw in Ex. 13 becomes (*a*) double- or (*b*) treble-threaded, what are now the angles of the thread?

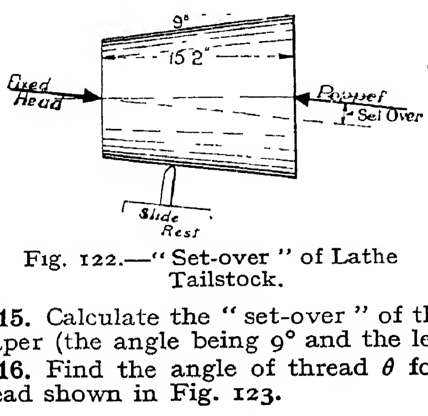


Fig. 122.—"Set-over" of Lathe Tailstock.

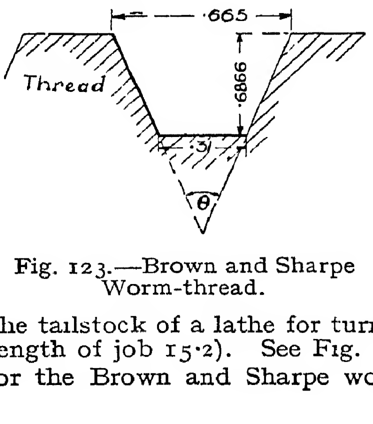


Fig. 123.—Brown and Sharpe Worm-thread.

15. Calculate the "set-over" of the tailstock of a lathe for turning a taper (the angle being 9° and the length of job 15·2). See Fig. 122.

16. Find the angle of thread θ for the Brown and Sharpe worm-thread shown in Fig. 123.

17. When using the Weldon Range Finder, one determines a length AB by comparison with a base AD. Find ratio of $\frac{AB}{AD}$ for the case illustrated (Fig. 124).

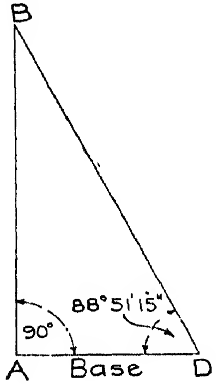


Fig. 124.

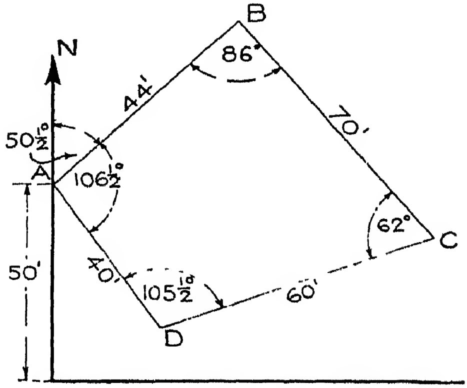


Fig. 125.

18. Determine the co-ordinates of the points A, B, C and D (Fig. 125) with references to the axes marked. Find the area of ABCD; and state also the "reduced bearings" of BC, CD and DA. The bearing of AB is 50·5° N.E.

19. In Fig. 126 calculate the co-ordinates of the points B and C, the reduced bearings of BC and CA, and the area of ABC, if the bearing of AB is 60° S.E.

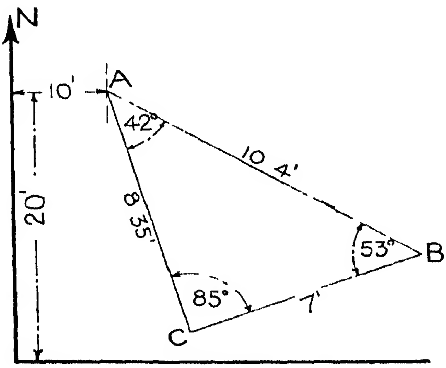


Fig. 126.

20. In finding the length of a line CB, a line CA was set out by means of the optical square at right angles to CB and the distance CA was chained and found to be 1·14 chains. The angle CAB was then observed by a box sextant and found to be 71° 54′. Calculate the length of CB.

21. The co-ordinates of two stations A and B are—

A. Latitude N 400 links; Departure W 700 links

B. Latitude S 160 links; Departure W 1500 links

Find the whole circle-bearing of AB.

22. You are 220 ft. horizontally away from the headgear of a mine. From a point on the same level as its base you find that the headgear subtends a vertical angle of 18° 30′. Find the height.

23. A ball fitting down to the taper sides was used to test the correctness of the cup-shaped check shown in Fig. 126*a*. The test was made by measurement of the distance AB. Calculate this length correct to $\frac{1}{10000}$″.

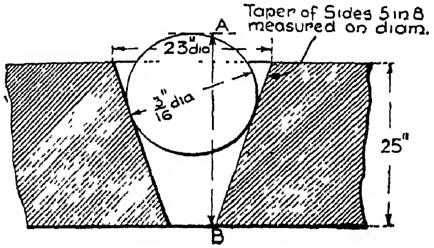


Fig. 126*a*.—Test for Gauge.

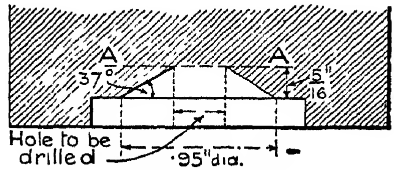


Fig. 126*b*.—Block for Jig.

24. Determine the diameter of the largest drill that could be used for the hole in the jig block shown in Fig. 126*b*, when you are told that the drilled hole, which is made first to clear away part of the metal, must cut the taper hole at the level AA.

Angles of any magnitude.—Up to this point our work has been confined to angles of 90° and under, whose trigonometrical ratios can easily be found from tables or by the use of the slide rule. **Angles greater than 90° must be reduced to those less than 90° by combination with 180° or 360°, i. e., they must be reduced to the equivalent acute angle made with some standard line, which in all this work will be taken as the N and S line.**

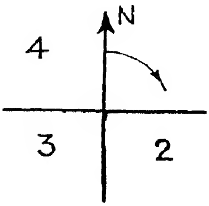


Fig. 127.

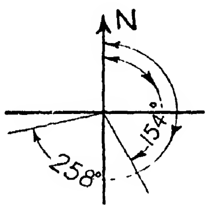


Fig. 128.

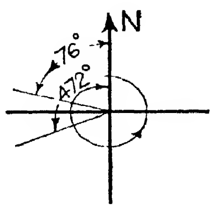


Fig. 129.

If the N. and S. line and the E. and W. line be drawn, they divide the space into four "quadrants," and the position of an angle can always be stated by reference to the quadrant in which it lies. Angles are measured in a right-hand direction from the N. and S. line, and the quadrants are numbered as shown in Fig. 127. A minus sign before an angle indicates a movement from the north in a left-hand direction.

e. g., referring to Figs. 128 and 129—

$154°$ is in the 2nd quadrant; and its equivalent acute angle $= 180°-154° = 26°$

$258°$ is in the 3rd quadrant; and its equivalent acute angle $= 258°-180° = 78°$

$-76°$ is in the 4th quadrant; and its equivalent acute angle $= 76°$

$-472°$ is in the 3rd quadrant; and its equivalent acute angle $= 68°$

To sum up, it will be seen that the equivalent acute angle (written e.a. angle) is always the angle made with the N. and S. line; *i. e.*, it is obtained by compounding with $180°$ or $360°$.

It is now necessary to find the algebraic signs to be prefixed to the trigonometric ratios of any angle. Thus although the sine of $-472°$ is *numerically* equal to the sine of $+68°$, since $68°$ is the e.a. angle for $-472°$ (see Fig. 129), it would not *necessarily* be correct to state that $\sin -472° = \sin 68°$, because we have not yet examined for the algebraic sign. As a matter of fact, $\sin -472° = -\sin 68°$.

Suppose that a line of unit length rotates in a right-hand direction, starting from the north, thus sweeping out the various angles.

Its "sense" will always be considered positive, whilst the usual convention will fix the signs for horizontal and vertical distances.

[*Note.*—In all that follows, be sure to measure every angle from the north point: thus in Fig. 130, the angle $(180 - A)$ is the angle *aod*, and the angle $(360 - A)$ is the angle *aoh* measured in a right-hand direction.]

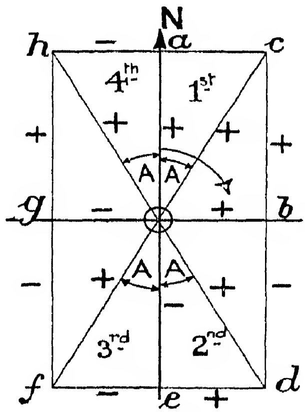


Fig. 130.

Let $\angle aoc$ (Fig. 130) represent the magnitude of the e.a. angle in all the four quadrants: *i. e.*, $\angle aoc = \angle eod = \angle eof = \angle aoh = A$, say.

In the 1st quadrant—

$$\sin A = \frac{+ac}{+oc} = \frac{+ac}{1} = +ac$$

$$\cos A = \frac{+oa}{+oc} = \frac{+oa}{1} = +oa$$

$$\tan A = \frac{+ac}{+oa} = +\frac{ac}{oa}$$

In the 2nd quadrant $\sin(180 - A) = \frac{+ed}{1} = +ed$: but $ed = ac$

so that sin $(180-A) = \sin A$. Hence the reason for compounding with $180°$ to find the e.a. angle is seen.

Again— $$\cos(180-A) = \frac{-oe}{1} = -oe = -oa$$

$-oe$ indicating that oe is a negative length, because measured downwards—

$$\tan(180-A) = \frac{+ed}{-oe} = -\frac{ed}{oe} = -\frac{ac}{oa}$$

In the 3rd quadrant—

$$\sin(180+A) = \frac{-ef}{1} = -ef = -ac$$

$$\cos(180+A) = \frac{-oe}{1} = -oe = -oa$$

$$\tan(180+A) = \frac{-ef}{-oe} = +\frac{ef}{oe} = +\frac{ac}{oa}$$

In the 4th quadrant—

$$\sin(360-A) = \frac{-ah}{1} = -ah = -ac$$

$$\cos(360-A) = \frac{+oa}{1} = +oa$$

$$\tan(360-A) = \frac{-ah}{+oa} = -\frac{ah}{oa} = -\frac{ac}{oa}$$

i. e., summarising for the equivalent acute angles in all four quadrants, the algebraic signs vary as follows—

Quadrant.	1st	2nd	3rd	4th.
sine and cosec .	+	+	−	−
cos and sec . .	+	−	−	+
tan and cot . .	+	−	+	−

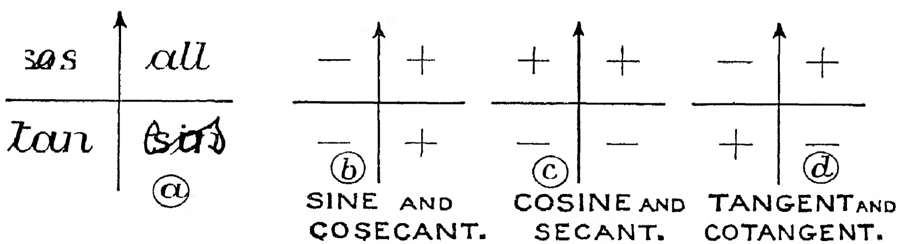


Fig. 131.—Variation in Sign of Ratios.

This variation in sign may be better or more plainly denoted by the diagrams (*a*), (*b*), (*c*) and (*d*), Fig. 131. Fig. (*a*) 131 may need an additional word of explanation. In each quadrant is written

the word to indicate which ratio or ratios is or are positive in that quadrant. Thus in the 3rd quadrant, the tangent alone is positive, and in the 4th quadrant the cosine alone. Fig. (*b*), (*c*) and (*d*) 131 are merely a representation of the table just given.

Hence, to find the trigonometric ratio of an angle of any magnitude: find first its e.a. angle and the quadrant in which the angle occurs, and then apply the sign of the quadrant for the ratio required. (*Numerically*, the ratio of any angle is that of its e.a. angle.) *In all cases it will be found that a diagram simplifies matters.*

Example 10.—Find the value of sin 172°.

$$\sin 172° = \sin(180-172) = \sin 8°, \text{ for } 8° \text{ is the e.a. angle}$$
$$= \underline{+\cdot 1392}$$

since 172° is in the 2nd quadrant, and the sine there is +.

Example 11.—Find cos 994°.

$$994° = [(2 \times 360°) + 274°]$$

$2 \times 360°$ brings us back to the starting line, and so we deal only with the 274°. Now 274° is in the 4th quadrant, and thus its cos is +; also the e.a. angle = 360 − 274 = 86°.

$$\therefore \qquad \cos 994° = +\cos 86° = \underline{+\cdot 0698.}$$

Example 12.—Find tan −327°.

The angle −327° is in the 1st quadrant, and hence its tan is +; also the e.a. angle = 33°.

$$\therefore \qquad \tan -327° = +\tan 33° = \underline{+\cdot 6494.}$$

Example 13.—Find the sin, cos and tan of 115°. What connection is there between them?

The angle is in the 2nd quadrant, hence— $\left.\begin{array}{l}\text{sine is } +\\ \text{cos is } -\\ \text{tan is } -\end{array}\right\}$

also the e.a. angle = 180°−115° = 65°

$$\therefore \qquad \sin 115° = +\sin 65° = \underline{+\cdot 9063}$$
$$\cos 115° = -\cos 65° = \underline{-\cdot 4226}$$
$$\tan 115° = -\tan 65° = \underline{-2\cdot 1445}$$

Now—
$$\frac{\sin 115°}{\cos 115°} = \frac{+\cdot 9063}{-\cdot 4226} = -2\cdot 1445$$
$$= \tan 115°.$$

This most important relation always holds, viz. that—

$$\tan A = \frac{\sin A}{\cos A}$$

The "reduced bearing," in surveying, may be regarded as identical with the "equivalent acute angle" here used.

In the general solution of triangles only angles up to 180° occur, hence we are concerned mainly with the 1st and 2nd quadrants.

Exercises 29.—On the Trigonometric Ratios of Angles of any Magnitude.

Find from the tables, the values of the sin, cos and tan of the following angles (Exs. 1 to 5).

1. 116°; 322°; 218°. **2.** − 82°; − 398°; 1562°.

3. 199·2°; 341° 5′; 984° 23′. **4.** 4; 11·62; ·85; 1·16 radians.

5. − 1194°; − 2·45 radians; 787° 11′.

6. Find values of cot 126°; cosec π; sec (−52°). [*Note.*—The angle π radians is that subtended at the centre by the half-circumference and is thus 180°.]

7. Find a value of A between 0 and 180° if—

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{and} \quad \begin{aligned} b &= 9{\cdot}8'' \\ c &= 6{\cdot}4'' \\ a &= 14{\cdot}45'' \end{aligned}$$

8. The equation $\cot \theta = \dfrac{2V^2 + u^2 - 2gH}{2Vu}$ relates to the design of water turbines. If $V = 53{\cdot}4$, $u = 10$, $H = 100$, $g = 32{\cdot}2$, find θ (between 0 and 180°).

9. As for the preceding question, but taking $V = \dfrac{16\sqrt{80}}{3}$, $u = \sqrt{80}$, $g = 32{\cdot}2$ and $H = 80$.

10. If $a°$ = angle of the crank of a steam engine from the dead centre, m = ratio of connecting rod length to length of crank and $f = -\,{\cdot}833$: find values of a to satisfy the equation—

$$\cos a = m - \sqrt{m^2 + 1 - 2mf} \quad \text{when } m = 4.$$

Solution of Triangles.—The "solution" of a triangle consists in the determination of the magnitudes of the six parts, viz. the three sides and the three angles. In many cases sufficiently accurate results can be obtained by careful drawing to scale, but for great precision the values of the parts of the triangle must be calculated. In such calculation extremely exact tables, giving the relations between the sides and angles, are employed, and the results obtained are superior to those given by even skilled draughtsmanship. Again, it sometimes happens that the triangle is difficult to construct: thus if in Fig. 136 the base AC was very small compared with the sides AB and BC, the intersection of AB and CB would not easily

be determined, and, therefore, the lengths of the sides as measured would only be approximate. The angle at B would under these circumstances be termed "badly conditioned."

There are a number of rules developed for the general solution of triangles, but of these the following will be found to be of the greatest service, while even this list may be reduced to the first two rules.

Adopting the usual notation for the triangle, viz. A, B and C for the angles, and a, b and c for the sides opposite these angles respectively, the rules for the solution of all triangles are—

(1) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, **usually referred to as the sine rule.**

(2) $a^2 = b^2 + c^2 - 2\,bc \cos A$, **usually referred to as the cosine rule.**

(3) (*a*) $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$ $\qquad \{2s = a + b + c\}$

(*b*) $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$

(*c*) $\tan \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

(4) $\tan \dfrac{B-C}{2} = \left(\dfrac{b-c}{b+c}\right) \cot \dfrac{A}{2}$

These may be employed under the following conditions—

I. *Given two sides and included angle:* use either rule (2) to find the third side and then either rule (1) or rule (3) to find another angle; or use rule (4) to find the remaining angles together with rule (2) for the third side.

e. g., suppose b, c and A are given.

Then from rule (2) the value of a can be found,

$$\text{also} \quad \frac{\sin B}{b} = \frac{\sin A}{a} \quad [\text{from rule (1)}]$$

$\therefore$ B is found

and $C = 180 - (A+B)$, since $A+B+C = 180°$;

or alternatively—

$$\tan \frac{B-C}{2} = \left(\frac{b-c}{b+c}\right) \cot \frac{A}{2}$$

$\therefore$ the angle $\dfrac{B-C}{2}$ is found, and hence also $(B-C)$.

But $(B+C)$, *i. e.*, $180 - A$ is known,

and therefore B and C are found by solving the simultaneous equations. Also a can be found from rule (2).

II. *Given two angles and a side,* say a, A and B—

Then— $$C = 180-(A+B)$$

From rule (1)—

$$\frac{c}{\sin C} = \frac{a}{\sin A}, \quad \text{and} \quad \frac{b}{\sin B} = \frac{a}{\sin A}$$

and therefore all the sides are found.

III. *Given two sides and an angle not included by them,* say b, c and B—

From rule (1)— $$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$\therefore$ C is found, and also A. $\{$For $A = 180-(B+C)\}$

and since $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$, a is found.

IV. *Given the three sides:* it is more convenient in this case to use rule (3) to find one of the angles; because logarithms can be applied.

From Rule (3) c—

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Then use Rule (1) to find B.

Otherwise— $$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad i.\ e., \text{ A is found}$$

and thence by the sine rule B may be found.

Thus if rules (1) and (2) are remembered, any triangle may be solved.

Proof of the "Sine" Rule.

Consider Figs. 132 and 133.

In both figures— $$\frac{p}{c} = \sin B$$

$$\therefore \quad p = c \sin B$$

also in Fig. 132— $p = b \sin C$

and in Fig. 133— $p = b \sin (180-C) = b \sin C$

Hence— $$c \sin B = b \sin C$$

$$\therefore \quad \frac{c}{\sin C} = \frac{b}{\sin B}$$

Similarly it could be proved that—

$$\frac{b}{\sin B} \text{ or } \frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\therefore \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

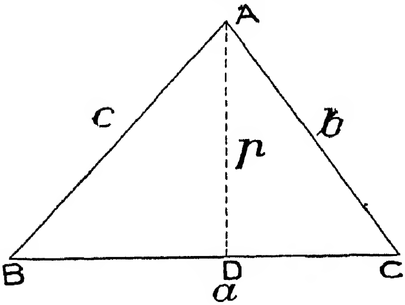


Fig 132.

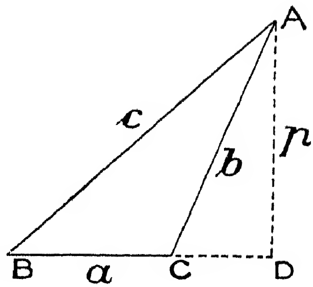


Fig 133.

Or, **the sides of a triangle are proportional to the sines of the opposite angles.**

Proof of the Cosine Rule.

In Figs. 135 and 136 let BD be perpendicular to AC. In Fig. 135 in the triangle ADB—

$$\begin{aligned} c^2 &= p^2+(b-n)^2 \\ &= p^2+b^2+n^2-2bn \quad \ldots\ldots\ldots \quad (1) \end{aligned}$$

and in the triangle BDC—

$$p^2+n^2 = a^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

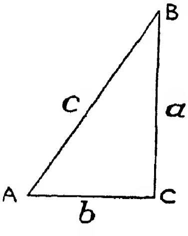


Fig. 134.

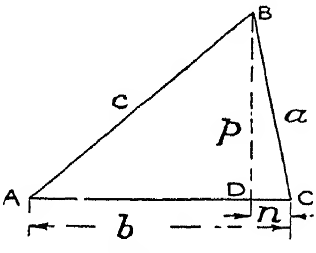


Fig. 135.

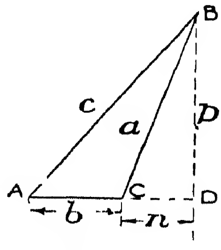


Fig 136.

Hence, by substitution from (2) into (1)—

$$c^2 = a^2+b^2-2bn \quad \ldots\ldots\ldots\ldots \quad (3)$$

Again in Fig. 136, in the triangle ADB—

$$\begin{aligned} c^2 &= p^2+(b+n)^2 \\ &= p^2+b^2+n^2+2bn \quad \ldots\ldots\ldots \quad (4) \end{aligned}$$

and in the triangle BDC—

$$p^2+n^2 = a^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

Hence, by substitution from (5) into (4)—

$$c^2 = a^2+b^2+2bn \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

Now in Fig. 135 $\frac{n}{a} = \cos C$ or $n = a \cos C$

so that, writing $a \cos C$ in place of n in (3)—

$$c^2 = a^2+b^2-2ab \cos C.$$

Also in Fig. 136—

$$\frac{n}{a} = \cos \angle BCD = \cos(180-C) = -\cos C \quad \text{or} \quad n = -a \cos C.$$

Substituting this value for n in (6)—

$$c^2 = a^2+b^2-2ab \cos C.$$

We have thus proved that the rule holds for the case in which C is an acute angle, and also for the case in which C is obtuse. When C is a right angle, as in Fig. 134, its cosine is zero and accordingly it is correct to write—

$$c^2 = a^2+b^2-0 = a^2+b^2-2ab \cos C.$$

Hence the rule is perfectly general.

The two other forms of the cosine rule can be written down by writing the letters one on in the sequence a, b, c, a.

i. e., $a^2 = b^2+c^2-2bc \cos A$
and $b^2 = c^2+a^2-2ca \cos B$.

By transposition—

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

the forms in which the rule must be used if the three sides are given and the angles are required.

In every case of a solution of a triangle the figure should be drawn to scale, for this serves as the best check on the results obtained by calculation.

The following examples should be carefully studied—

Examples on the use of the Sine Rule.

Example 14.—Solve the $\triangle$ ABC completely when $c = 1916$ ft., $b = 1748$ ft., and $C = 59°$. [This triangle is drawn to scale in Fig. 137.]

To find B— $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

and hence— $\sin B = \dfrac{b \sin C}{c} = \dfrac{1748 \times \sin 59^\circ}{1916}$

Taking logs throughout—

$$\begin{aligned}\log \sin B &= \log 1748 + \log \sin 59^\circ - \log 1916 \\ &= 3{\cdot}2425 + \bar{1}{\cdot}9331 - 3{\cdot}2823 \\ &= \bar{1}{\cdot}8933 = \log \sin 51^\circ 28'\end{aligned}$$

$$\therefore \quad B = \underline{51^\circ 28'}$$

Then— $$\begin{aligned}A &= 180^\circ - (59^\circ + 51^\circ 28') \\ &= \underline{69^\circ 32'}\end{aligned}$$

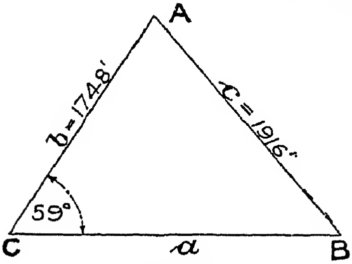


Fig. 137.

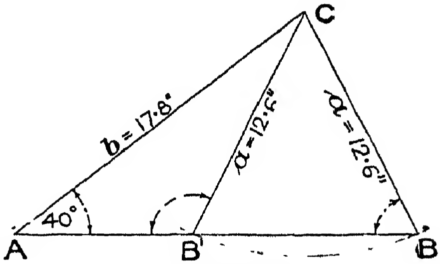


Fig. 138.

To find a— $$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$a = \frac{b \sin A}{\sin B}$$

In the log form—

$$\begin{aligned}\log a &= \log 1748 + \log \sin 69^\circ 32' - \log \sin 51^\circ 28' \\ &= 3{\cdot}2425 + \bar{1}{\cdot}9717 - \bar{1}{\cdot}8933 \\ &= 3{\cdot}3209\end{aligned}$$

$$\therefore \quad a = \underline{2093''}$$

Example 15.—Solve the $\triangle$ ABC completely when $a = 12{\cdot}6''$, $b = 17{\cdot}8''$, $A = 40^\circ$. (This is similar to the last Example up to a certain point.)

To draw this to scale (see Fig. 138).—Make the angle 40° with a horizontal line and along AC mark off a length to represent 17·8″; this is the side b. With centre C and radius = 12·6″ (to scale) strike an arc to cut the horizontal; and two points of section being found, call them B and B′. Both the $\triangle$ ABC and the $\triangle$ AB′C satisfy the given conditions, because $AC = b = 17{\cdot}8$, $CB = CB' = a = 12{\cdot}6$ and $A = 40^\circ$, so that in this case there are *two* solutions. **This case is known as the "ambiguous" case** in the solution of triangles.

Since— $$CB = CB', \quad \angle CBB' = \angle CB'B$$

$$\therefore \quad \angle CB'A = 180 - \angle CBA$$

or $$B' = 180 - B$$

and the two values of the angle B, which are indicated on the figure, are supplementary, *i. e.*, together they add to 180°. AB and AB′ are the two different lengths for c for the different cases, while ACB and ACB′ give the two values for the angle at C.

To solve by calculation.—Two sides and one opposite angle are given, and therefore the sine rule is to be used. Taking the same diagram—

To find B—

$$\sin B = \frac{b \sin A}{a} = \frac{17{\cdot}8 \sin 40^\circ}{12{\cdot}6}$$

In the log form—

$$\begin{aligned} \log \sin B &= \log 17{\cdot}8 + \log \sin 40^\circ - \log \sin 12{\cdot}6 \\ &= 1{\cdot}2504 + \bar{1}{\cdot}8081 - 1{\cdot}1004 \\ &= \bar{1}{\cdot}9581 = \log \sin 65^\circ 13' \\ \therefore \quad B &= 65^\circ 13' \end{aligned}$$

The value of B′, which is alternative to B must be $180 - B = 114^\circ 47'$. The mode of calculation would be unchanged, for—

$$\sin 114^\circ 47' = \sin 65^\circ 13'.$$

To find C—

In the first case $\quad C = 180^\circ - (40^\circ + 65^\circ 13')$
$\qquad = 74^\circ 47'$

In the second case $\quad C = 180^\circ - (40^\circ + 114^\circ 47')$
$\qquad = 25^\circ 13'$

To find c.—This is the base, which is AB′ *or* AB. Either the sine or the cosine rule can be here used, but the sine rule is more adapted for logarithmic computation.

$$c = \frac{a \sin C}{\sin A}$$

In the first case—

$$\begin{aligned} \log c &= \log 12{\cdot}6 + \log \sin 74^\circ 47' - \log \sin 40^\circ \\ &= 1{\cdot}1004 + \bar{1}{\cdot}9845 - \bar{1}{\cdot}8081 \\ &= 1{\cdot}2768 \\ \therefore \quad c &= 18{\cdot}91'' \end{aligned}$$

In the second case—

$$\begin{aligned} \log c &= \log 12{\cdot}6 + \log \sin 25^\circ 13' - \log \sin 40^\circ \\ &= 1{\cdot}1004 + \bar{1}{\cdot}6295 - \bar{1}{\cdot}8081 \\ &= {\cdot}9218 \\ \therefore \quad c &= 8{\cdot}352 \end{aligned}$$

Grouping the results—

$B = 65^\circ 13'$ or $114^\circ 47'$, $C = 74^\circ 47'$ or $25^\circ 13'$, $c = 18{\cdot}91''$ or $8{\cdot}352''$ all respectively.

The sine scale on the slide rule could be used with advantage in this example. To multiply or divide by sines of angles, multiply

or divide by the angles, as marked on the scale, in the ordinary way. *E. g.*—

$$c = \frac{12{\cdot}6 \times \sin 74^\circ 47'}{\sin 40^\circ}$$

Set the cursor over 12·6 on the A scale, move the sine scale until 40° is level with the cursor; then place the cursor over 74° 47′ on the S scale. The value of c is read off on the A scale, and = 18·9.

A little confusion may arise regarding the graduations on the S and T scales. The markings usually shown are not for decimals of a degree, but for minutes. As regards the S scale: up to 10°, a line is shown at every 5′, *i. e.*, there are 12 divisions for each degree. From 10° to 20° every 10′ is shown, from 20° to 40° every 30′, from 40° to 70° each degree, and thence 70°, 72°, 74°, 76°, 78°,

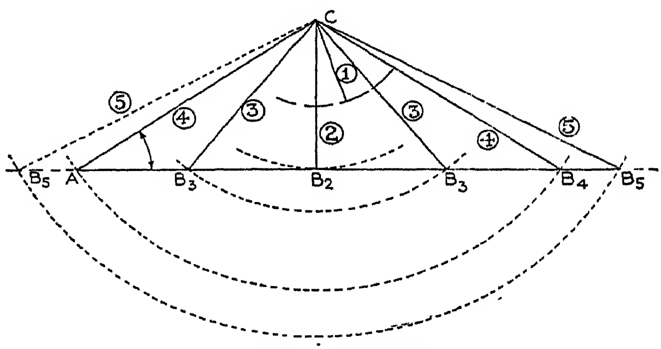


Fig. 139.—Solutions of Triangles.

80°, 85° and 90°. On the T scale, up to 20°, markings are at each 5′ and then at every 10′.

Whenever two sides and an opposite angle are given, we must consider the possibility of the two solutions.

The drawing to scale is an excellent test, for the arc B′B in Fig. 138 must either cut or touch the base if the triangle is to be possible.

The various cases that arise are illustrated in Fig. 139: in which the sides a and b, and the angle A are given. Drawing a horizontal line of unlimited length to serve as a base, the angle A can be set out and the point C fixed, since the length of AC is given. Then an arc of radius equal to b is described from the centre C. If b is very small, the arc does not cut the base and case (1) arises; there being no triangle to satisfy the conditions. If the radius of the circle, *i. e.*, the length of the side b, is increased, we arrive at case (2), in which the arc just touches the base and so gives one

triangle only, viz. the right-angled triangle ACB_2. By further increasing the length of b cases (3), (4) and (5) are found, in which there are two, one, and one, solutions respectively.

It will thus be seen that there may be two solutions if two sides of a triangle and an angle opposite the shorter of these is given. In all cases, however, the triangle should be drawn to scale before any trigonometrical rules are applied.

Example 16.—A mill chimney stands on the even slope of a hill, which has a gradient of 4° (Fig. 140). Two points are chosen on the same side of the hill and in the same vertical plane as that including the chimney. These points are 75 ft. apart measured up the slope, and, viewed from the points, the chimney subtends angles of 48° and 59° from the horizontal. Find the height of the chimney above the ground on which it stands.

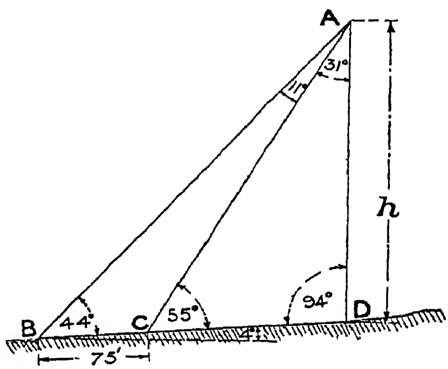


Fig. 140.

It should be noted that the angles of elevation are measured from the horizontal, since the scale of the theodolite vertical circle reads zero when the telescope is horizontal.

Hence— $\angle ABC = 48° - 4° = 44°$

and $\angle ACD = 55°$

Thus— $\angle ACB = 180° - 55° = 125°$, $\angle BAC = 11°$

$\angle ADC = 94°$, $\angle CAD = 31°$

Here we have two triangles, viz. ACB and ACD, one containing the known length and one containing the unknown length; and these must be connected up through a side common to both, viz. AC.

Let the required height AD $= h$

Then, in the $\triangle$ ACB—

$$\frac{AC}{\sin 44°} = \frac{75}{\sin 11°}$$

$$\therefore \quad AC = \frac{75 \sin 44°}{\sin 11°}$$

In the $\triangle$ ACD—

$$\frac{h}{\sin 55°} = \frac{AC}{\sin 94°} = \frac{AC}{\sin 86°} \qquad \text{since } \sin 86° = \sin 94°$$

$$\therefore \quad h = \frac{AC \times \sin 55°}{\sin 86°}$$

Substituting for AC its value—

$$h = \frac{75 \times \sin 44^\circ \times \sin 55^\circ}{\sin 11^\circ \times \sin 86^\circ}$$

$$= \underline{225 \text{ ft.}} \text{ (from the slide rule).}$$

Example 17.—The elevation of the top P of a mountain (see Fig. 141) at a point A on the ground is 32°. The surveying instrument is directed to another station B, also on the ground, and 4600 ft. distant from A, the angle PAB being found to be 48°; also ∠PBA is 77°. Find the height of the mountain.

The sloping triangle PAB is shown laid flat on the ground in Fig. 142. From this ground plan—

$$\frac{PA}{\sin 77^\circ} = \frac{4600}{\sin 55^\circ}$$

$$\therefore \qquad PA = \frac{4600 \times \sin 77^\circ}{\sin 55^\circ}$$

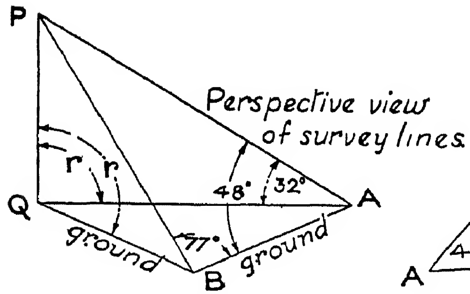


Fig. 141.

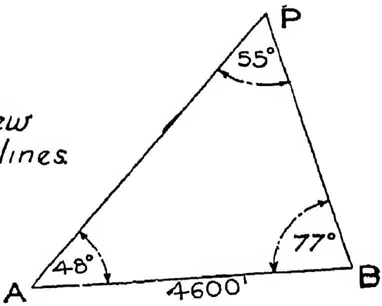


Fig. 142.

In the right-angled △ PAQ—

$$\frac{PQ}{AP} = \sin 32^\circ$$

and $\qquad PQ = AP \times \sin 32^\circ$

Substituting for AP—

$$= \frac{4600 \times \sin 77^\circ \times \sin 32^\circ}{\sin 55^\circ}$$

$\therefore \qquad$ Height of mountain $= \underline{2900 \text{ ft.}}$

Example 18.—It is required to lay out a circular arc to connect the two straight roads AB and CD (Fig. 143): the radius *r* of the arc is known, but the meeting point E of AB and CD is inaccessible.

Select two convenient stations F and G, and by directing a theodolite first along F*e* and then along FG the angle EFG is measured. Similarly measure ∠EGF.

Let the sum of $\angle$ EFG and $\angle$ EFG $= 2a$

Then— $\angle \text{AED} = 180 - 2a$

and $\angle \text{AEO} = \frac{1}{2}\angle \text{AED} = 90 - a$

$\therefore$ $\angle \text{EOS} = a$

Now— $\frac{\text{ES}}{\text{OS}} = \tan a$, since $\angle$ ESO is a right angle

$\therefore$ $\text{ES} = \text{OS} \tan a = r \tan a$ . . . . (1)

and also $\text{ET} = r \tan a$, since ET = ES . . . . (1)

To find FS and GT, EF and EG must first be found.

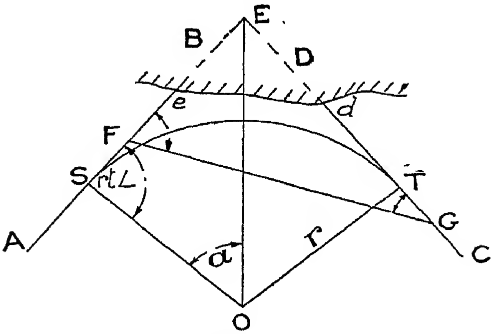


Fig. 143.

In the $\triangle$ EFG—

$$\frac{\text{EF}}{\sin \text{EGF}} = \frac{\text{FG}}{\sin \text{FEG}} \quad \text{or} \quad \text{EF} = \text{FG}\frac{\sin \text{EGF}}{\sin \text{FEG}}$$

$\therefore$ EF is known . . . . (2)

Also, in the same way—

$$\text{EG} = \text{FG}\frac{\sin \text{EFG}}{\sin \text{FEG}}$$

$\therefore$ EG is known . . . . (3)

Finally, FS is found from (1) and (2) since FS = ES − EF, and GT is also found from (1) and (3) since GT = EG − ET. Thus the points F and G having been taken at random, S and T can now be plotted therefrom, which show the starting-points of the curved road

Examples on the use of Cosine Rule.

Example 19.—In the triangle ABC (Fig. 144) find $\angle$ C when $a = 4{\cdot}45'$, $b = 7{\cdot}85'$, and $c = 11{\cdot}94'$.

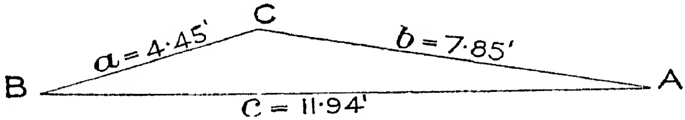


Fig. 144.

The longest side is always opposite the largest angle; and therefore C is the largest angle.

Now—

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(4\cdot45)^2 + (7\cdot85)^2 - (11\cdot94)^2}{2 \times 4\cdot45 \times 7\cdot85}$$
$$= \frac{19\cdot8 + 61\cdot5 - 142\cdot5}{69\cdot8}$$
$$= \frac{-61\cdot2}{69\cdot8} = -\cdot877$$
$$= -\cos 28^\circ 43'$$
$$= \cos (180 - 28^\circ 43') = \cos 151^\circ 17'$$
$$\therefore \quad C = \underline{151^\circ 17'}.$$

It will be seen that a negative value for the cosine implies that the angle is obtuse.

To avoid remembering too many rules the reader is advised to work entirely with the sine or cosine rules: this example, however, is worked out, in addition to the above, by another rule, to demonstrate its usefulness and ease of application.

$$a = 4\cdot45,\ b = 7\cdot85,\ c = 11\cdot94$$
$$2s = a + b + c = 24\cdot24$$
$$\therefore \quad s = 12\cdot12$$
$$\tan\frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}} = \sqrt{\frac{4\cdot27 \times 7\cdot67}{12\cdot12 \times \cdot18}}$$
$$\therefore \quad \log\tan\frac{C}{2} = \frac{1}{2}\{(\log 4\cdot27 + \log 7\cdot67) - (\log 12\cdot12 + \log \cdot18)\}$$
$$= \frac{1}{2}\{(\cdot6304 + \cdot8848) - (1\cdot0835 + \bar{1}\cdot2553)\}$$
$$= \cdot5882$$
$$= \log\tan 75^\circ 32'$$
$$\therefore \quad \frac{C}{2} = 75^\circ 32' \quad \text{and} \quad C = \underline{151^\circ 4'}$$

i. e., an error of 13′ was made when using the slide rule.

[*Note* that if this rule is used and the angle is required correct to the nearest minute, we must work throughout correct to a half-minute since the rule gives as the direct result the value of a half-angle.]

Example 20.—If in the triangle ABC: $a = 5\cdot93''$, $c = 2\cdot94''$, $B = 65^\circ$, find the side b (Fig. 145).

Using the cosine rule—

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$= (5\cdot93)^2 + (2\cdot94)^2 - (2 \times 5\cdot93 \times 2\cdot94 \times \cos 65^\circ)$$
$$= 35\cdot1 + 8\cdot62 - (2 \times 5\cdot93 \times 2\cdot94 \times \cdot4226)$$
$$= 43\cdot72 - 14\cdot72$$
$$= 29$$
$$\therefore \quad b = \underline{5\cdot39''}$$

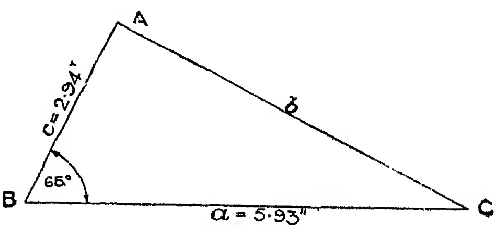


Fig. 145.

Example 21.—Two forces, of 47·2 lbs. and 98·4 lbs. respectively, making an angle of 63° with one another, act on a small body at A. Find the magnitude of their resultant, or single equivalent force.

If AB and AD in Fig. 146 represent the given forces, AC represents their resultant, as shown in Mechanics.

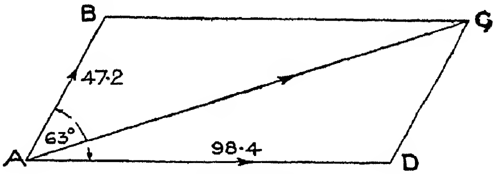


Fig. 146.

Then—

$$CD = 47{\cdot}2,\ AD = 98{\cdot}4,\ \angle ADC = 180° - 63° = 117°$$

$$\begin{aligned}\therefore\quad (AC)^2 &= (AD)^2 + (DC)^2 - (2 \times AD \times DC \times \cos 117°)\\ &= (98{\cdot}4)^2 + (47{\cdot}2)^2 - (2 \times 98{\cdot}4 \times 47{\cdot}2 \times -{\cdot}4540)\\ &= 9670 + 2230 + 4220\\ &= 16120\\ \therefore\quad AC &= \underline{127 \text{ lbs.}} \text{ and this is the resultant.}\end{aligned}$$

Area of a Triangle.—The following rule gives the area when two sides and the included angle are given; it is simply an extension of the $\frac{1}{2}$ base × height rule, for—

$$\begin{aligned} AD &= AB \sin B \quad \text{or} \quad AC \sin C \qquad\qquad \text{(Fig. 147)}\\ &= c \sin B \quad \text{or} \quad b \sin C\\ \therefore\quad \text{Area} &= \tfrac{1}{2} \times \text{base} \times \text{height}\\ &= \tfrac{1}{2} \times a \times c \sin B \quad \text{or} \quad \tfrac{1}{2} \times a \times b \sin C\\ &= \tfrac{1}{2}\, ac \sin B \quad \text{or} \quad \tfrac{1}{2}\, ab \sin C\end{aligned}$$

or, generally, **area of triangle = $\frac{1}{2}$ product of two sides × sine of included angle.**

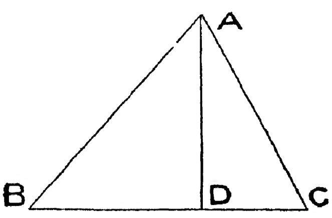


Fig. 147.

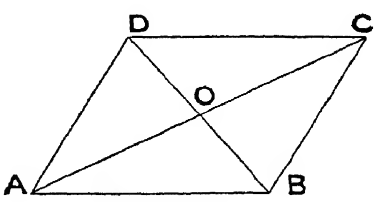


Fig 148.

This gives a rule for the area of a parallelogram—
Area of ABCD—

$$\begin{aligned}&= 2\{\text{area AOB} + \text{area AOD}\} \qquad\qquad \text{(Fig. 148)}\\ &= 2\{\tfrac{1}{2}\, AO.OB \sin \angle AOB + \tfrac{1}{2}\, AO.OD \sin \angle AOD\}\end{aligned}$$

$= \sin \angle AOB \{AO.OB + AO.OD\}$ since $\sin \angle AOB = \sin (180° - AOB) = \sin \angle AOD$

$= \sin \angle AOB \times AO \{OB + OD\}$

$= AO.BD \sin \angle AOB$

$= \frac{1}{2} AC.BD \sin \angle AOB$

$= \frac{1}{2}$ product of diagonals $\times$ sine of angle included between them.

Example 22.—Find the area of $\triangle ABC$ in which $a = 5{\cdot}93''$, $c = 2{\cdot}94''$ and $B = 65°$.

$$\text{Area} = \tfrac{1}{2} ac \sin B = \tfrac{1}{2} \times 5{\cdot}93 \times 2{\cdot}94 \times \sin 65°$$
$$= \tfrac{1}{2} \times 5{\cdot}93 \times 2{\cdot}94 \times {\cdot}9063$$
$$= \underline{7{\cdot}91 \text{ sq. ins.}}$$

This result should agree with that found by the "s" rule given in Chapter III; it being possible to apply this rule since the three sides are known and are $5{\cdot}93$, $5{\cdot}39$ and $2{\cdot}94$ respectively (compare *Example* 20).

Thus—
$$s = \frac{5{\cdot}93 + 5{\cdot}39 + 2{\cdot}94}{2} = 7{\cdot}13$$

and
$$\text{area} = \sqrt{7{\cdot}13 \times 1{\cdot}20 \times 1{\cdot}74 \times 4{\cdot}19} = 7{\cdot}91 \text{ sq. ins.}$$

Proof of the "s" Rule for the Area of a Triangle.

It has been demonstrated in the previous paragraph that—

$$\text{Area of triangle} = \tfrac{1}{2}\, ab \sin C.$$

Now for any angle it is true that $(\text{sine})^2 + (\text{cosine})^2 = 1$: hence

$$\sin^2 C + \cos^2 C = 1, \quad \text{or} \quad \sin C = \sqrt{1 - \cos^2 C}$$

Also—
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

then
$$\cos^2 C = \frac{(a^2 + b^2 - c^2)^2}{4a^2b^2}$$

and
$$1 - \cos^2 C = \frac{(2ab)^2 - (a^2 + b^2 - c^2)^2}{4a^2b^2}$$

[Factorising difference of two squares]—

$$= \frac{(2ab - a^2 - b^2 + c^2)(2ab + a^2 + b^2 - c^2)}{4a^2b^2}$$
$$= \frac{\{c^2 - (a - b)^2\}\{(a + b)^2 - c^2\}}{4a^2b^2}$$

[Factorising difference of two squares]—

$$= \frac{(c-a+b)(c+a-b)(a+b-c)(a+b+c)}{4a^2b^2}$$

$$= \frac{2(s-a)\times 2(s-b)\times 2(s-c)\times 2s}{4a^2b^2}$$

i. e., $$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore \text{ area of triangle ABC} = \tfrac{1}{2}ab\times\frac{2}{ab}\times\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

In all of these worked examples the results have been given to as great a degree of accuracy as four-figure log tables or the slide rule allow.

When extremely careful observations have been made it is advisable to employ five- or even seven-figure log tables in any necessary calculations; but it should be remembered that the results must not be given to a greater degree of accuracy than the observations or measurements warrant. Thus it would be useless to express a length "correct" to eight figures when the least possible error in measurement was $\frac{1}{2}\%$.

The rules used in such cases are those stated in this chapter, except that the cosine rule is put into a form more *adapted for logarithmic computation* by means of the following artifice—

$$a^2 = b^2+c^2-2bc\cos A$$

In place of this rule we may write—

$$a = (b+c)\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

provided that θ is found from—

$$\sin\theta = \frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Both (1) and (2) can be solved by the aid of logs; and the angle θ thus introduced is known as a *subsidiary angle.*

Let us illustrate this by taking the figures of *Example* 20.

Given $a = 5{\cdot}93$, $c = 2{\cdot}94$, $B = 65°$.

To find b—

From the above—

$$b = (c+a)\cos\theta$$

and $$\sin\theta = \frac{2\sqrt{ca}}{c+a}\cos\frac{B}{2}$$

i. e., $$\sin\theta = \frac{2\sqrt{2{\cdot}94\times 5{\cdot}93}}{8{\cdot}87}\cos 32\tfrac{1}{2}°$$

In the log form—

$$\log \sin \theta = \log 2 + \tfrac{1}{2}\{\log 2{\cdot}94 + \log 5{\cdot}93\} + \log \cos 32\tfrac{1}{2}^\circ - \log 8{\cdot}87$$
$$= \bar{1}{\cdot}8998 = \log \sin 52^\circ 33'$$
$$\therefore \quad \theta = 52^\circ 33'$$

Then— $\quad b = 8{\cdot}87 \times \cos 52^\circ 33'$

In the log form—

$$\log b = \log 8{\cdot}87 + \log \cos 52^\circ 33' = {\cdot}7318$$
$$\therefore \quad \underline{b = 5{\cdot}393''}.$$

Exercises 30.—On the Solution of Triangles.

In Exs. 1 to 14 solve the triangle ABC completely, being given that—

1. $a = 3''$, $b = 5{\cdot}2''$, $B = 78\frac{1}{2}^\circ$. 2. $a = 79{\cdot}5''$, $C = 51^\circ 32'$, $B = 47^\circ 36'$.
3. $C = 26^\circ 50'$, $b = 8{\cdot}86''$, $c = 5{\cdot}68''$. 4. $b = 5{\cdot}97''$, $C = 64^\circ 18'$, $A = 75^\circ$.
5. $c = 9{\cdot}2$, $a = 10{\cdot}31$, $C = 46^\circ$. 6. $b = 6{\cdot}1$ ft., $c = 9{\cdot}3$ ft., $A = 73^\circ 16$.
7. $a = 124{\cdot}4$, $b = 93{\cdot}7$, $c = 99{\cdot}3$. 8. $a = 13{\cdot}7''$, $b = 10{\cdot}5''$, $C = 130^\circ$.
9. $a = 4{\cdot}27''$, $A = 29^\circ$, $b = 5{\cdot}86''$. 10. $c = 6880$, $B = 30^\circ$, $b = 5141$.

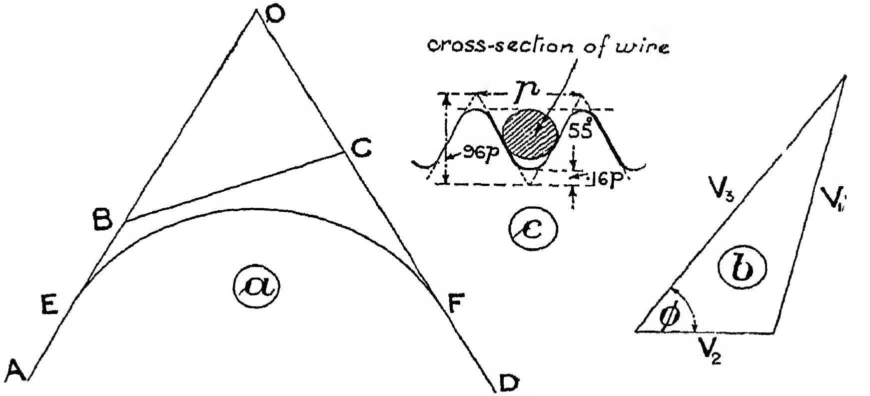


Fig. 149.—Solution of Triangles.

11. $A = 50^\circ 50'$, $b = 922{\cdot}4$, $c = 1003{\cdot}8$.
12. $B = 35^\circ 30'$, $b = 38{\cdot}6$, $c = 43{\cdot}57$. 13. $a = 21{\cdot}8$, $b = 15{\cdot}7$, $C = 47^\circ 32'$.
14. $c = 32{\cdot}7$, $b = 39{\cdot}4$, $B = 55^\circ 30'$. Find also the area.

15. The area of a triangle is 120 sq. ft. and the angles are 75°, 60° and 45°. Find the longest side.

16. The connecting rod of an engine is 8 ft. in length and the crank 1′-6″. Find the inclination of the connecting rod to the line of stroke when the crank has moved 52° from its inner dead centre position.

17. The sides of a "triangle of forces" represent the forces 3·7 tons, 2·275 tons and 3·025 tons respectively. Find the angles of this triangle.

18. Forces of 21·6 and 19·7 lbs., making an angle of 126° with one another act at a point. Find the magnitude of their resultant and its inclination to the larger force.

19. In setting out a railway curve to connect the lines AO and OD ((a) Fig. 149), a line CB was measured and found to be 1·474 chains.

If $\angle ABC = 171° 10' 30''$, and $\angle BCD = 145° 15'$, find the lengths of OB and OC; also if the radius of the curve is to be 5 chains and the starting-points are E and F find the lengths of BE and CF.

20. The diagram, (*b*) Fig. 149, is necessary for the calculation of lag by the 3-voltmeter method. If ϕ is the angle of lag, find its value for the case illustrated. $\{V_3 = 107,\ V_1 = 90,\ V_2 = 48.\}$

21. The jib of a crane is inclined at 57° to the horizontal; the post is 12 ft. high and the tie rod makes 35° with the horizontal. Find the lengths of the jib and the tie rod.

22. In a triangle the shortest side is 290 ft. and the adjacent angles are 43° 30′ and 125°; find the length of the longest side.

23. The tangents to a curve meet at 120°. On the bisector of this angle is a point 100 ft. distant from the point of meeting of the tangents, and through which the curve must pass. Find the radius of the required curve and also the tangent distances.

24. It is required to find the height of a house on the opposite bank of a river. The elevation of the top of the house is read at a certain point as 17°; approaching 86 ft. nearer to the bank, towards the house, the elevation is found to be 31°. Find the height of the house.

25. A theodolite is set up at two stations A and B at the water's edge of a lake which is 1240 ft. above sea-level. A staff on a hill at C is sighted from each station. From A the elevation of C is 15° 14′ and the horizontal angles CAB and CBA are 59° 10′ and 71° 48′ respectively. If AB = 820 yds., find the height of C above sea-level.

26. From a station C on a hill, two stations A and B, on opposite sides of the hill are observed. The horizontal projection of $\angle ACB$ is 43° 23′, the horizontal projection of CA is 3633 links and of CB is 4275 links. The angle of elevation of C at A is 44° 37′ and at B is 33° 24′. Determine the horizontal distance between A and B and the difference of level between them.

27. It is required to set out a curve of $\frac{1}{2}$ mile radius between two straight portions of a railway, AB and DC, which intersect in an inaccessible point E. Rods are set up at points B and C on the two straight portions and the angles ABC and BCD are measured and found to be 110° 20′ and 120° 30′ respectively.

If BC = 830 links, determine the distances of the tangent points G and H from B and C respectively.

28. In a theodolite survey to find the positions of two visible but inaccessible points B and C, the following measurements were made—AD = 517·75 links, $\angle BAC = 70° 44' 10''$, $\angle BAD = 108° 9'$, $\angle ADB = 36° 18' 30''$, and $\angle ADC = 101° 18' 30''$. Find the lengths of AB, DC, AC and BC in order.

29. When setting out the centre line for a tunnel between the two ends A and B, an observatory station C is chosen on the top of a hill from which both A and B are visible, but it is not on the centre line of the tunnel. Let D be a point on a vertical through C. The horizontal projection of $\angle ACB = 45° 58'$, the vertical angle ACD = 49° 45′ and the vertical angle BCD = 57° 42′. The horizontal projection of CA is 750 yards, and of CB is 800 yards. Find the horizontal distance between A and B and the difference of level.

30. A light railway is to be carried round the shoulder of a hill, and

its centre line is to be tangential to each of the three lines AB, BC and CD as follows—

Line	Bearing.	Length.
AB	E. 30° N.	—
BC	E	600 feet
CD	S	—

Calculate the radius of the curve and the lengths required for setting out the tangent points. [*Note.*—E. 30° N. means 30° north of east.]

31. In taking soundings from a boat the position is fixed by observations taken to three stations A, B and C on the shore. The lines AB and BC have been measured by the following traverse: A to B, 542 ft., bearing 70° 14′; B to C, 714 ft., bearing 110° 33′. From the boat in a certain position P, the angles APB and BPC were read as 32° 16′ and 44° 21′ respectively. Calculate the distances AP, BP and CP.

32. The speed of the blades of a turbine is 600 ft. per sec., the velocity of the steam at entrance to the wheel is 1780 ft. per sec., and the nozzle is inclined at 20° to the blades. Find the relative velocity of the steam at discharge, and the inclination of the direction of this velocity to the line of motion of the blades.

33. Find the diameter of the wire, whose section is shown in (*c*) Fig. 149, in terms of the pitch p of the V-threaded screw. This wire is used as a gauge to test the accuracy of the form of the thread.

Further Mensuration Examples.

34. A circular arch has a rise of 20 ft. and a span of 80 ft. Find the angle at the centre of the circle which is subtended by this arc, and also the length of the curved portion of the arch.

35. A wooden core, having as section an equilateral triangle, is placed in the tubes (internal diameter $\frac{3}{4}''$) of a surface condenser. Find the ratio of the tube surface to the water-carrying section.

36. A roof is in the form of the surface of a segment of a sphere of 6 ft. radius. The tangents at the eaves make 48° with the horizontal. Find the area of the roof surface, and the weight of sheet lead required to cover it at 7 lbs. per sq. ft.

37. Find the diagonals of a rhombus in which one side is 6·5″ and one angle is 70°.

38. A quadrilateral has two adjacent sides equal and containing a right angle. The other pair of sides are equal and contain 60°. The area is 1 sq. ft. Find the lengths of the sides.

39. A quadrilateral has two adjacent angles each 120°. The side between them is 24 ft., and the perpendiculars on this side from the other angular points are 7 ft. and 10 ft. respectively. Find the area of the quadrilateral.

40. A trapezoid has its parallel sides 82″ and 38″ and two of its angles each 60°. Find its area and the area of the triangle obtained by producing the non-parallel sides.

41. A quadrilateral with two opposite angles right angles and one of the remaining angles 60° is described about a circle of 2″ radius. Find its area.

The Addition Formulæ.—It is sometimes necessary, more particularly in electrical work, to express the ratio of a compound angle in terms of the ratios of the simpler angles, or vice versa; *e. g.*, it might be easier to state tan (A+B) in terms of tan A and tan B, and then evaluate, than to evaluate directly. The following rules must be committed to memory for this purpose—

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$
$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Considering sin (A+B) one might be tempted at a first glance to apply the ordinary rules of brackets, and write the expansion as sin A+sin B. That this is not correct may be readily seen by referring to any angles.

e. g., suppose $A = 46°$, and $B = 15°$, then $(A+B) = 61°$

i. e., $\sin (A+B) = \sin 61° = \cdot 8746$

whereas— $\sin A+\sin B = \sin 46°+\sin 15°$
$= \cdot 7193+\cdot 2588$
$= \cdot 9781$

and ·9781 does not equal ·8746.

It will be observed, however, that the above rule holds, at any rate for these particular values of A and B.

Thus—

$$\sin 46° \cos 15° + \cos 46° \sin 15° = (\cdot 7193 \times \cdot 9659) + (\cdot 6947 \times \cdot 2588)$$
$$= \cdot 8745$$
$$= \sin 61°$$
$$= \sin (46°+15°).$$

A more general proof is necessary to establish the truth of these rules for *all* angles; and the proofs are here given for the simplest cases only.

To prove that sin (A+B) = sin A cos B + cos A sin B.

Taking the simplest case, when A and B are both acute—

In Fig. 150 let $\angle PQR = A$, and $\angle RQM = B$, then $\angle PQM = (A + B)$; also let QR be perpendicular to PM.

Then— $\triangle PQM = \triangle PQR + \triangle QRM$

$$\therefore \tfrac{1}{2}PQ \cdot QM \sin (A + B) = \tfrac{1}{2}PQ \cdot QR \cdot \sin A + \tfrac{1}{2}QR \cdot QM \cdot \sin B.$$

Dividing through by $\frac{1}{2}$PQ . QM—

$$\sin(A+B) = \frac{QR}{QM}\sin A + \frac{QR}{PQ}\sin B$$
$$= \cos B \sin A + \cos A \sin B$$
$$\text{or } \sin A \cos B + \cos A \sin B.$$

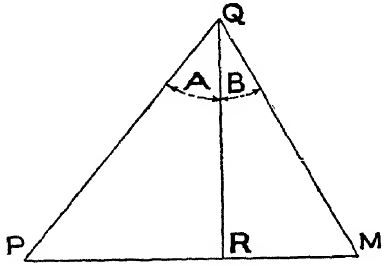


Fig. 150.

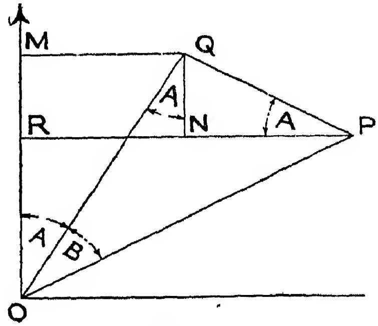


Fig. 151.

To prove that $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

In Fig. 151 let $\angle MOQ = A$, and $\angle QOP = B$
$\angle PQO =$ right angle.

Drop QN perpendicular to RP, RP being perpendicular to OM. Then $\angle OQN = A$, $\angle NQP = 90 - A$, and therefore $\angle QPN = A$.

Now— $$\cos(A+B) = \cos \angle ROP = \frac{OR}{OP} = \frac{OM-MR}{OP}$$
$$= \frac{OM}{OP} - \frac{NQ}{OP} \quad [\text{since } NQ = MR]$$
$$= \frac{OM}{OQ}\cdot\frac{OQ}{OP} - \frac{NQ}{QP}\cdot\frac{QP}{OP}$$
$$= \cos A \cos B - \sin A \sin B.$$

To prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Assume the rules for sin (A+B) and cos (A+B).

Then— $$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$
$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

[Dividing both numerator and denominator by cos A cos B.]
$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}}$$
$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Example 23.—Verify the rules for sin (A − B), cos (A + B) and tan (A − B) for the case when A = 164° and B = 29°.

$$\sin(A - B) = \sin(164 - 29) = \sin 135^\circ$$
$$= \sin 45^\circ$$
$$= \cdot 707.$$

Also $\sin A \cos B - \cos A \sin B = \sin 164^\circ \cos 29^\circ - \cos 164^\circ \sin 29^\circ$

$[\cos 164^\circ = -\cos 16^\circ]$ $= \sin 16^\circ \cos 29^\circ + \cos 16^\circ \sin 29^\circ$

$$= (\cdot 2756 \times \cdot 8746) + (\cdot 9613 \times \cdot 4848)$$
$$= \cdot 241 + \cdot 465$$
$$= \cdot 706.$$

For brevity we shall denote the side containing (A + B) by L.H.S. (left-hand side); the other by R.H.S. (right-hand side).

$\therefore$ L.H.S. = R.H.S.

For cos (A + B)—

L.H.S. $= \cos(A + B) = \cos(164^\circ + 29^\circ) = \cos 193^\circ = -\cos 13^\circ = -\cdot 9744$

R.H.S. $= \cos A \cos B - \sin A \sin B = \cos 164^\circ \cos 29^\circ - \sin 164^\circ \sin 29^\circ$

$$= -\cos 16^\circ \cos 29^\circ - \sin 16^\circ \sin 29^\circ$$
$$= (-\cdot 9613 \times \cdot 8746) - (\cdot 2756 \times \cdot 4848)$$
$$= -\cdot 841 - \cdot 133$$
$$= -\cdot 974$$

$\therefore$ L.H.S. = R.H.S.

For tan (A—B)—

L.H.S. $= \tan(164^\circ - 29^\circ) = \tan 135^\circ = -\tan 45^\circ = -1$

$$\text{R.H.S.} = \frac{\tan 164^\circ - \tan 29^\circ}{1 + \tan 164^\circ \tan 29^\circ} = \frac{-\tan 16^\circ - \tan 29^\circ}{1 - \tan 16^\circ \tan 29^\circ}$$
$$= \frac{-\cdot 2867 - \cdot 5543}{1 - (\cdot 2867 \times \cdot 5543)}$$
$$= \frac{-\cdot 841}{1 - \cdot 159} = \frac{-\cdot 841}{\cdot 841} = -1$$

$\therefore$ L.H.S. = R.H.S.

Example 24.—Find the value of cos (A + B) when sin A = ·5, cos B = ·23. (Tables are not to be used.)

Before proceeding with this example, a little preliminary investigation is necessary.

In the right-angled triangle ABC (Fig. 152)—

$$b^2 + a^2 = c^2$$
$$\frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{c^2}{c^2} = 1$$

$\therefore$ $\mathbf{Cos^2 A + sin^2 A = 1,}$ since $\cos A = \frac{b}{c}$ and $\sin A = \frac{a}{c}$

This is a most important relation between these ratios; and it holds for every value given to the angle A.

Two other rules obtained by similar methods are—

$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

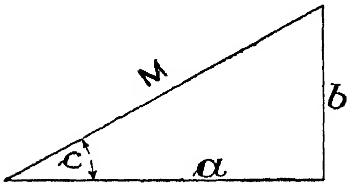


Fig. 153.

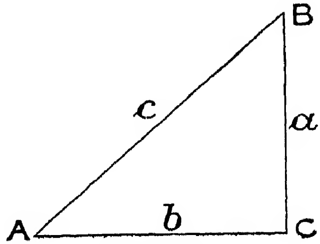


Fig. 152.

Returning to *Example* 24:—

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

Values must first be found for—

$$\cos A \text{ and } \sin B.$$

Now— $\cos^2 A + \sin^2 A = 1$

from which— $\cos^2 A = 1 - \sin^2 A$, or $\sin^2 A = 1 - \cos^2 A$

or $\cos A = \sqrt{1 - \sin^2 A}$, $\sin A = \sqrt{1 - \cos^2 A}$

Then— $\cos A = \sqrt{1 - (\cdot 5)^2} = \sqrt{\cdot 75} = \cdot 866$

and $\sin B = \sqrt{1 - (\cdot 23)^2} = \sqrt{\cdot 947} = \cdot 973$

$$\begin{aligned} \therefore \quad \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= (\cdot 866 \times \cdot 23) - (\cdot 5 \times \cdot 973) \\ &= \cdot 1991 - \cdot 4865 \\ &= -\cdot 2874. \end{aligned}$$

It is often necessary to change the binomial or two-term expression $a \sin qt + b \cos qt$ into an expression of the form $M \sin(qt + c)$, where c is an angle. We must therefore find the values of M and c in terms of a and b, so that—

$$M \sin(qt + c) = a \sin qt + b \cos qt \quad \ldots \ldots \quad (1)$$

Take the addition formula, viz.—

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

Replacing A by qt and B by c, this statement becomes—

$$\sin(qt + c) = \sin qt \cos c + \cos qt \sin c.$$

Multiplying through by M—

$$M \sin(qt + c) = M \sin qt \cos c + M \cos qt \sin c \quad \ldots \ldots \quad (2)$$

Since the right-hand sides of (1) and (2) are equal in total value, they can be made equal term for term by choosing suitable values for the coefficients.

Thus— $\quad M \sin qt \cos c = a \sin qt$

and $\quad M \cos qt \sin c = b \cos qt$

$$\therefore \quad M \cos c = a \quad \text{and } M \sin c = b$$

$$i.e., \quad \cos c = \frac{a}{M} \quad \text{and } \sin c = \frac{b}{M}$$

If, now, a triangle be drawn (Fig. 153) with sides a, b and hypotenuse M, it will be seen that the angle opposite the side b is the angle c, for its adjacent side is a, and therefore its cosine $= \frac{a}{M}$

Hence c is found, for $\quad \tan c = \frac{b}{a}$

Knowing the values of b and a, the value of c is read off from the table of tangents and is usually expressed in radians.

Also— $\quad M^2 = a^2 + b^2$

$$\therefore \quad M = \sqrt{a^2 + b^2}, \text{ so that M is found.}$$

This investigation is valuable in cases of harmonic motion.

Example 25.—The voltage necessary to produce an alternating current C, after any particular period of time t, and in a circuit of resistance 2 ohms in which the current varies, being given by $C = 100 \sin 600t$, can be expressed as—

$$V = 200 \sin 600t + 300 \cos 600t$$

Find a simpler expression for V.

Let $200 \sin 600t + 300 \cos 600t = M \sin (600t + c)$. Then by the previous work—

$$M = \sqrt{200^2 + 300^2} = 360{\cdot}6$$

and $\quad \tan c = \frac{300}{200} = 1{\cdot}5 = \tan 56{\cdot}3°$

$$\therefore \quad c = 56{\cdot}3° = \frac{56{\cdot}3}{57{\cdot}3} \text{ radians} = {\cdot}983 \text{ radian}$$

$$\therefore \quad V = \underline{360{\cdot}6 \sin (600t + {\cdot}983)}$$

or, as it might be written, $V = 360{\cdot}6 \sin 600\,(t + {\cdot}00164)$.

Note.—If the current were continuous, then—

$$\text{voltage} = \text{current} \times \text{resistance}$$
$$= 200 \sin 600t.$$

When the current is interrupted, "inertia" or "induction" effects set up another current to oppose that due to the impressed voltage, and therefore the amperes are not a maximum when the voltage is,

i. e., the current lags behind the E.M.F.; in this case to the extent of ·00164 second. The coefficient 600 in the formulæ $= 2\pi f$ where $f =$ frequency: thus in this case the frequency $= \frac{600}{2\pi} = 95{\cdot}6$ cycles per second.

As a further example of transformation consider the following case:—

Example 26—

Let $S_m =$ displacement of the main steam valve of an engine from its central position

$S_e =$ displacement of the expansion plate from its central position

for the case of an engine with Meyer valve gear.

Then— $S_m = r_m \cos(\theta + a_1)$, and $S_e = r_e \cos(\theta + a_2)$.

To find a simple expression for the displacement of the expansion plate relative to the main valve.

This relative displacement $= S_m - S_e = r_m \cos(\theta + a_1) - r_e \cos(\theta + a_2)$.

Then—

$$\begin{aligned}
S_m - S_e &= r_m \cos(\theta + a_1) - r_e \cos(\theta + a_2) \\
&= r_m \cos\theta \cos a_1 - r_m \sin\theta \sin a_1 - r_e \cos\theta \cos a_2 + r_e \sin\theta \sin a_2 \\
&= \cos\theta(r_m \cos a_1 - r_e \cos a_2) + \sin\theta(r_e \sin a_2 - r_m \sin a_1) \\
&= A\cos\theta + B\sin\theta \\
&= \sqrt{A^2 + B^2}\sin(\theta + c) \qquad \text{as before proved} \\
&= \sqrt{A^2 + B^2}\cos\left(\frac{\pi}{2} - (\theta + c)\right) \\
&= \sqrt{A^2 + B^2}\cos(\theta + p)
\end{aligned}$$

where—

$$A = r_m \cos a_1 - r_e \cos a_2, \quad B = r_e \sin a_2 - r_m \sin a_1$$

and $p = c - \frac{\pi}{2} = \tan^{-1}\frac{A}{B} - \frac{\pi}{2}$ $\left\{\tan^{-1}\frac{A}{B} \text{ is the } \textit{angle} \text{ whose tan is } \frac{A}{B}\right\}$

We have thus reduced the expression for the relative displacement to a form of a simple character which shows that this displacement is equivalent to that caused by an imaginary eccentric of radius $\sqrt{A^2 + B^2}$ and of angular advance p.

Exercises 31.—On the Addition Formulæ in Trigonometry.

1. If $\sin A = {\cdot}45$, find $\cos A$ and $\tan A$ (without reference to the tables).

2. If $\sin B = {\cdot}16$, $\cos A = {\cdot}29$, find the value of
$$\sin A \cos B - \cos A \sin B.$$

3. Find the values of $\cos(A + B)$, and $\sin(A - B)$, when—
$$\sin A = {\cdot}65, \ \sin B = {\cdot}394.$$

4. Tan A = 1·62, tan B = ·58; find the values of tan (A + B) and tan (A − B).

5. The horizontal force P necessary to just move a weight W down a rough plane inclined at a to the horizontal, the coefficient of friction between the plane and the weight being μ, can be obtained from the formula—

$$\frac{P}{W} = \tan(\phi - a)$$

If $\tan\phi = \mu$, find an expression for $\frac{P}{W}$ in terms of μ and $\tan a$. Hence find the value of P when W = 48, μ = ·21, and $a = 8°$

6. The effort P required to raise a load W by means of a screw, of pitch p and radius r, is given by

$$P = W \tan(\phi + a)$$

where a = angle of screw and $\tan\phi = \mu$ = coefficient of friction. Find an expression for P in terms of W, p, r and μ.

7. If $\frac{P}{W} = \frac{\sin(\phi + a)}{\cos\phi}$, and $\tan\phi = \mu$, find a simple expression for P.

8. Given that tan (A−B) = ·537 and tan B = ·388, find tan A.

9. Express $4·2 \cos 5t + 2·7 \sin 5t$ in the form $M \sin(5t + c)$.

10. Express $200 \sin 50t - 130 \cos 50t$ in the form $M \sin(50t + c)$.

11. If a bullet be projected from a point on ground sloping at an angle A to the horizontal, the elevation being θ to the incline, the range R is given by the formula $R = ut - \frac{1}{2}gt^2 \sin A$. Find a simpler expression for R, if $u = V\cos\theta$ and $t = \frac{2V\sin\theta}{g\cos A}$

12. The efficiency of a screw jack $= \frac{\tan\theta}{\tan(\theta + \phi)}$ where θ is the angle of the screw and ϕ is the angle of friction. In a certain experiment the efficiency was found to be 3, and by measurement of the pitch and the mean circumference of the screw $\tan\theta$ was calculated as ·083. Find $\tan\phi$, which is the coefficient of friction between the screw and nut, and thence find ϕ.

13. If the E.M.F. in an inductive circuit is given by—

$$E = RI \sin 2\pi ft + 2\pi fLI \cos 2\pi ft$$

find a simpler expression for E, *i.e.*, one having the form $M \sin(2\pi ft + c)$, when R = 4·6, f = 60, L = ·02 and I = 13·8.

Formulæ for the Ratios of the Multiple and Sub-multiple Angles.—In the addition formulæ let B be replaced by A; by so doing, expressions may be found for the ratios of 2A.

Thus—
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\therefore \quad \sin(A+A) = \sin A \cos A + \cos A \sin A$$
or
$$\sin 2A = 2 \sin A \cos A.$$

Also—
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\therefore \quad \cos(A+A) = \cos A \cos A - \sin A \sin A$$
or
$$\cos 2A = \cos^2 A - \sin^2 A.$$

If for $\cos^2 A$ we write $1-\sin^2 A$, which is permissible since $\cos^2 A + \sin^2 A = 1$,

then—
$$\cos 2A = 1-\sin^2 A-\sin^2 A$$
$$= 1-2\sin^2 A.$$

Also—
$$\cos 2A = \cos^2 A-(1-\cos^2 A)$$
$$= 2\cos^2 A-1.$$

Again—
$$\tan (A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$$
$$\therefore \quad \tan (A+A) = \frac{\tan A + \tan A}{1-\tan A \tan A}$$
$$= \frac{2\tan A}{1-\tan^2 A}$$

Grouping the results—

$$\mathbf{\sin 2A = 2\sin A \cos A}$$
$$\left.\begin{aligned}\mathbf{\cos 2A} &= \mathbf{\cos^2 A - \sin^2 A} \\ &= \mathbf{2\cos^2 A - 1} \\ &= \mathbf{1 - 2\sin^2 A}\end{aligned}\right\}$$
$$\mathbf{\tan 2A = \frac{2\tan A}{1-\tan^2 A}}$$

If the ratios of the half-angles are required they can be obtained from the foregoing by dividing all the angles by 2.

E.g.,
$$\left.\begin{aligned}\cos A &= \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \\ &= 2\cos^2 \frac{A}{2} - 1 \\ &= 1 - 2\sin^2 \frac{A}{2}\end{aligned}\right\}$$
$$\sin A = 2\sin \frac{A}{2} \cos \frac{A}{2}$$
$$\tan A = \frac{2\tan \frac{A}{2}}{1-\tan^2 \frac{A}{2}}$$

Similarly, by multiplying all the angles by 2, expressions can be found for the ratios of the angle 4A—

$$e.g., \quad \sin 4A = 2\sin 2A \cos 2A$$

and this expansion can be further developed if necessary.

Formulæ for ratios of 3A can be obtained by writing 2A in place of B in the (A+B) formulæ, and using the rules for the ratios of 2A—

E.g.,
$$\begin{aligned}\sin 3A &= \sin(2A+A)\\ &= \sin 2A\cos A + \cos 2A \sin A\\ &= 2\sin A\cos^2 A + (1-2\sin^2 A)\sin A\\ &= 2\sin A\cos^2 A + \sin A - 2\sin^3 A\\ &= 2\sin A(1-\sin^2 A) + \sin A - 2\sin^3 A\\ &= 3\sin A - 4\sin^3 A.\end{aligned}$$

In like manner— $\cos 3A = 4\cos^3 A - 3\cos A$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$$

Example 27.—Verify the rules for cos 2A, tan 2A, and sin 3A for the case when $A = 24°$.

$$2A = 48°,\quad 3A = 72°$$

For cos 2A—

$$\begin{aligned}\text{L.H.S.} &= \cos 2A = \cos 48° = \cdot 669\\ \text{R.H.S.} &= \cos^2 A - \sin^2 A = \cos^2 24° - \sin^2 24°\\ &= (\cdot 9135)^2 - (\cdot 4067)^2\\ &= \cdot 835 - \cdot 165\\ &= \cdot 670\end{aligned}$$

$$\therefore \quad \text{L.H.S.} = \text{R.H.S.}$$

For tan 2A—

$$\text{L.H.S.} = \tan 2A = \tan 48° = 1\cdot 1106$$

$$\text{R.H.S.} = \frac{2\tan A}{1-\tan^2 A} = \frac{2\tan 24°}{1-\tan^2 24°} = \frac{2\times \cdot 4452}{1-(\cdot 4452)^2} = \frac{\cdot 89}{\cdot 802} = 1\cdot 108$$

$$\therefore \quad \text{L.H.S.} = \text{R.H.S.}$$

(the small differences being due to slide-rule working).

For sin 3A—

$$\begin{aligned}\text{L.H.S.} &= \sin 3A = \sin 72° = \cdot 9511.\\ \text{R.H.S.} &= 3\sin A - 4\sin^3 A = 3\sin 24° - 4\sin^3 24°\\ &= 3\times \cdot 4067 - 4(\cdot 4067)^3\\ &= 1\cdot 220 - \cdot 269\\ &= \cdot 951.\end{aligned}$$

Example 28.—If $\sin A = \cdot 85$, find $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ and $\tan\frac{A}{2}$, without the use of the tables. (Examples are set in this manner so that the reader may become familiar with the formulæ and the method of using them; but in practice the tables would be used.)

To find $\sin \frac{A}{2}$. The formula that contains $\sin \frac{A}{2}$, only, of the ratios of the half-angles is—

$$\cos A = 1 - 2\sin^2 \frac{A}{2}$$

and, therefore, to use this, $\cos A$ must first be found.

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - (\cdot 85)^2} = \cdot 526.$$

Then—

$$\cdot 526 = 1 - 2\sin^2 \frac{A}{2}$$

$$2\sin^2 \frac{A}{2} = 1 - \cdot 526 = \cdot 474$$

or

$$\sin^2 \frac{A}{2} = \cdot 237$$

$$\therefore \quad \sin \frac{A}{2} = \underline{\cdot 487} \qquad \{\text{the positive root only being taken}\}.$$

To find $\cos \frac{A}{2}$

$$\cos A = 2\cos^2 \frac{A}{2} - 1 \qquad \left\{ \begin{array}{l} \text{or alternatively—} \\ \cos \frac{A}{2} = \sqrt{1 - \sin^2 \frac{A}{2}} \end{array} \right\}$$

$$\therefore \quad \cdot 526 = 2\cos^2 \frac{A}{2} - 1$$

$$2\cos^2 \frac{A}{2} = 1 \cdot 526$$

$$\cos^2 \frac{A}{2} = \cdot 763$$

$$\cos \frac{A}{2} = \underline{\cdot 875.}$$

To find $\tan \frac{A}{2}$

$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{\cdot 487}{\cdot 875} = \underline{\cdot 557.}$$

Example 29.—Find the value of $\tan 2A$ if $\cos A = \cdot 96$.

$$\begin{aligned} \sin A &= \sqrt{1 - \cos^2 A} \\ &= \sqrt{1 - (\cdot 96)^2} \\ &= \cdot 2795 \end{aligned}$$

Then—

$$\tan A = \frac{\sin A}{\cos A} = \frac{\cdot 2795}{\cdot 96} = \cdot 292$$

and

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{2 \times \cdot 292}{1 - (\cdot 292)^2} = \frac{\cdot 584}{\cdot 915} = \underline{\cdot 638.}$$

Example 30.—It was required to find, to an accuracy of ·0001″, the dimension marked c in Fig. 154; the figure representing part of a gauge for the shape of a boring tool. There is a radius of ·5″ at the top of the sloping side, which is tangential to an arc of 3·4″ radius at the bottom; and other dimensions are as shown.

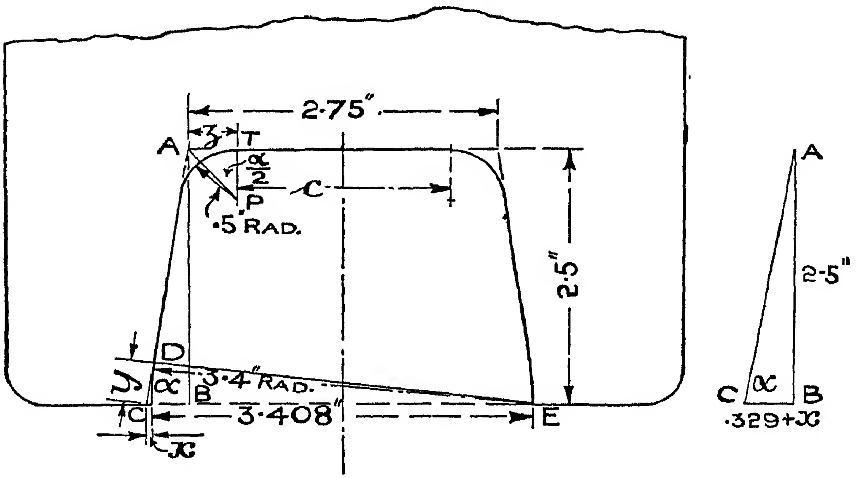


Fig. 154.—Gauge for Boring Tool.

Introduce the three unknowns x, y and z as indicated on the figure, y being the distance along the slant side from the point of contact with the arc to the base.

Let the angle ACB $= a$, then $\angle$ APT $= \frac{a}{2}$

$$CB = 1{\cdot}704 + x - 1{\cdot}375 = {\cdot}329 + x$$

In the triangle ACB— $\tan a = \dfrac{2{\cdot}5}{{\cdot}329 + x}$ . . . . (1)

In the triangle APT— $\tan \dfrac{a}{2} = \dfrac{z}{{\cdot}5} = 2z$ . . . . (2)

also $\tan a = \dfrac{3{\cdot}4}{y}$ . . . . (3) } From the properties of the circle.

and $y^2 = (6{\cdot}8 + x)x$ . . . . (4) } ($\angle$ EDC is a right angle)

Connecting (1) with (3)—

$$\frac{2{\cdot}5}{{\cdot}329 + x} = \frac{3{\cdot}4}{y}$$

$$\therefore \quad y = 1{\cdot}36\,({\cdot}329 + x)$$

$$i.\,e., \quad y^2 = 1{\cdot}8496\,({\cdot}329 + x)^2.$$

Hence from (4)—

$$(6{\cdot}8 + x)x = 1{\cdot}8496\,({\cdot}329 + x)^2$$

whence $\quad {\cdot}8496x^2 - 5{\cdot}583x + {\cdot}20021 = 0$

so that $$x = \frac{+5\cdot583 \pm \sqrt{31\cdot1699 - \cdot6804}}{1\cdot6992}$$

or the required value of $x = \dfrac{\cdot06134}{1\cdot6992} = \cdot0361''$.

Now—
$$\frac{3\cdot4}{y} = \frac{2\cdot5}{\cdot329 + x}$$

so that
$$y = \cdot4965$$

also
$$\tan a = \frac{3\cdot4}{\cdot4965} = 6\cdot8482.$$

It would be unwise to use the tables to find a from the previous equation, for in the neighbourhood of the required value the change in the value of the tangent is extremely rapid; hence it is a good plan to make use of the rule for tan 2A or its modification.

Thus—
$$\tan a = \frac{2 \tan \frac{a}{2}}{1 - \tan^2 \frac{a}{2}}$$

i. e., $6\cdot8482 = \dfrac{4z}{1 - 4z^2}$ for $\tan \dfrac{a}{2} = 2z$ from (2).

This is a quadratic in terms of z, and the solution applicable to this case is $z = \cdot4323$.

$$\therefore \quad c = 2\cdot75 - 2 \times \cdot4323 = \underline{\underline{1\cdot8854''}}.$$

Further Transpositions of the Addition Formulæ—

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Hence, by addition—

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

and by subtraction—

$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$$

Also—
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\therefore \quad \cos(A+B) + \cos(A-B) = 2 \cos A \cos B$$
and
$$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B.$$

[Note the change in the order on the left-hand side in this last formula.]

Now—

$$A = \frac{(A+B)+(A-B)}{2} \quad i.e., = \tfrac{1}{2} \text{ sum of the two angles,}$$

and $$B = \frac{(A+B)-(A-B)}{2} \quad i.e., = \tfrac{1}{2} \text{ difference of the two angles.}$$

Hence, the first of these formulæ could be written—

Sine (one angle) + sine (another angle)

= 2 sine ($\frac{1}{2}$ their sum) × cos ($\frac{1}{2}$ their difference)

A substitution is very often of great service; thus,

$$\text{let} \quad (A+B) = C \quad \text{and} \quad (A-B) = D$$

Then—

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}, \text{ etc.,}$$

and we have the summary—

If the change is to be made from a sum or difference to a product, use the (C+D) formulæ—

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \quad \ldots \ldots \quad (1)$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \quad \ldots \ldots \quad (2)$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \quad \ldots \ldots \quad (3)$$

$$\cos D - \cos C = 2 \sin \frac{C+D}{2} \sin \frac{C-D}{2} \quad \ldots \ldots \quad (4)$$

If, however, the change to be made is from a product to a sum or difference, use the A and B formulæ, which follow—

$$\sin A \cos B = \tfrac{1}{2}\{\sin(A+B) + \sin(A-B)\} \quad \ldots \ldots \quad (5)$$

$$\cos A \sin B = \tfrac{1}{2}\{\sin(A+B) - \sin(A-B)\} \quad \ldots \ldots \quad (6)$$

$$\cos A \cos B = \tfrac{1}{2}\{\cos(A+B) + \cos(A-B)\} \quad \ldots \ldots \quad (7)$$

$$\sin A \sin B = \tfrac{1}{2}\{\cos(A-B) - \cos(A+B)\} \quad \ldots \ldots \quad (8)$$

In later work it will be found that certain operations can be performed on a sum or difference of two trigonometric ratios that cannot be done with products; hence the great importance of this last set of formulæ.

It may appear to the reader that his memory will be severely taxed by the above long list of formulæ, but a second thought will convince him that all are derived from the original (A + B) and (A − B) formulæ, which must be committed to memory to serve as the first principles from which all the later formulæ are developed.

Example 31.—Express 17 sin 56° sin 148° as a sum or difference.

$$\sin 56^\circ \sin 148^\circ = \tfrac{1}{2}\{\cos(148^\circ - 56^\circ) - \cos(148^\circ + 56^\circ)\} \quad . \quad . \quad \text{from (8)}$$

$$\{A = 148^\circ,\ B = 56^\circ\}$$

$$\therefore \quad 17 \sin 56^\circ \sin 148^\circ = \underline{8{\cdot}5\ \{\cos 92^\circ - \cos 204^\circ\}.}$$

To check by the use of tables—

$$\begin{aligned} \text{L.H.S.} &= 17 \sin 56^\circ \sin 148^\circ = 17 \sin 56^\circ \sin 32^\circ \\ &= 17 \times \cdot 8290 \times \cdot 5299 \\ &= 7{\cdot}47. \end{aligned}$$

$$\begin{aligned} \text{R.H.S.} &= 8{\cdot}5\{\cos 92^\circ - \cos 204^\circ\} \\ &= 8{\cdot}5\{-\cos 88^\circ + \cos 24^\circ\} \\ &= 8{\cdot}5\{-\cdot 0349 + \cdot 9135\} = 8{\cdot}5 \times \cdot 8786 = 7{\cdot}47. \end{aligned}$$

$$\therefore \quad \text{L.H.S.} = \text{R.H.S.}$$

Example 32.—The voltage V in an A.C. circuit, after a time t, is given by $V = 200 \sin 360t$, and the current by $C = 3{\cdot}5 \sin(360t + c)$. Find an expression for the watts at any time, expressing it as a sum or difference.

$$\begin{aligned} \text{Watts} &= \text{amps} \times \text{volts} \\ &= 3{\cdot}5 \sin(360t + c) \times 200 \sin 360t \\ &= 700 \sin(360t + c) \sin 360t \\ &= \frac{700}{2}\{\cos c - \cos(720t + c)\} \quad \ldots\ldots \text{ from (8)} \\ &= \underline{350\{\cos c - \cos(720t + c)\}}. \end{aligned}$$

Example 33.—Express $(4 \sin 5t)(5 \cos 3t)$ as a sum or difference.

$$\begin{aligned} (4 \sin 5t)(5 \cos 3t) &= 20 \sin 5t \cos 3t \\ &= \underline{10\ \{\sin 8t + \sin 2t\}} \quad \ldots \text{ from (5)} \end{aligned}$$

Exercises 32.—On Transpositions of the Addition Formulæ.

1. If $\sin 2A = \cdot 824$, find $\cos 2A$ and $\tan 2A$.

2. If $\sin A = \frac{4}{5}$, find $\sin 2A$ and $\cos 2A$.

3. Express $\cos^2 14^\circ$ in terms of $\cos 28^\circ$.

4. Find an expression for $\sin 2B$ in terms of $\cos B$ alone. Hence find the value of $\sin 2B$ when $\cos B = \cdot 918$.

5. If $\sin A = \cdot 317$, find $\sin \frac{A}{2}$, $\cos \frac{A}{2}$ and $\sin 3A$.

6. If $\sin 2A = \cdot 438$, find $\cos 4A$ and $\tan \frac{A}{2}$

7. Change $5 \sin^2 2t$ into a form containing the first power only of the trigonometric function.

8. Express $15{\cdot}7 \cos 160^\circ \sin 29^\circ$ as a sum or difference.

9. Simplify $\sin 15t + \sin 3t + \cos 11t - \cos 7t$.

10. $\sin 2A = \cdot 504$. Find $\sin A$, $\tan A$ and $\cos \frac{A}{2}$

11. A rise of level is given by $100 \sin a \cos a \times s$ where s = difference between the readings of the top and bottom hairs of a tacheometric telescope. Express this statement in a more convenient form.

If the angle of elevation a is $11^\circ\ 37'\ 30''$, and the staff readings are $5{\cdot}72$ and $8{\cdot}41$, find the rise.

12. Express as products, and in forms convenient for computation: (*a*) $\sin 48° - \sin 17°$; (*b*) $\cos 99° + \cos 176°$; (*c*) $12 \cos 365° - 12 \cos 985°$.

13. When using a tacheometer and a staff it is found that, if C and K are the constants of the instrument, θ is the angle of depression, s the difference of the staff readings, then depth of point below level of station $= \frac{CS}{2} \sin 2\theta + K \sin \theta - E + Q$, and distance of point from station $= CS \cos^2 \theta + K \cos \theta$. Find the depth and the distance when $C = 98{\cdot}87$, $\sin \theta = {\cdot}2753$, $K = {\cdot}75$, $S = {\cdot}69$, $E = 4{\cdot}88$ and $Q = 9{\cdot}55$.

14. If $\tan a = \frac{2{\cdot}5}{1{\cdot}375 - z}$ and $\tan \frac{a}{2} = 2z$, find values of z to satisfy the equations. [Refer to Fig. 154 and the worked *Example* 30.]

15. If $V = 94 \sin 2\pi ft$ and $A = {\cdot}2 \sin (2\pi ft - {\cdot}117)$, express the product AV as a sum or difference.

Trigonometric Equations.—Occasionally one meets with an equation involving some trigonometric ratios; if only these ratios occur, *i. e.*, if no algebraic terms are present in addition, the equations may be solved by the methods here to be detailed.

The relations between the ratios themselves, already given, must be borne in mind, so that the whole expression can be put into terms of one unknown quantity, and the equation solved in terms of that quantity.

For emphasis, the relations between the ratios are here repeated—

$$\tan A = \frac{\sin A}{\cos A}, \quad \cot A = \frac{1}{\tan A}, \quad \sec A = \frac{1}{\cos A}, \quad \operatorname{cosec} A = \frac{1}{\sin A}$$

$$\sin^2 A + \cos^2 A = 1, \quad \text{whence } \sin^2 A = 1 - \cos^2 A$$

$$\text{or} \quad \cos^2 A = 1 - \sin^2 A$$

$$\sec^2 A = 1 + \tan^2 A, \quad \operatorname{cosec}^2 A = 1 + \cot^2 A.$$

The idea in the solution of these trigonometric equations is to eliminate all the unknowns except one, by the use of the above relations, and then to apply the ordinary rules of equations to determine the value of that unknown.

Example 34.—Solve the equation $4 \sin \theta = 3{\cdot}5$.

$$4 \sin \theta = 3{\cdot}5$$

$$\text{and} \qquad \sin \theta = \frac{3{\cdot}5}{4} = {\cdot}875.$$

Hence one value of θ, viz. the simplest, is $61° \, 3'$,

$$\text{since} \qquad \sin 61° \, 3' = {\cdot}875$$

$$\text{but} \qquad \sin (180 - 61° \, 3'), \ \textit{i. e.}, \sin 118° \, 57', \text{ also} = {\cdot}875,$$

so that a possible solution is $118° \, 57'$.

Again, $360° + 61° 3'$ or $360° + 118° 57'$ would also satisfy, and so an infinite number of solutions could be found; but whilst these could all be included in one formula, it is not at all necessary from the engineer's standpoint that they should be, for, at the most, the angles of a circle, viz. $0°$ to $360°$, are all that occur in his problems.

Hence, throughout this part of the work the range of angles will be understood to be $0°$ to $360°$.

$\therefore$ The solutions in this example are $\underline{61° 3' \text{ and } 118° 57'}$.

Example 35.—If $\tan\theta = 5\sin\theta$, determine values of θ to satisfy the equation.

Apparently, in this one equation two unknowns occur, or the data are insufficient, but in reality two equations are given, for $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Thus—
$$\frac{\sin\theta}{\cos\theta} = 5\sin\theta.$$

Dividing through by $\sin\theta$ [and in doing this we must put $\sin\theta = 0$ as a possible solution, since $\dfrac{0}{\cos\theta} = 0$ and $5 \times 0 = 0$]—

Then—
$$\frac{1}{\cos\theta} = 5$$
$$\cos = \cdot 2 = \cos 78° 28'$$

$\therefore$ $\underline{\theta = 0°;\ \text{or } 180°;\ \text{or } 78° 28';\ \text{or } 360° - 78° 28',\quad i.e.,\ 281° 32'.}$

Example 36.—Solve the equation $\sin\theta + \tan\theta = 3\cos\theta\sin\theta$.

$$\sin\theta + \tan\theta = 3\cos\theta\sin\theta$$

By substituting for $\tan\theta$ its value—

$$\sin\theta + \frac{\sin\theta}{\cos\theta} = 3\cos\theta\sin\theta$$

or
$$\frac{\sin\theta}{\cos\theta}\{\cos\theta + 1\} = 3\cos\theta\sin\theta.$$

Dividing through by $\sin\theta$ $\{\sin\theta = 0$ thus being one solution$\}$ and multiplying through by $\cos\theta$—

$$\cos\theta + 1 = 3\cos^2\theta$$

or
$$3\cos^2\theta - \cos\theta - 1 = 0.$$

It may appear easier to solve this equation if X is written for $\cos\theta$—

i. e.,
$$3X^2 - X - 1 = 0$$

whence
$$X = \frac{+1 \pm \sqrt{1 + 12}}{6}$$
$$= \frac{1 \pm 3\cdot 606}{6}$$
$$= \frac{4\cdot 606}{6} \quad \text{or} \quad \frac{-2\cdot 606}{6}$$
$$= \cdot 7677 \quad \text{or} \quad -\cdot 4343$$

$$\therefore \quad \cos\theta = \cdot 7677 \quad \text{or} \quad \cos\theta = -\cdot 4343.$$

Now for the cosine to be positive, the angle lies in the first and fourth quadrants; and, since the smallest angle having its cosine $= \cdot 7677$ is $39° 51'$, the values of θ are $39° 51'$ or $360° - 39° 51'$, *i. e.*, $320° 9'$.

For the cosine to be negative, the angle lies in the second and third quadrants. Now $\cos 64° 15' = \cdot 4343$, and therefore the values of θ are $180° - 64° 15'$, *i. e.*, $115° 45'$, or $180° + 64° 15'$, *i. e.*, $244° 15'$.

Hence the solutions are—

$$\underline{\theta = 0°;\ 180°;\ 39° 51';\ 320° 9';\ 115° 45' \text{ or } 244° 15'.}$$

Example 37.—The velocity of the piston of a reciprocating engine is given by the formula—

$$v = 2\pi nr\left(\sin\theta + \frac{r\sin 2\theta}{2l}\right)$$

where r = crank radius; n = R.P.M., θ = crank angle from dead centre position, and l = length of connecting-rod.

The velocity is a maximum when $\cos\theta + \frac{r}{l}\cos 2\theta = 0$; find the crank angles for the maximum velocity when $l = 8r$.

We require to solve the equation—

$$\cos\theta + \frac{\cos 2\theta}{8} = 0.$$

To change into terms of $\cos\theta$ write $2\cos^2\theta - 1$ in place of $\cos 2\theta$.

Then—

$$\cos\theta + \frac{2\cos^2\theta - 1}{8} = 0$$

$$8\cos\theta + 2\cos^2\theta - 1 = 0$$

or

$$2X^2 + 8X - 1 = 0 \quad \text{where } X = \cos\theta.$$

The solutions of this equation are given by—

$$\begin{aligned} X &= \frac{-8 \pm \sqrt{64 + 8}}{4} \\ &= \frac{-8 \pm 8{\cdot}485}{4} \\ &= \frac{-16{\cdot}485}{4} \text{ or } \frac{\cdot 485}{4} \\ &= -4{\cdot}1212 \text{ or } \cdot 1212, \end{aligned}$$

which are the values of $\cos\theta$.

But $\cos\theta$ cannot $= -4{\cdot}1212$, since $\cos\theta$ is never greater than 1, hence the first root is disregarded.

$\therefore$ $\cos\theta = \cdot 1212$, which gives the required solutions,

i. e., $\theta = 83° 2'$ or $276° 58'$.

If a skeleton diagram is drawn it will be observed that when θ has these values the crank and connecting-rod are very nearly at right angles to one another.

U

Exercises 33.—On the Solution of Trigonometric Equations.

Solve the equations (for angles between 0° and 360°).

1. $\sin^2 A + 2 \sin A = 2 - \cos^2 A$. **2.** $2 \sin^2 \theta + 4 \cos^2 \theta = 3$.

3. $\cos \theta + 6 \cos^2 \frac{\theta}{2} = 1$. **4.** $\cot \theta - 14 \tan \theta = 5$.

5. $2 \sin^2 \theta - 5 \cos \theta = 4$. **6.** $15 \cos^2 \theta + 9 \sin \theta = 12{\cdot}6$.

7. $\tan x \tan 2x = 1$. **8.** $\tan A + 3 \cot A = 4$.

9. $\cos^2 A + 2 \sin^2 A - 2{\cdot}5 \sin A = 0$.

10. $\cos x \tan x = {\cdot}5842$. **11.** $3 \tan^2 B - 2 \tan B - 1 = 0$.

12. $\tan x + \cot x = 2$. **13.** $3 \tan^2 \theta + 1 = 4 \tan \theta$.

14. $\cos^2 x = 3 \sin^2 x$. **15.** $\cos 2x + \sin 2x = 1$.

16. $\cos x + \sqrt{3} \sin x = 1$. **17.** $\cos x - \sin x = \frac{7}{16}$.

18. $2{\cdot}35 \sin x - 1{\cdot}72 \cos x = {\cdot}64$.

19. The velocity of a valve actuated by a particular Joy valve gear is maximum when—

$$1{\cdot}2p^2 \cos pt + 1{\cdot}8p^2 \sin pt = 0$$

where p = angular velocity of the crank shaft.

Find the values of the angle pt for maximum velocity.

20. To find the maximum bending moment on a circular arch it is necessary to solve the equation—

$$-wR^2 \sin \theta \cos \theta + {\cdot}934wR^2 \sin \theta = 0.$$

Find values of θ to satisfy this equation.

21. The following equation occurred when taking soundings from a boat, the position of the boat being fixed by reference to three points on the shore. [Compare Exercise 31, p. 272.]

$$\sin (70° 14' + x) = 1{\cdot}195 \sin (48° 56' + x).$$

Find the value of x to satisfy this equation, x being an acute angle.

Hyperbolic Functions.—Consider the circle of unit radius (Fig. 155) and the rectangular hyperbola whose half-axes are also unity (Fig. 156), *i. e.*, OA in either case = 1.

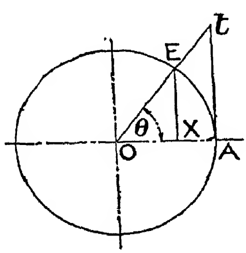


Fig. 155.

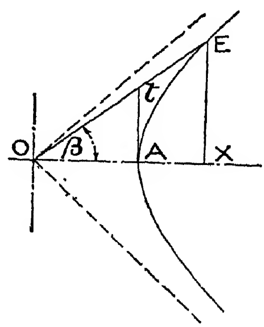


Fig. 156.

Draw any angle EOA in each diagram, and let the "circle angle" EOA = θ, and let the "hyperbola angle" EOA = β.

The angle is defined in either "circular" or "hyperbolic" radians by—

$$\frac{\text{length of circular or hyperbolic arc}}{\text{mean length of radius vector}}; \text{ or by } 2 \times \text{area of sector OAE}$$

Now in Fig. 155 $\frac{EX}{OE} = EX = \sin\theta$, and the corresponding ratio, viz. EX, of the hyperbolic angle is termed **sinh** β.*

Similarly—

$OX = \cos\theta$ in Fig. 155 and $OX = \mathbf{cosh}\ \beta$ in Fig. 156
$At = \tan\theta$ in Fig. 155 and $At = \mathbf{tanh}\ \beta$ in Fig. 156.

In Fig. 155— $(EX)^2 + (OX)^2 = 1$

i. e., $\sin^2\theta + \cos^2\theta = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

In Fig. 156 $(OX)^2 - (EX)^2 = 1$, since the equation of the rectangular hyperbola is $x^2 - y^2 = 1$ if the semi-axes are each equal to unity and the centre is taken as the origin.

Hence— $\cosh^2\beta - \sinh^2\beta = 1$

or $\cosh^2\beta + (-1 \times \sinh^2\beta) = 1$

i. e., $\cosh^2\beta + (\sqrt{-1} \times \sinh\beta)^2 = 1$

or $\cosh^2\beta + (j \sinh\beta)^2 = 1$

where j is written to indicate $\sqrt{-1}$.

Comparing the last equation with equation (1), we see that we may change from circular to hyperbolic functions if we write $j \sinh\beta$ for $\sin\theta$, and $\cosh\beta$ for $\cos\theta$, and hence $j \tanh\beta$ for $\tan\theta$.

If these substitutions are made, the ordinary rules for circular functions follow.

E. g., $\sin(x + y) = \sin x \cos y + \cos x \sin y$

and the corresponding expansion with hyperbolic functions is—

$j \sinh(X+Y) = j \sinh X \cosh Y + \cosh X \,.\, j \sinh Y$

or $\sinh(X+Y) = \sinh X \cosh Y + \cosh X \sinh Y$

or again, $\cos 2x = -\sin^2 x + \cos^2 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad$ see p. 280

i. e., $\cosh 2X = -(j \sinh X)^2 + (\cosh X)^2$
$= \sinh^2 X + \cosh^2 X$, since $j^2 = (\sqrt{-1})^2 = -1$.

It can be shown that these hyperbolic functions can be expressed in terms of the exponentials in the forms—

$$e^{-x} = \cosh x - \sinh x$$
$$e^{x} = \cosh x + \sinh x$$

i. e., $$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \ldots\ldots$$

and $$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \ldots\ldots$$

* To avoid confusing with the circular functions, *sinh* is usually pronounced "shine," and *tanh* "tank."

The corresponding relations for the circular functions are—

$$e^{jx} = \cos x + j \sin x$$
$$e^{-jx} = \cos x - j \sin x$$
$$\cos x = \frac{e^{jx} + e^{-jx}}{2} = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots\ldots$$
$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots\ldots$$

Hyperbolic functions occur frequently in engineering theory; *e. g.*, in connection with the whirling of shafts the equation—

$$y = A \cos mx + B \sin mx + C \cosh mx + D \sinh mx$$

plays a most important part: the equation of the catenary is $y = \cosh x$; and so on.

It is in electrical work that these functions occur most frequently; thus, for a long telegraph line having a uniform linear leakage to earth by way of the poles the diminishing of the voltage is represented by a curve of the form $y = \cosh x$, whilst the curve $y = \sinh x$ represents the current.

Example 38.—A cable weighing 3 lbs. per foot hangs from two points on the same level and 60 feet apart; and it is strained by a horizontal pull of 300 lbs. The form taken by the cable is a catenary. Find the length of the cable from the formula—

$$\text{Length} = 2c \sinh \frac{L}{2c}$$

$$\text{where } L = \text{span and } c = \frac{\text{horizontal tension}}{\text{weight of 1 foot of cable}}$$

Here we have $L = 60$ and $c = \frac{300}{3} = 100$, hence $\frac{L}{2c} = \frac{60}{200} = \cdot 3$

Thus— length of cable $= 2 \times 100 \sinh \cdot 3$

Table XI at the end of the book may be utilised to find the value of sinh ·3, in the following manner: Look down the first column until ·3 is seen as the value for x: follow the line in which this value occurs until the column headed sinh x is reached. The value there shown is that of sinh ·3 and is ·3045.

Hence length of cable $= 200 \times \cdot 3045 = \underline{60 \cdot 9 \text{ ft.}}$

This rule gives the exact length of the cable, but in practice the form of the cable is assumed to be parabolic, and the approximate length is given by—

$$\text{Length} = \text{span} + \frac{8\,(\text{sag})^2}{3 \text{ span}}$$

the sag also being calculated on the assumption of the parabolic form of the cable. In this case the sag is found to be 4·5 ft. and hence—

$$\text{length of cable} = 60 + \frac{8 \times (4\cdot5)^2}{3 \times 60} = 60\cdot9$$

In this instance the result obtained by the true and approximate methods agree exactly: and in the majority of cases met with in practice the approximate rule gives results sufficiently accurate.

Example 39.—The resistance of the conductor of a certain telegraph line is 8·3 ohms per kilometre and the insulation resistance is 600 megohms per km. The difference in potential E between the line and earth at distance L kms. from the sending end is found from the formula—

$$E = A \cosh \sqrt{rl}\,.\,L + B \sinh \sqrt{rl}\,.\,L$$

where A and B are constants, r = resistance of unit length of the conductor and l = conductance of unit length of the path between the line and earth.

If the total length of the line is 100 kms., the voltage at the sending end is 110, and at the receiving end is 85, find the values of A and B.

We have two unknowns and we must therefore form two equations. At the sending end— $L = 0$

and then— $$\begin{aligned} 110 &= A \cosh \sqrt{rl}\,.\,0 + B \sinh \sqrt{rl}\,.\,0 \\ &= A \cosh 0 + B \sinh 0 = A \times 1 = A \end{aligned}$$

Hence— $\underline{A = 110.}$

Now $rl = \dfrac{8\cdot3}{600 \times 10^6} = 1\cdot383 \times 10^{-8}$ and $\sqrt{rl} = \cdot0001176$; also at a distance of 100 kms from the sending end the value of E is to be 85. Substituting these numerical values in the original equation—

$$\begin{aligned} 85 &= 110 \cosh (\cdot0001176 \times 100) + B \sinh (\cdot0001176 \times 100) \\ &= 110 \cosh \cdot01176 + B \sinh \cdot01176 \quad \ldots\ldots\ldots\ldots \quad (1) \end{aligned}$$

In order to solve this equation for B the values of cosh ·01176 and sinh ·01176 must first be found; and as the given tables of values of cosh and sinh are not convenient for this purpose we proceed according to the following plan—

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}; \quad e^x = e^{\cdot01176};$$

and to evaluate we must take logs.

Let $y = e^{\cdot01176}$ and then—

$$\begin{aligned} \log y &= \cdot01176 \times \log e \\ &= \cdot01176 \times \cdot4343 = \cdot0051 \end{aligned}$$

so that— $y = 1\cdot012.$

Thus $e^{\cdot01176} = 1\cdot012$ and $e^{-\cdot01176}$ which is the reciprocal of $e^{\cdot01176}$ is ·9883.

Then— $\cosh \cdot 01176 = \dfrac{1 \cdot 012 + \cdot 988}{2} = 1$

and $\sinh \cdot 01176 = \dfrac{1 \cdot 012 - \cdot 988}{2} = \cdot 012$

Substituting these values in equation (1)—

$$85 = (110 \times 1) + (B \times \cdot 012)$$

whence $\cdot 012\ B = -25$

or $\underline{B = -2083}$

Hence— $\underline{E = 110 \cosh \cdot 0001176\ L - 2083 \sinh \cdot 0001176\ L.}$

Complex Quantities.—Algebraic quantities generally may be divided into two classes, *real* and *imaginary*, and the former of these

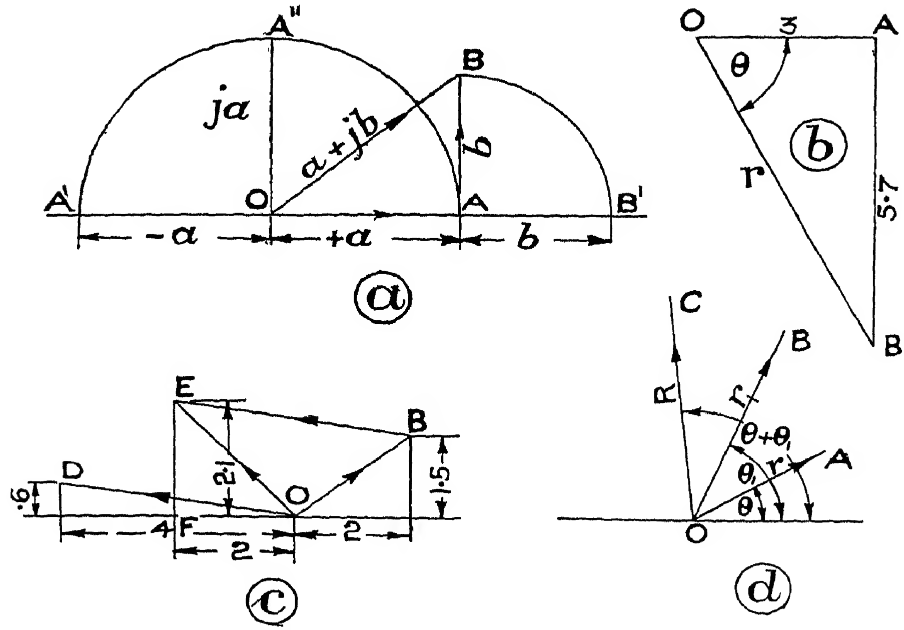


Fig. 157.—Complex and Vector Quantities.

may be further subdivided into *rational* and *irrational or surd* quantities. Thus, $\sqrt{5}$ and also $7a$ are real, whilst $\sqrt{-15}$ is imaginary; indeed, all quantities involving the square root of a negative quantity are classed as imaginary. An expression that is partly real and partly imaginary is spoken of as a *complex quantity*; thus $4 + 7\sqrt{-9}$, and $\sqrt{2x}+16\sqrt{-2y}$ are complex quantities. The first expression might be written as $4 + (7 \times \sqrt{9} \times \sqrt{-1})$, *i. e.*, $4+21j$ where j stands for $\sqrt{-1}$. The general form for

these complex expressions is usually taken as $a + jb$, where a and b may have any real values.

According to the ordinary convention of signs, if OA ((*a*) Fig. 157) represents $+ a$ units, then OA′ would stand for $-a$ if the length of OA′ were made equal to that of OA; in other words, to multiply by -1, revolution has been made through two right angles. Now $a \times \sqrt{-1} \times \sqrt{-1} = -a$, so that the multiplication by $\sqrt{-1}$ must involve a revolution one-half of that required for the multiplication by -1; or OA″ must represent ja. Accordingly a meaning has been found for the imaginary quantity j, and that is: If $+a$ is measured to the right and $-a$ is measured to the left, from a given origin, then ja must be measured upward, and differs from the other quantities only in direction, which is 90° from either $+a$ or $-a$.

To represent $a + jb$ on a diagram, therefore, we must set out a distance OA to represent a, erect a perpendicular AB making AB equal to b, choosing the same scale as that used for the horizontal measurement, and then join OB; then $OB = a + jb$. For OB′ would represent $a + b$ and $AB = j \,.\, AB'$, so that OB must be the result of the addition of a to jb. The addition is not the simple addition with which we have been familiar, but is spoken of as **vector addition, *i. e.*, addition in which attention is paid to the direction in which the quantity is measured as well as to its magnitude.**

It is often necessary to change from the form $a + jb$ to the form $r(\cos\theta + j \sin\theta)$; and this can be done in the following way—

If $r(\cos\theta + j \sin\theta)$ is to be *identically* the same as $a+jb$ then the *real* parts of each must be equal, and also the *imaginary*.

$$i.\,e., \qquad r\cos\theta = a \qquad \dots\dots (1)$$

$$rj\sin\theta = jb$$

$$\text{whence} \qquad r\sin\theta = b \qquad \dots\dots (2)$$

By division of (2) by (1) $\quad \tan\theta = \dfrac{b}{a}$

and by squaring both (1) and (2) and adding—

$$r^2\cos^2\theta + r^2\sin^2\theta = a^2+b^2$$

$$\text{or} \qquad r^2 = a^2+b^2 \qquad \text{since } \cos^2\theta + \sin^2\theta = 1$$

Thus, at (*b*) Fig. 157—

$$OB = r$$

$$\text{and} \qquad \angle BOA = \theta$$

Example 40.—To change $3 - 5{\cdot}7j$ into the form $r(\cos\theta + j\sin\theta)$.

From the above, since $a = 3$, and $b = -5{\cdot}7$

$$r^2 = 3^2 + (-5{\cdot}7)^2 = 9 + 32{\cdot}4 = 41{\cdot}4$$

$$\therefore \quad r = 6{\cdot}44$$

and $\quad \tan\theta = -\frac{5{\cdot}7}{3} = -1{\cdot}9 \quad$ or $\quad \theta = -62^\circ 14\tfrac{1}{2}'$.

This case is illustrated at (*b*) Fig. 157, in which OB represents r; and the angle BOA is the angle θ.

Vector quantities, such as forces, velocities, electrical currents and pressures, may be combined by either graphic or algebraic methods; in the algebraic addition, for example, the components along two directions at right angles are added to give the components along these axes of the resultant. Thus if the vector $2+1{\cdot}5j$ were added to the vector $-4+{\cdot}6j$, the resultant vector would be $2-4+1{\cdot}5j+{\cdot}6j$, i.e., $-2+2{\cdot}1j$. The addition is really simpler to perform by the graphic method, thus: OB at (*c*) Fig. 157 represents the vector quantity $2+1{\cdot}5j$ and OD represents $-4+{\cdot}6j$. Through B draw BE parallel and equal to OD and join OE; then OE is the resultant of OB and OD. It will be seen that OE is the vector $-2+2{\cdot}1j$ since OF $= 2$ units measured in a negative direction and FE $= 2{\cdot}1$ units.

To multiply complex quantities.—Let OA ((*d*) Fig. 157) represent—

$$a + jb, \quad \text{i. e.,} \quad r(\cos\theta + j\sin\theta)$$

and let OB represent— $\quad a_1 + jb_1, \quad$ i. e., $\quad r_1(\cos\theta_1 + j\sin\theta_1)$

Then:

$$\begin{aligned}
\text{OA} \times \text{OB} &= (r\cos\theta + rj\sin\theta)(r_1\cos\theta_1 + r_1 j\sin\theta_1) \\
&= rr_1\cos\theta\cos\theta_1 + rr_1 j\sin\theta_1\cos\theta + \\
&\qquad rr_1 j\sin\theta\cos\theta_1 + rr_1 j^2\sin\theta\sin\theta_1 \\
&= rr_1(\cos\theta\cos\theta_1 - \sin\theta\sin\theta_1) + \\
&\qquad rr_1 j(\sin\theta\cos\theta_1 + \cos\theta\sin\theta_1) \\
&= rr_1\left\{\cos(\theta + \theta_1) + j\sin(\theta + \theta_1)\right\}
\end{aligned}$$

To divide complex quantities.—Let $\dfrac{a + jb}{a_1 + jb_1} = \dfrac{r(\cos\theta + j\sin\theta)}{r_1(\cos\theta_1 + j\sin\theta_1)}$

Rationalise the denominator by multiplying by $(\cos\theta_1 - j\sin\theta_1)$

Then

$$\begin{aligned}
\frac{a + jb}{a_1 + jb_1} &= \frac{r(\cos\theta + j\sin\theta)(\cos\theta_1 - j\sin\theta_1)}{r_1(\cos^2\theta_1 + \sin^2\theta_1)} \\
&= \frac{r}{r_1}\left\{\cos(\theta + \theta_1) + j\sin(\theta - \theta_1)\right\}
\end{aligned}$$

which can be expressed in the form A + jB if desired.

These results might have been arrived at by expressing $r(\cos\theta + j\sin\theta)$ as $re^{j\theta}$ and $r_1(\cos\theta_1 + j\sin\theta_1)$ as $r_1e^{j\theta_1}$.

Thus

$$\begin{aligned}
(a + jb)(a_1 + jb_1) &= re^{j\theta} \times r_1e^{j\theta_1} = rr_1e^{j(\theta+\theta_1)} \\
&= rr_1\left\{\cos(\theta + \theta_1) + j\sin(\theta + \theta_1)\right\}
\end{aligned}$$

Example 41.—The electric current C in a star-connected lighting system was measured by the product—Potential P $\times$ admittance y. If P = $(\cdot 068 - \cdot 0015j)(28 + 30j)$ and $y = \cdot 9 + \cdot 18j$ find C.

$$P = (\cdot 068 - \cdot 0015j)(28 + 30j) = 1\cdot 949 + 1\cdot 998j \text{ (by actual multiplication).}$$
$$= 2\cdot 791(\cos 45^\circ\ 43' + j \sin 45^\circ\ 43')$$
$$y = \cdot 9 + \cdot 18j = \cdot 9179\ (\cos 11^\circ\ 19' + j \sin 11^\circ\ 19')$$
$$\therefore C = Py = \underline{2\cdot 563\ (\cos 57^\circ\ 2' + j \sin 57^\circ\ 2')}$$ or, alternatively, $\underline{1\cdot 395 + 2\cdot 149j.}$

Inverse Trigonometric Functions.—If $\sin x = y$, then x is the angle whose sine is y, and this statement may be expressed in the abbreviated form $x = \sin^{-1} y$. (Note that $\sin^{-1} y$ does not mean $\frac{1}{\sin y}$, but the -1 indicates a converse statement, y being the value of the sine and not the angle.)

$\sin^{-1} y$ is called an *inverse circular function.*

Similarly $\sinh^{-1} y$ is called an *inverse hyperbolic function*

Angles are sometimes expressed in this way instead of in degrees; *e. g.*, when referring to the angle of friction for two surfaces: if the coefficient of friction between the surfaces is given, that is the value of the tangent of the angle of friction, and the angle of friction $= \tan^{-1}\mu$, where μ is the coefficient of friction.

Example 42.—Given $\sin^{-1}x = y$, find the values of $\cos y$ and $\tan y$.

$$\sin^{-1} x = y$$
i. e.,
$$\sin y = x$$
$\therefore$
$$\cos y = \sqrt{1 - \sin^2 y} = \underline{\sqrt{1 - x^2}}$$
and
$$\tan y = \frac{\sin y}{\cos y} = \underline{\frac{x}{\sqrt{1 - x^2}}}$$

Example 43—The transformation from the hyperbolic to the logarithmic form occurs when concerned with a certain integration If $\cosh y = x$, show that $\cosh^{-1}x = \log_e(x + \sqrt{x^2 - 1})$.

$$\cosh y = \frac{e^y + e^{-y}}{2}$$
i e.,
$$x = \frac{e^y + e^{-y}}{2}$$
or
$$2x = e^y + e^{-y} = e^y + \frac{1}{e^y}$$
whence
$$e^{2y} - 2xe^y + 1 = 0$$
$\therefore$
$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} \quad \text{(Solving the quadratic)}$$
$$= x \pm \sqrt{x^2 - 1}$$
$$= x + \sqrt{x^2 - 1} \quad \text{or} \quad \frac{1}{x + \sqrt{x^2 - 1}}$$

$$\left\{\begin{array}{l}\text{since } x - \sqrt{x^2 - 1} = \dfrac{1}{x + \sqrt{x^2 - 1}} \\ \text{as is seen if we multiply across}\end{array}\right\}$$

$$\therefore \quad \log_e (x + \sqrt{x^2 - 1}) = y, \quad \text{or} \quad \log_e \left(\frac{1}{x + \sqrt{x^2 - 1}}\right) = y$$

$$i.e., \qquad y = \pm \log_e (x + \sqrt{x^2 - 1})$$

or if only the positive root is taken—

$$y = \log_e (x + \sqrt{x^2 - 1})$$

$$\therefore \qquad \cosh^{-1} x = \log_e (x + \sqrt{x^2 - 1})$$

In like manner it can be proved that—

$$\sinh^{-1} \frac{x}{a} = \log_e \frac{x + \sqrt{x^2 + a^2}}{a}$$

$$\cosh^{-1} \frac{x}{a} = \log_e \frac{x + \sqrt{x^2 - a^2}}{a}$$

Exercises 34.—On Hyperbolic and Inverse Trigonometric functions.

1. Read from the tables the values of $\cosh \cdot 7$ and $\sinh 1 \cdot 5$.

2. Evaluate $5 \cosh \cdot 015 + \cdot 1 \sinh \cdot 015$.

3. Find the true length of a cable weighing 1·8 lbs. per foot, the ends being 120 ft. apart horizontally and the straining force being 90 lbs. weight. [Refer to worked *Example* 38, p. 292.]

4. Calculate the sag of the cable in Question 3, from the rule—

$$\text{sag} = c\left(\cosh \frac{\text{span}}{2c} - 1\right)$$

$$\text{where—} \qquad c = \frac{\text{straining force in lbs. wt.}}{\text{wt. of 1 ft. of cable}}$$

Hence find the approximate length of the cable, from the rule—

$$\text{length} = \text{span} + \frac{8 \times (\text{sag})^2}{3 \times \text{span}}$$

5. The E.M.F. required at the transmission end of a track circuit used for signalling can be found from—

$$E_1 = \frac{E}{2}\left(e^{x\sqrt{\frac{r}{r_1}}} + e^{-x\sqrt{\frac{r}{r_1}}}\right) + \frac{rr_1}{2\left(\frac{r}{r_1}\right)^{\frac{1}{2}}}\left(e^{x\left(\frac{r}{r_1}\right)^{\frac{1}{2}}} - e^{-x\left(\frac{r}{r_1}\right)^{\frac{1}{2}}}\right)$$

Put this expression into a simpler form, viz. one involving hyperbolic functions.

6. If the "angle of friction" for iron on iron is $\tan^{-1} \cdot 19$, find this angle.

7. The lag in time between the pressure and the current in an alternating current circuit is given by $\dfrac{\text{period}}{360} \times \tan^{-1} \dfrac{2\pi n L}{R}$ where n = number of cycles per second, L = self-induction of circuit and R = resistance of circuit, the angle being expressed in degrees. If the frequency is 60 cycles per second, L = ·025 and R = 1·2, find the lag in seconds.

8. If $\cosh y = 1{\cdot}4645$, find the positive value of y.

9. A block is subjected to principal stresses of [illegible]55 lbs. and 171 lbs., both tension. The inclination of the resultant stress on a plane inclined at $27°$ to the plane of the greater stress is $\tan^{-1}\left(\frac{f_2}{f_1}\tan\theta\right)$ where f_1 and f_2 are the greater and lesser stresses respectively and θ is the inclination of the plane. Find the inclination of the resultant stress for this case.

10. The solution of a certain equation by two different methods gave as results—

$$s = -\frac{5}{53}\sin\left(7t + \tan^{-1}\frac{28}{45}\right) \quad \text{and } s = \frac{5}{53}\sin\left(7t - 2\tan^{-1}\frac{7}{2}\right)$$

respectively. By finding the numerical values of the angles $\tan^{-1}\frac{28}{45}$ and $\tan^{-1}\frac{7}{2}$, show that the two results agree.

11. The following equation occurred in connection with alternator regulation—

$$a = \theta + \phi$$

If $\theta = \sin^{-1}\frac{342}{6180}$ and $\phi = \cos^{-1}{\cdot}55$, find $\sin a$.

12. The speed V knots of waves over the bottom in shallow water is calculated from—

$$V^2 = 1{\cdot}8L\tanh\frac{6{\cdot}3d}{L}$$

where d = depth in feet
L = wave length in feet.

If $d = 40$ ft., and $L = 315$ ft., calculate the value of V.

13. By calculating the values of the angles (in radians) prove the truth of the following relations :—

$$\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3} = \frac{\pi}{4}$$

$$4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239} = \frac{\pi}{4}$$

$$4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{70} + \tan^{-1}\tfrac{1}{99} = \frac{\pi}{4}$$

Illustrate the first of these by a diagram.

14. An equation occurring in the calculation of the arrival current in a telegraph cable contained the following :—

$$N = \frac{9b}{10} - \frac{3}{2}\cos 2a \sinh 2b - \frac{b}{10}\cos 2a \cosh 2b.$$

Find N when $a = 4{\cdot}5$ and $b = 2$.

15. If $C = 5\ 4\,(\cos 62° + j\sin 62°)$ and $y = 1{\cdot}8\,(\cos 12° + j\sin 12°)$ find P (in the form $a + jb$). The letters have the same meanings as in Example 41, p. 297.

CHAPTER VII

AREAS OF IRREGULAR CURVED FIGURES

Areas of Irregular Curved Figures.—Rules have already been given (see Chapter III) for finding the areas of irregular figures bounded by straight sides; if, however, the boundaries are not straight lines, such rules only apply to a limited extent.

The mean pressure of a fluid such as steam or gas on a piston is found from the area of the "indicator diagram," the figure automatically drawn by an engine "indicator," correlating pressure and volume. By far the quickest and most accurate method of determining the area of this diagram is (*a*) to use an instrument called the **planimeter** or integrator. Other methods are (*b*) the averaging of boundaries, (*c*) the counting of squares, (*d*) the use of the computing scale, (*e*) the trapezoidal rule, (*f*) the mid-ordinate rule, (*g*) Simpson's rule and (*h*) graphic integration.

To deal with these methods in turn:—

(*a*) **The Planimeter.**—The Amsler planimeter is the instrument most frequently employed, on account of its combination of simplicity and accuracy. It consists essentially of two arms, at the end of one of which is a pivot O (see Fig. 158), whilst at the end of the other is the tracing-point P. By unclamping the screw B the length of the arm AP can be varied, fine adjustment being made by the adjusting screw C: and this length AP determines the scale to which the area is read. The rim of the wheel W rotates or partially glides over the paper as the point P is guided round the outline of the figure whose area is being measured; the pivot O being kept stationary by means of a weight. The motion of the wheel W is measured on the wheel N in integers, and on the wheel D in decimals, further accuracy being ensured by the use of the vernier V.

To use in the ordinary manner, the pivot O being outside the figure. By rough trial find a position for the pivot so that the figure can be completely traversed in a comfortable manner. Mark some convenient starting-point on the boundary of the figure and

place the tracer on that spot. Note the reading; say 2 on the scale of integers, 48 on the scale of decimals and 3 on the vernier; or 2483 altogether. Trace carefully round the boundary in a right-handed direction until the starting-point is again reached. Again note the reading; let it be 3327. Then subtract the initial reading from the final and the area of the plane figure is found. In this case the area would be 844 sq. units.

Along the arm AP are marks for adjustment to different scales. If A is set at one of these marks the area will be in sq. ins., at

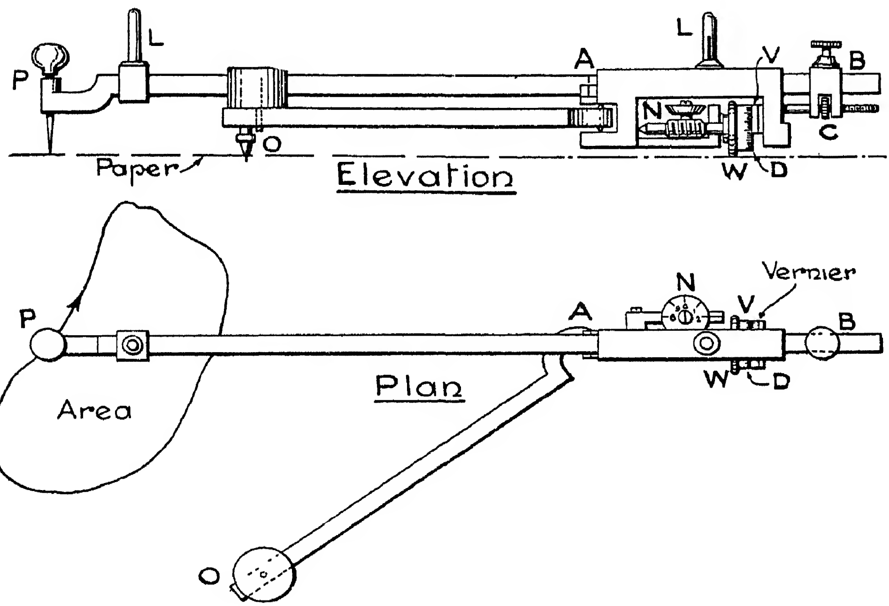


Fig 158.—The Amsler Planimeter.

another, in sq. cms., etc.; but if there be any doubt about the scale, a rectangle, say 3″ × 2″ should be drawn, and the tracer guided round its boundary. Whatever reading is thus obtained must represent 6 sq. ins. so that the reading for 1 sq. in. can be calculated therefrom.

If, in the tracing for which the figures are given above, the zero mark A had been set at the line at which "·01 sq. in." is found on the long bar, then the area would be 8·44 sq. ins., since the divisions on the vernier scale represent ·01 sq in. each.

For large areas it may be found necessary to place the pivot O inside the area; and in such cases the difference between the first and last readings will at first occasion surprise, for it may give an

area obviously much less than the true one. This is accounted for by the fact that under certain conditions, illustrated in Fig. 159, the tracer P traces out a circle, called the **zero circle**, for which the area as registered by the instrument is zero, since the wheel does not revolve at all. For a large area, then, the reading of the instrument may be either the *excess* of the required area over that of the zero circle, or the amount by which it falls short of the zero circle area. These areas are shown respectively at EEE and III in Fig. 159, while the zero circle is shown dotted. [*Note.*—For the ordinary Amsler planimeter the area of the zero circle is about 220 sq. ins., but it is indicated for other units by figures stamped on the bar AP.]

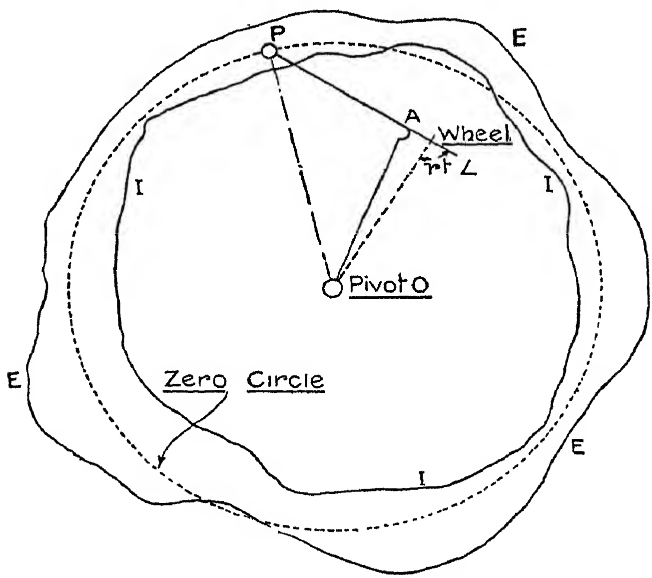


Fig. 159.—Zero Circle of Planimeter.

To use in the special manner.—By means of a set square, adjust the instrument so that the axis of the tracing arm is perpendicular to a line joining the fixed centre O in Fig. 159 to the point of contact of the wheel and the paper. Measure the radius from the fixed centre O to the tracing-point P, and draw a circle with this radius on a sheet of tracing-paper. Place this over the plot whose area is being measured, and endeavour to estimate whether the figure is larger or smaller than the zero circle. If this is at once apparent trace round the figure in the ordinary way and add the area of the zero circle to the reading, or subtract the reading from the zero

circle area as the case may demand. If not apparent, proceed thus—

Set the planimeter to some convenient reading, say 2000 and trace the area in a right-handed direction. Then, if the final reading is greater than 2000, the area is greater than that of the zero circle and *vice versa*. Then to obtain the area—

(1) If the area is *greater* than the zero circle, trace in a right-handed direction and *add* the excess of the last reading over the first to the area of the zero circle; *i. e.*, if x is the excess of the final reading over the initial reading, the true area $= x +$ zero circle area.

(2) If the area is *less* than the area of the zero circle, trace in a *left-handed* direction and *subtract* the difference of the first and last readings from the area of the zero circle. For $-x =$ excess of the last reading over the first, if the tracing is in a right-handed direction, and this becomes $+x$ if the tracing is in a *left-handed* direction. The true area $= -x +$ zero circle area, and the tracing is performed in a left-handed direction in order to get a positive value for x.

If the instrument is to be used as an averager, as would be the case if the mean height of an indicator diagram was required, LL in Fig. 158 must be set to the width of the diagram and the outline must be traced as before. Then the difference of the readings gives the mean height of the diagram. Further reference to the planimeter is made in Volume II of *Mathematics for Engineers*.

The Coffin Averager and Planimeter (Fig. 160) is somewhat simpler in construction as regards the instrument itself, but there are in addition some attachments. It is, in fact, the Amsler instrument with the arm AO (Fig. 158) made infinitely long so that A, or its equivalent, moves in a straight line and not along an arc of a circle.

Referring to Fig. 160 it will be seen that the pointer O is constrained to move along the slot GH.

To use the instrument to find the mean height of a diagram: Trace the diagram on paper and draw a horizontal line and two perpendiculars to this base line to touch the extreme points on the boundary of the figure. Place the paper in such a way that the base line is parallel to the edge of the clip B and set the clips AE and CD along the perpendiculars already drawn. Then start from F, the reading of the instrument being noted, and trace the outline of the figure until F is again reached. Next move the tracing-point along the vertical through F, *i. e.*, keep the tracer P against

the clip, until it arrives at M at which stage the instrument records the initial reading. FM is then the mean height of the diagram.

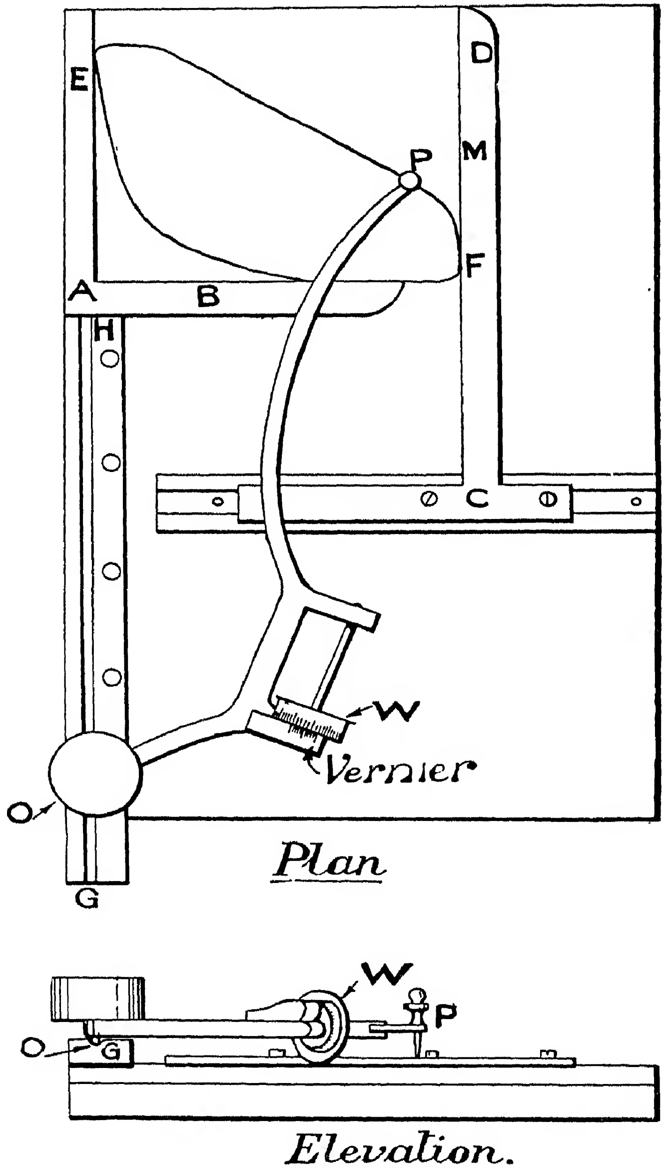


Fig. 160—The Coffin Averager.

If the area of the figure is required, the reading of the instrument must be made when the tracing-point is at F. The outline

of the figure is then traced until F is again reached and the reading is again noted. Then the difference between the two readings is the area of the figure.

(*b*) **Method of averaging Boundaries.**—The area of a figure of the shape bounded by the wavy line in Fig. 161 being required, proceed as follows: Draw the polygon ABCRED so that it shall occupy the same area as the original figure, viz. the portions added are to be equal to those subtracted, as nearly as can be estimated.

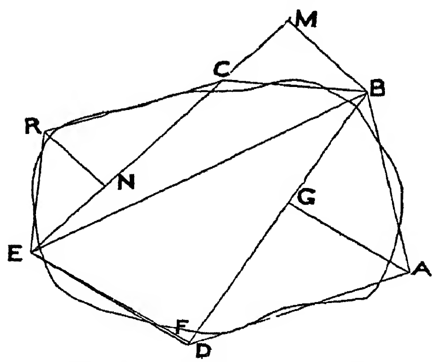


Fig. 161.—Area by Averaging Boundaries.

Then, by joining BD, BE, CE, etc., the polygon is divided into a number of triangles and the area of each is : $\frac{1}{2} \times$ base $\times$ height. Therefore draw the necessary perpendiculars, scale off the lengths of the bases and the heights, and tabulate as follows :—

Triangle.	Base.	Height.	Sum of Heights.	Area of the two triangles = $\frac{1}{2}$ base × sum of heights
ABD BED	BD BD	AG EF	} AG + EF	$\frac{1}{2}$ BD (AG + EF)
CBE CRE	CE CE	BM RN	} BM + RN	$\frac{1}{2}$ CE (BM + RN)

The triangles are thus grouped in pairs and the area of the figure is the sum of the quantities shown in the last column.

(*c*) **Method of counting Squares.**—Draw the figure, whose area is required, on squared paper, choosing some convenient scales. Then count the squares, taking all portions of a square greater than one-half as one, and neglecting all portions smaller than a half-square.

If 1 linear inch represents x units horizontally and y units vertically and the paper is divided into n squares to the linear inch : each square is $\frac{1}{n^2}$ sq. ins., and 1 sq. in. on the paper represents xy sq. units of area, so that each square represents $\frac{xy}{n^2}$ sq. units.

If the total number of squares = N

$$\text{Area of figure} = \frac{Nxy}{n^2} \text{ sq. units}$$

(*d*) **The Computing Scale** is often employed in the drawing office to find the areas of plots of land. It consists of two main parts, viz. a slider A, Fig. 162, and a fixed scale C. The slider can

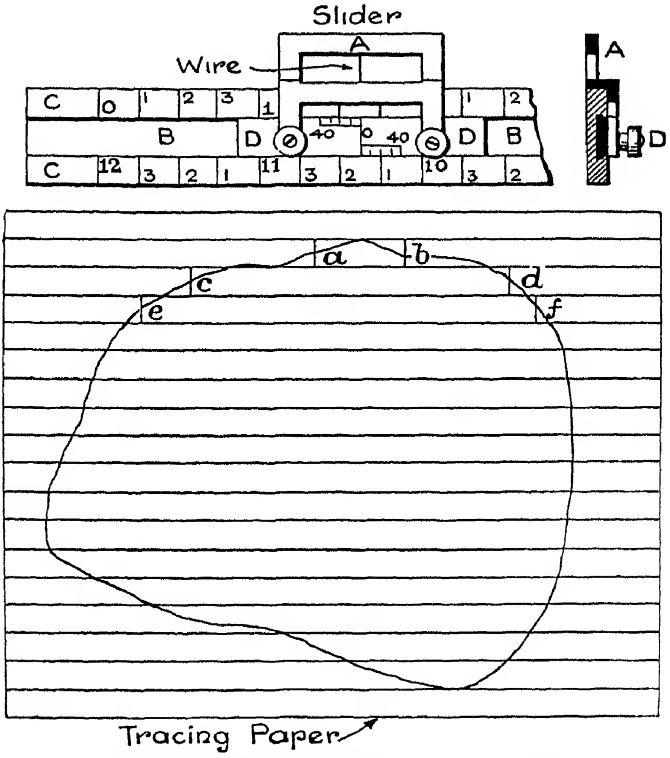


Fig. 162.—Area by the Computing Scale.

be moved along the slot B by means of the handles D; and it carries a vertical wire, which is kept tightly in position by means of screws.

Along the fixed scales are graduations for acres and roods according to a linear scale of 4 chains to 1″, and a scale of square poles, 40 of which make up 1 rood, is indicated on the slider.

To use the instrument.—Rule a sheet of tracing-paper with a number of parallel lines exactly $\frac{1}{4}''$ apart, *i. e.*, 1 chain apart according to the particular scale chosen. Place the tracing-paper

over the plot in such a way that the whole width in any one direction is contained between two of these parallel lines.

Place the slider at the zero mark, and move the whole instrument bodily until the wire at *a* cuts off as much from the area as it adds to it. Next move the slider from left to right until *b* is reached.

Remove the instrument and without altering the position of the wire, place the scale so that the wire is in the position *c*: then run the slider along the slot until the wire arrives at *d*, and so on. Take the final reading of the instrument, and this is the total area of the plot. If the slider reaches the end of the top scale before the area has been completed, the movement can be reversed, *i. e.*, it becomes from right to left and the lower scale must be used.

It will be observed that by the movement of the slider the mean widths of the various strips are added. Now the strips are each 1 chain deep, so that if the mean lengths of the strips measured in chains are multiplied by 1 chain, the total area of the plot is found in square chains. But 10 sq. chains = 1 acre, and the scale to which the plan is drawn is $1'' = 4$ chains. Hence $2\frac{1}{2}'' =$ 10 chains, and the scale must be so divided that $2\frac{1}{2}'' = 1$ acre, since the strip depth is 1 chain.

If the plot is drawn to a scale other than the one for which the scale is graduated the method of procedure is not altered in any way, but a certain calculation must be introduced. Thus if the figure is drawn to a scale of 3 chains to the inch and the computing scale is graduated according to the scale $1'' = 4$ chains, then the true area $= (\frac{3}{4})^2$ or $\frac{9}{16}$ of the registered area.

(*e*) **The Trapezoidal Rule.**—When using this rule divide the base of the figure into a number of equal parts and erect ordinates through the points of division. The strips into which the figure is thus divided are approximately trapezoids. For a figure with a very irregular outline the ordinates should be drawn much closer than for one with a smooth outline. Then the area of the figure is the sum of the areas of the trapezoids, *i. e.*, in Fig. 163,

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}h(y_1+y_2) + \tfrac{1}{2}h(y_2+y_3) + \cdots\cdots\cdots \tfrac{1}{2}h(y_{10}+y_{11})\\ &= h\{\tfrac{1}{2}y_1+y_2+y_3+\cdots\cdots\cdots\cdots y_{10}+\tfrac{1}{2}y_{11}\}\\ &= h\{\tfrac{1}{2}(y_1+y_{11})+y_2+y_3+\cdots\cdots\cdots\cdots y_{10}\}\end{aligned}$$

Or, the area is equal to the length of one division multiplied by the sum of half the first and last ordinates, together with all the remaining ordinates.

Example 1.—Find the area of the figure ABCD in Fig. 163, which is drawn to the scale of half full size.

The base is divided into 10 equal parts and the ordinates are measured. Then the calculation for the area is set down thus—

$y_1 = 2{\cdot}5$	$y_2 = 4{\cdot}40$
$y_{11} = 2{\cdot}0$	$y_3 = 5{\cdot}10$
$\frac{1}{2}$ sum of first and last $= 2{\cdot}25$	$y_4 = 5{\cdot}34$
	$y_5 = 4{\cdot}85$
	$y_6 = 4{\cdot}13$
	$y_7 = 3{\cdot}83$
	$y_8 = 3{\cdot}80$
	$y_9 = 3{\cdot}63$
	$y_{10} = 2{\cdot}55$
	$\frac{1}{2}(y_1 + y_{11}) = 2{\cdot}25$
	Sum $= 39{\cdot}88$

Width of one division of the base $= h = 1''$

$\therefore$ Area $= 1 \times 39{\cdot}88 = \underline{39{\cdot}88}$ sq. ins.

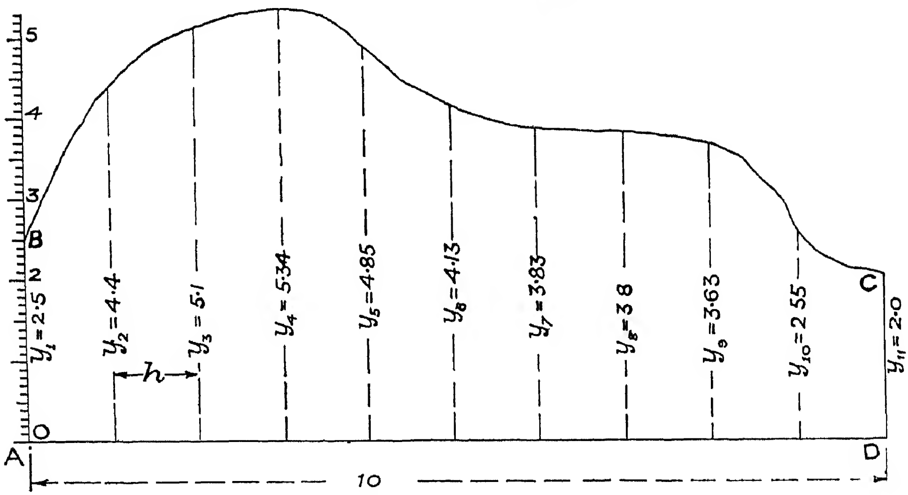


Fig. 163.—Area by Trapezoidal Rule.

(*f*) **The Mid-ordinate rule** is very frequently used and is similar to the trapezoidal rule. The base of the figure is divided into a number of equal parts or strips, and ordinates are erected at the middle points of these strips; such ordinates being called *mid-ordinates* as distinct from the extreme ordinates through the actual points of section. The average of the mid-ordinates multiplied by the length of the base is the area of the figure.

Example 2.—Find the area of the figure ABCD in Fig. 164, which is an exact copy of Fig. 163, and is drawn to the scale of half full size.

The lengths of the mid-ordinates are 3·66, 4·90, 5·24, 5·24, 4·4, 3·92, 3·8, 3·73, 3·3 and 2·13 ins. respectively, and the average $= \frac{40{\cdot}32}{10} = 4{\cdot}032$.

Hence the area = 40·32 sq. ins., as against the previous result of 39·88 sq. ins., showing a difference of 1%.

The mid-ordinate rule is much in vogue on account of its simplicity. As a modification of this method we may ascertain the total area by the addition of the separate strip areas. It is not necessary to divide the base into equal portions: but the divisions may be chosen according to the nature of the bounding curve. If

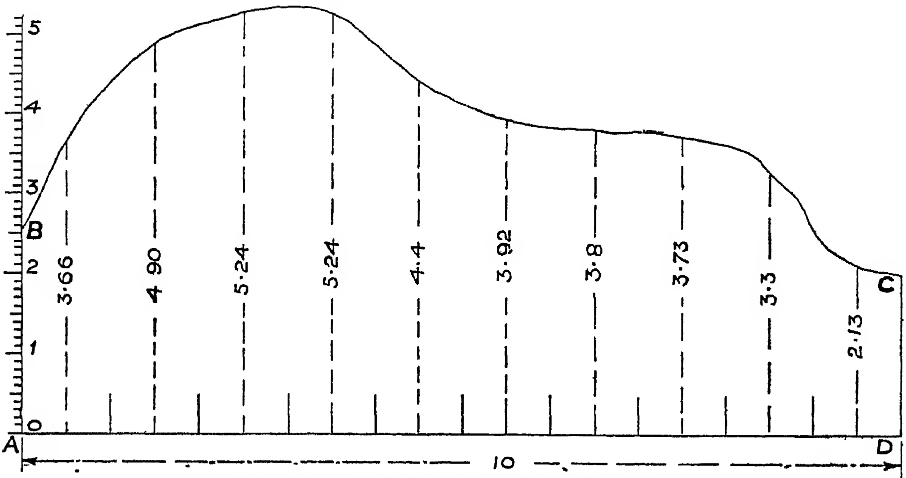


Fig. 164.—Area by Mid-ordinate Rule.

the latter is pretty regular for a large width of base, the division may be correspondingly wide; but sudden changes in curvature demand narrower widths. Assuming that the area has been divided into strips in the manner suggested, find the lengths of the mid-ordinates and the widths of the separate strips and tabulate as in the following example.

Example 3.—Calculate the area of the figure ABMP (Fig. 165).

Strip.	Width (inches)	Length of Mid-ordinate (ins.).	Area of Strip (sq. ins.)	Sum of Areas of Strip (sq. ins.)
AB	1·8	3·3	5·94	5·94
BC	·4	5·02	2·01	7·95
CD	·3	4·35	1·31	9·26
DE	·5	3·78	1·89	11·15
EF	·4	3·85	1·54	12·69
FG	1·3	3·54	4·60	17·29
GH	·6	2·53	1·52	18·81
HJ	1·3	3·6	4·68	23·49
JK	·6	4·54	2·72	26·21
KL	·8	3·4	2·72	28·93
LM	1·0	2·31	2·31	31·24

Thus the total area = 31·24 sq. ins.

(g) **Simpson's Rule** is the most accurate of the strip methods and is scarcely more difficult to remember or more complicated in its application than the trapezoidal rule.

In this rule, the base must be divided into an *even* number of equal divisions; the ordinates through the points of section being added in a particular way, viz.—

$$\text{Area} = \frac{\text{width of one division of base}}{3}\left\{\begin{array}{l}\text{first ordinate} + \text{last ordinate} + \\ 4\,(\text{sum of even ordinates}) + \\ 2\,(\text{sum of odd ordinates,} \\ \text{excluding the first and last}).\end{array}\right\}$$

If the portions of the curve joining pairs of ordinates are straight or parabolic, *i. e.*, if the equations to these portions are of the form $y = a + bx + cx^2$, the ordinates being vertical, the rule gives

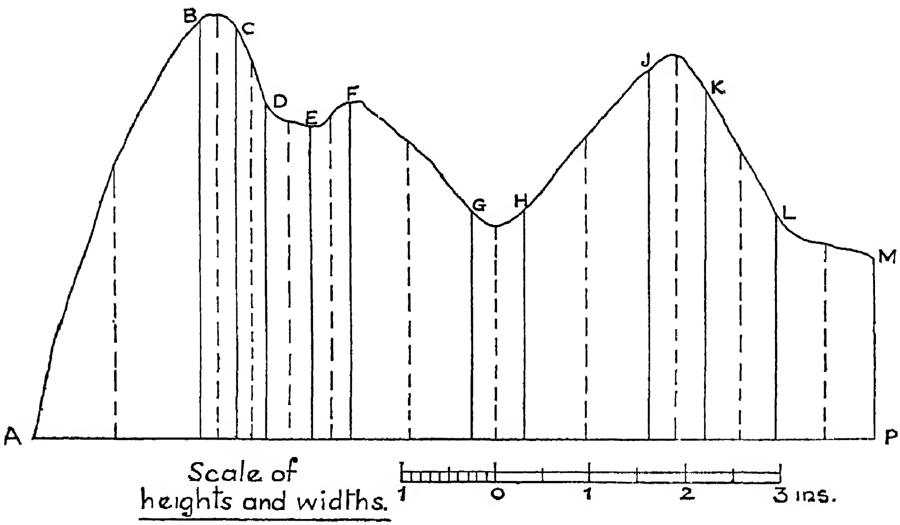


Fig. 165.—Modification of Mid-ordinate Rule.

perfectly correct results; and the strip width should be chosen to approximately satisfy these conditions.

Taking an example—

Example 4.—Find the area of the indicator diagram shown in Fig. 166.

A convenient horizontal line is selected to serve as a base and, in this instance, is divided into 10 equal parts. The ordinates are numbered y_1, y_2, y_3, etc., and their heights are measured, being those between the boundaries of the figure, and not down to the base line.

The working is set out thus—
Width of 1 division = 1 foot.

	Ordinates. Even.	*Odd.*
1st = y_1 = 0	y_2 = 80	y_3 = 66
last = y_{11} = 0	y_4 = 55·5	y_5 = 48
	y_6 = 43·5	y_7 = 38·5
Sum = 0	y_8 = 34·5	y_9 = 30·5
	y_{10} = 24	Sum = 183
	Sum = 237·5	
and	(Even) 4×sum = 950	2×sum = 366 (Odd)

∴ Area = $\frac{1}{3}\{0 + 950 + 366\}$ = 439 sq. units, which in this case would represent to some scale the work done per stroke on the piston.

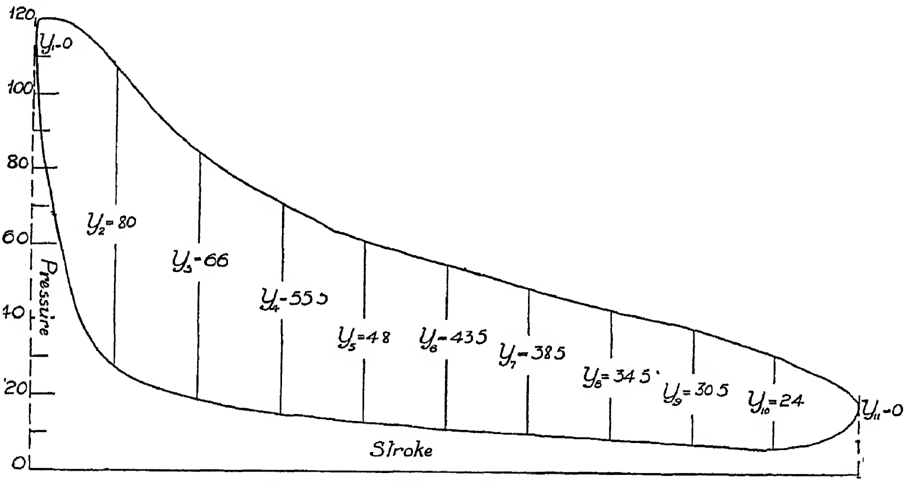


Fig. 166.—Area by Simpson's Rule.

Notice that this rule agrees with our notion of
" average height × base " for—

Number of ordinates considered = 1+1+4(5)+2(4) = 30
and if A = sum of ordinates according to the particular scheme—

$$\text{Area} = \text{base} \times \text{average ordinate} = 10h \times \frac{A}{30} = \frac{h}{3} \times A$$

If the area is of such a character that *two* divisions of the base are sufficient—

$$\text{Area} = \frac{h}{3}\{\text{1st} + \text{last} + (4 \times \text{mid.})\}$$

$$= \frac{\text{length}}{6}\{\text{1st} + \text{last} + (4 \times \text{mid})\} \text{ since length} = 2h$$

(*h*) **Graphic Integration** is a means of summing an area with the aid of tee- and set-square, by a combination of the principles of the "addition of strips" and "similar figures." An area in Fig. 167 is bounded by a curve $a'b'z'$, a base line az and two vertical ordinates aa' and zz'. The base is first divided as in method (*f*), where the widths of the strips are taken to suit the changes of curvature between a' and z', and are therefore not necessarily equal; and mid-ordinates (shown dotted) are erected for every division. Next the tops of the mid-ordinates are projected horizontally on

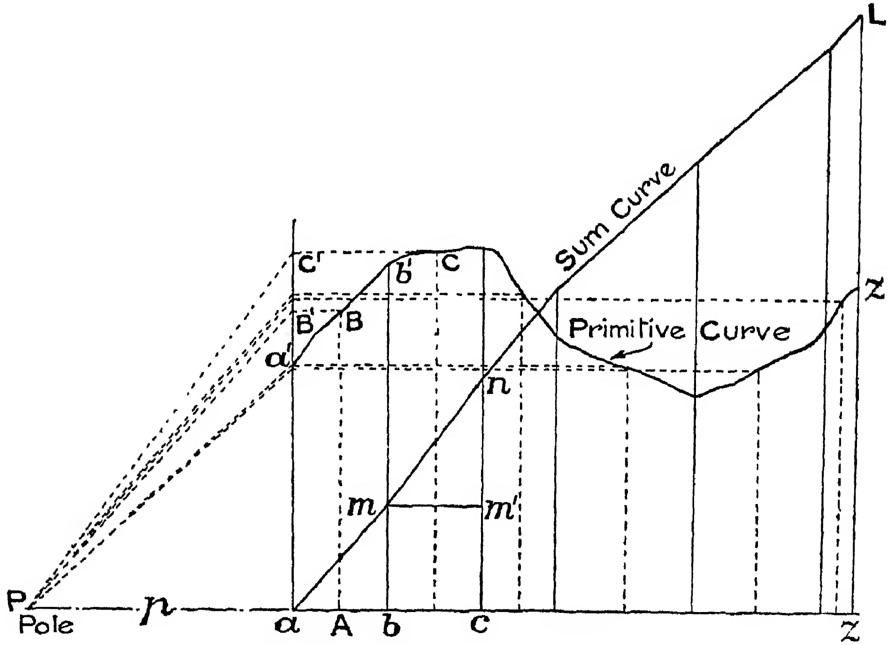


Fig. 167.—Graphic Integration.

to a vertical line, as BB′. A *pole* P is now chosen to the left of that vertical; its distance from it, called the polar distance p being a round number of horizontal units. The pole is next joined to each of the projections in turn and parallels are drawn across the corresponding strips so that a continuous curve results, known as the **Sum Curve.** Thus *am* parallel to PB′ is drawn from *a*, across the first strip; *mn* parallel to PC′ is drawn from *m* across the second strip, and so on.

The ordinate to the sum curve through any point in the base gives the area under the original or *primitive* curve from *a* up to the point considered.

Referring to Fig. 167—

$$\text{Area of strip } abb'a' = ab \times \text{AB}$$

but, by similar figures—

$$\frac{\text{B}'a \text{ or BA}}{p} = \frac{bm}{ab}$$

whence $$\text{AB} \times ab = p \times bm$$

i. e., $$bm = \frac{\text{area of strip}}{p} \quad \text{or} \quad \text{area of strip} = p \times bm$$

i. e., bm measures the area of the first strip to a particular scale, which depends entirely on the value of p.

In the same way $nm' = \dfrac{\text{area of second strip}}{p}$

and by the construction nm' and bm are added, so that—

$$cn = \frac{\text{area of 1st and 2nd strips}}{p}$$

or— $$\text{area of 1st and 2nd strips} = p \times cn$$

Thus, summing for the whole area—

$$\text{Area of } aa'z'z = p \times z\text{L}$$

Thus the scale of area is the old vertical scale multiplied by the polar distance; and accordingly the polar distance should be selected in terms of a number convenient for multiplication.

e. g., if the original scales are—

$$1'' = 40 \text{ units vertically}$$

and $$1'' = 25 \text{ units horizontally}$$

and the polar distance is taken as $2''$, *i. e.*, 50 horizontal units; then the new vertical scale—

$$= \text{old vertical scale} \times \text{polar distance}$$
$$= 40 \times 50 = 2000 \text{ units per inch.}$$

If the original scales are given and a *particular scale* is desired for the sum curve, then the polar distance must be *calculated* as follows—

$$\text{Polar distance in horizontal units} = \frac{\text{new vertical scale}}{\text{old vertical scale}}$$

e. g., if the primitive curve is a "velocity-time" curve plotted to the scales, $1'' = 5$ ft. per sec. (vertically) and $1'' = \cdot 1$ sec. (horizontally) and the scale of the sum curve, which is a "displacement-time" curve, is required to be $1'' = 2\cdot 5$ ft., then—

$$\text{Polar distance (in horizontal units)} = \frac{2\cdot 5}{5} = \cdot 5$$

and since $1'' = \cdot 1$ unit along the horizontal, the polar distance must be made $5''$.

The great advantages of graphic integration are—

(*a*) Its ease of application and its accuracy.

(*b*) The *whole* or *part* of the area is determined without separate calculation; the growth being indicated by the change in the sum curve. Thus, if the load curve on a beam is known, the sum curve indicates the shear values, because the shear at any section is the sum of the loads to the right or left of that section.

Example 5.—Draw the sum curve for the curve of acceleration given in Fig. 168. Find the velocity gained in 20 seconds from rest, and also in 35 seconds; find also the average acceleration.

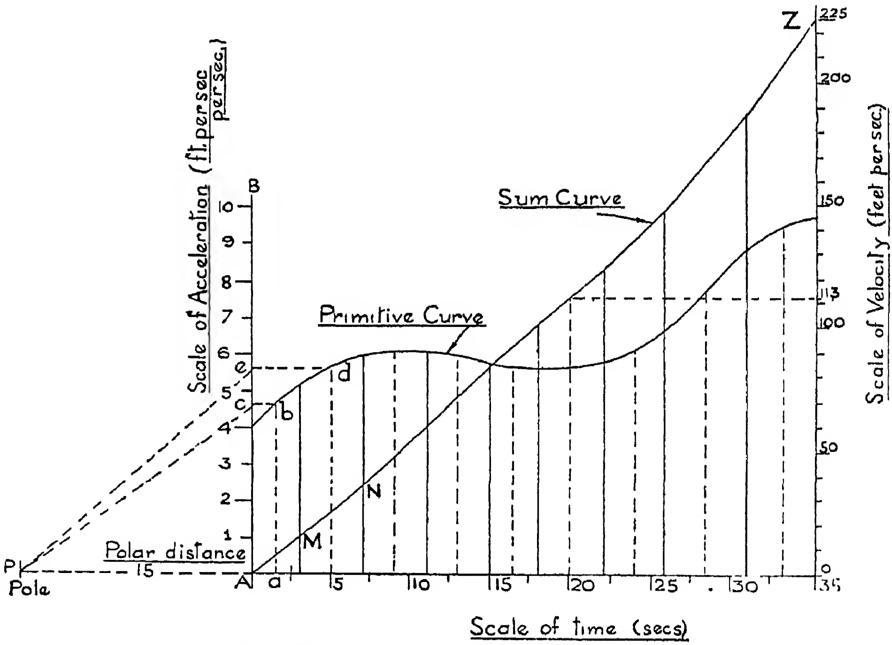


Fig. 168.—Construction of Sum Curve.

Method of procedure.—Project *bc* horizontally to meet the vertical AB in *c*. Draw AM parallel to P*c* to cut the second ordinate in M. Project *de* horizontally and draw MN parallel to P*e*. Continue the construction till Z is reached on the last ordinate.

The polar distance was chosen as $3''$, or 15 horizontal units, so that, whilst the old vertical scale was $1'' = 2$ units of acceleration, the sum curve vertical scale (in this case a scale of velocity) will be $1'' = 2 \times 15 = 30$ units. This new scale is indicated on the extreme right by the title "scale of velocity." Note that Z is at the point

marked 225. Hence the area of the figure is 225 sq. units, which gives the total velocity gained in the 35 seconds as 225 ft. per sec.

Also the velocity gained in 20 secs. from rest = 113 ft. per sec., this being the length of the ordinate through 20 on the scale of time.

The average acceleration = 225 ÷ 35 = 6·46 ft. per sec. per sec.; this being the average height of the figure.

Graphic integration can only be immediately applied when the base is a straight line. If it is otherwise, the figure must be reduced to one with a straight base by stepping off the ordinates

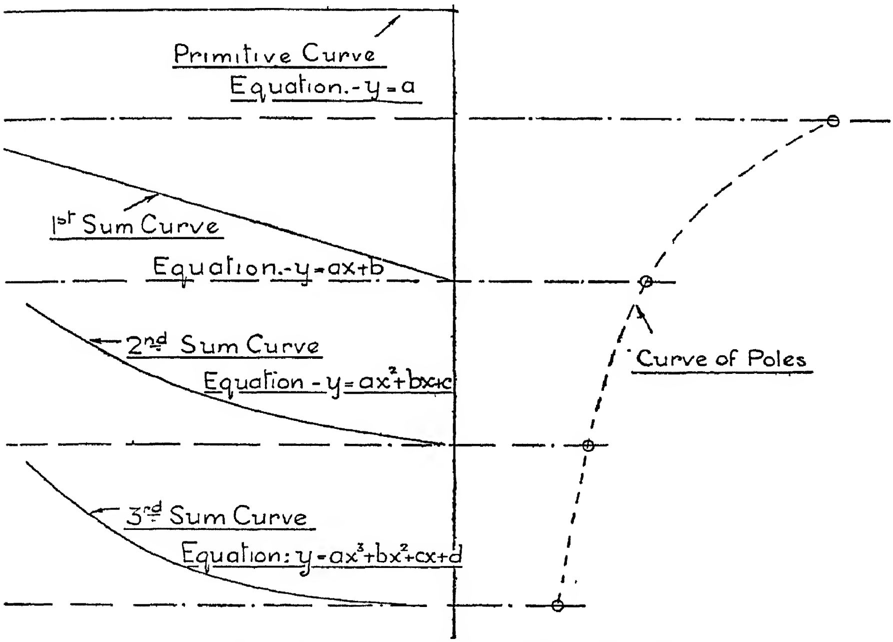


Fig. 169.—Comparison of Sum Curves.

with the dividers. Therefore, if the full area is required, as in the case of an indicator diagram, the additional complication would neutralize any other gain; but if separate portions of the area are wanted the method is the most efficient.

It is of interest to note that if the original curve is a horizontal line—

The first sum curve is a sloping straight line,

The second sum curve is a parabola of the second degree, or a "square" parabola.

The third sum curve is a parabola of the third degree, or a "cubic" parabola.

These cases are illustrated in Fig. 169, the poles being chosen to bring the curves to about equal scales for comparison.

Graphic Integration will be again referred to when dealing with the Calculus generally.

Calculation of Volumes.—All the rules for finding areas can be extended to the calculation of volumes. The area of the figure should then *represent the volume*: e.g., if the cross-section at various distances through an irregular solid be noted or estimated, and ordinates be erected to represent these cross-sections at the proper distances along the base of the diagram; the area of the figure on the paper will represent the volume of the solid. Thus—

If 1″ represents x feet of length
and 1″ represents y sq. ft. of cross-section, then
1 sq. in. of area represents xy cu. ft. of volume.

Example 6.—Find the capacity of a conical tub of oval cross-section, the axes of the upper oval being 28″ and 20″, those of the base being 21″ and 15″, and the height being 12″.

In this case the rule for the three sections may be applied; the axes of the mid-section are $24\frac{1}{2}''$ and $17\frac{1}{2}''$ and the areas of the three sections are—

$$A = \pi \times 14 \times 10 = 140\pi \text{ sq. ins.}$$
$$B = \pi \times 10{\cdot}5 \times 7{\cdot}5 = 78{\cdot}75\pi \text{ ,, ,,}$$
$$M = \pi \times 12{\cdot}25 \times 8{\cdot}75 = 107{\cdot}2\pi \text{ ,, ,,}$$

$$\therefore \quad \text{Volume} = \frac{\text{length}}{6}\{A + B + 4M\} = \frac{12}{6}\pi\{140 + 78{\cdot}75 + 428{\cdot}8\}$$
$$= 2\pi \times 647{\cdot}6 = 4070 \text{ cu. ins.}$$

$$\therefore \quad \text{Capacity} = \frac{4070}{277{\cdot}2}\text{ gallons} = \underline{14{\cdot}7 \text{ gallons.}}$$

Other worked Examples on the calculation of volumes will be found in Chapter VIII.

Exercises 35.—On the Areas of Irregular Curved Figures.

1. A gas expands according to the law $pv = 150$, from volume 3 to volume 25. Find the work done in this expansion.

2. An indicator card for a steam cylinder is divided into 10 equal parts by 9 vertical ordinates which have the respective values of 100, 84, 63, 50, 42, 36, 32, 28 and 26 lbs. per sq. in.; and the extreme ordinates are 100 and 25 lbs. per sq. in. respectively. Find the mean pressure of the steam.

3. The end areas of a prismoid are 62·8 and 20·5 sq. ft., the section mid-way between is 36·7 sq. ft. and the length of the prismoid is 15 ft. Find the average cross-section and the volume.

4. The mid-ship section of a vessel is given, the height from keel to deck being $19\frac{1}{2}$ ft.; and the horizontal widths, at intervals of 3·25 ft., are respectively 46·8, 46·2, 45·4, 43, 36·2, 26·2 and 14·4 ft., the first being measured at deck level and the last at the keel. Calculate the total area of the section.

5. To measure the area of the horizontal water plane, at load line, of a ship, the axial length of the ship was divided into nine abscissæ whose half-ordinates from bow to stern were ·6, 2·85, 9·1, 15·54, 18, 18·7, 18·45, 17·6, 15·13 and 6·7 ft. respectively; while the length of the ship at load line was 270 ft. Find the area of the water plane.

6. The velocity of a moving body at various times is as given in the table—

Time (secs.) .	0	1·5	2·8	3·6	5	6·2	7·7	8·9	10·3	12
Velocity (ft. per sec.) .	37·3	31·5	27·5	25·4	22·4	20·3	18·2	16·9	15·8	15

Find the total distance covered in the period of 12 seconds (*i. e.*, find the area under the velocity curve plotted to a time base.)

7. To find the cross-section of a river 90 ft. in breadth, the following depths, marked y, in feet, were taken across the river; x, in feet, being the respective horizontal distances from one bank.

x	0	10	20	30	40	50	60	70	80	90
y	3	4·5	5·6	6	5·7	4·8	4·7	4·5	4	3

Find the area of the cross-section. If the average velocity of the water normal to the cross-section is 5·1 ft. per sec., find the flow in cu. ft per sec.

8. A series of offsets was measured from a straight line to a river bank. Find, by Simpson's rule, the area between the line and the river bank.

Offsets (ft) .	0	7	9	8	5	2	3	7	9	11	15	20	13	5	0
Dist along line (ft)	0	100	200	300	400	450	500	600	700	725	750	775	800	900	1000

9. The mean spherical candle-power (M.S C.P.) of a lamp can be determined by calculating the mean height of a Rousseau diagram (candle power plotted to any linear base). Find the M.S C.P. for the arc lamp for which the Rousseau diagram is constructed from the following figures :—

Dist. from one end of base (ins.)	0	1	1·8	2·5	3·5	4·2	4·9	5·4	5·8	6
Candle power . .	0	115	350	650	1100	1350	1500	1200	400	0

10. Reproduce (*a*) Fig. 12 to scale and then determine its area.

11. Fig. 170 is a reproduction of an indicator card taken during a test on a 10 H.P. Diesel engine. Calculate the mean pressure for this case, *i. e.*, find the mean height of the diagram.

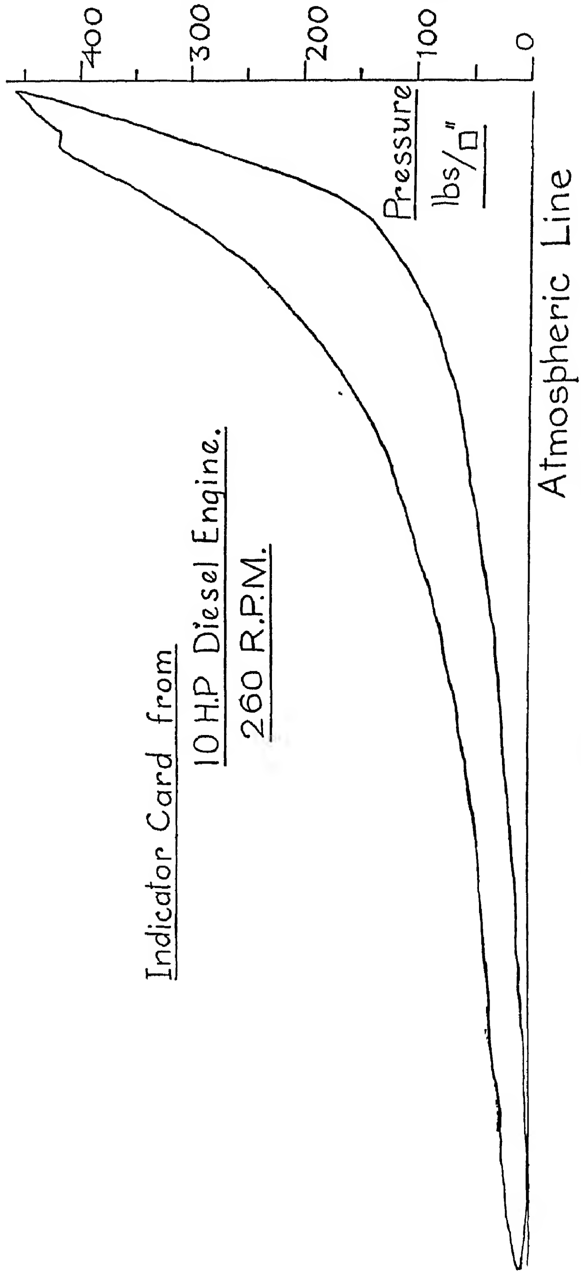


Fig. 170.—Mean Pressure from Indicator Card.

CHAPTER VIII

CALCULATION OF EARTHWORK VOLUMES

In this chapter a series of examples will be worked out to illustrate the method of calculating volumes of earthwork, such as railway cuttings, embankments, and other excavation work, mostly for the purpose of estimating the cost of earth removal.

Definitions of Terms introduced in these Examples.

The *formation surface* is the surface at the top of an embankment or at the bottom of a cutting, and in all the cases here considered it will be regarded as horizontal. The line in which the formation surface intersects the transverse section of the cutting or embankment is spoken of as the *formation width.*

The *natural surface* of the ground is the surface existing before the cutting or embankment is commenced.

The sides of a cutting or an embankment slope at an angle which is less than that of sliding for the particular earth; and the *slope* is usually expressed as x horizontal to one vertical.

A few typical values of the slope are given for various soils :—

Soil.	Compact Earth.	Gravel.	Dry Sand	Vegetable Earth	Damp Sand.	Wet Clay.
Angle with Horizontal	50°	40°	38°	28°	22°	16°
Slope (*i. e.*, x horizontal to 1 vertical) . . .	·8391	1·192	1·28	1·88	2·475	3·487

The unit of volume usually adopted in questions of earth removal is one cubic yard, and accordingly the weights in the following table are expressed in terms of that unit :—

Material.	Slate.	Granite	Sandstone	Chalk	Clay	Gravel	Mud.
Weight (cwts. per cu. yd.)	43	42	39	36	31	30	25

Volumes of Prismoidal Solids.—To find the volume of any irregular solid having two parallel faces or ends, find the average

cross-section parallel to these faces and multiply by the axial distance between them.

Then—
$$\textbf{volume} = \frac{L}{6}\{A + B + 4M\}$$

and
$$\textbf{average section} = \frac{1}{6}\{A + B + 4M\}$$

where L = axial length; and A, B, and M are the end sections and the middle section respectively.

Example 1.—**A solid with vertical sides.**

Let the base be horizontal and all the sides be vertical as in excavating foundations for a house. Referring to Fig. 171—

$$A = \frac{(12 + 10)}{2} \times 20 = 220 \text{ sq. ft.}$$
$$B = \frac{(4 + 8)}{2} \times 20 = 120 \text{ sq. ft.}$$
$$M = \frac{(7 + 10)}{2} \times 20 = 170 \text{ sq. ft.}$$
$$L = 50$$

$$\therefore \text{ Volume} = \frac{50}{6}\{220 + 120 + (4 \times 170)\} = 8500 \text{ cu. ft.} = \underline{314{\cdot}8 \text{ cu. yds.}}$$

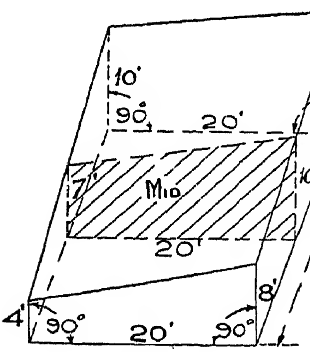


Fig. 171.

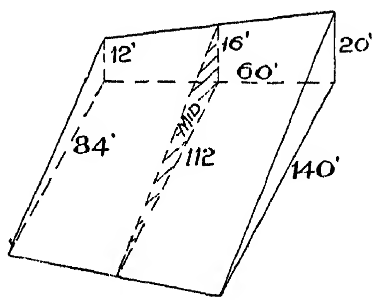


Fig. 172.

Example 2.—Calculate the weight of clay removed in making the **simple wedge-shaped excavation** shown in Fig. 172.

In this case— $A = \frac{1}{2} \times 12 \times 84 = 504$ sq. ft.
$B = \frac{1}{2} \times 20 \times 140 = 1400$ sq. ft.
$M = \frac{1}{2} \times 16 \times 112 = 896$ sq. ft.
and $L = 60$ ft.

then
$$\text{volume} = \frac{60}{6}\{504 + 1400 + (4 \times 896)\} = 54880 \text{ cu. ft.}$$
$$= 2032{\cdot}6 \text{ cu. yds.}$$

$$\text{and weight of clay removed} = \frac{2032{\cdot}6 \times 31}{20} \text{ tons}$$
$$= \underline{3151 \text{ tons.}}$$

Example 3.—**A more difficult wedge-shaped excavation,** which is shown in Fig. 173. To calculate the volume of earth removed.

The earth removed is represented by a wedge figure ADEF and a triangular pyramid AFBC.

The volume of the pyramid can be found if the area of the base is first obtained.

$$(AD)^2 = (60)^2 + (18)^2$$

whence $AD = 62{\cdot}65$

$$\sin \angle BAD = \frac{60}{62{\cdot}65}$$

but $\sin \angle BAC = \sin(180 - BAD) = \sin \angle BAD$

and hence—

$$\sin \angle BAC = \frac{60}{62{\cdot}65}$$

Also—

$$AC = 250 - 62{\cdot}65 = 187{\cdot}35.$$

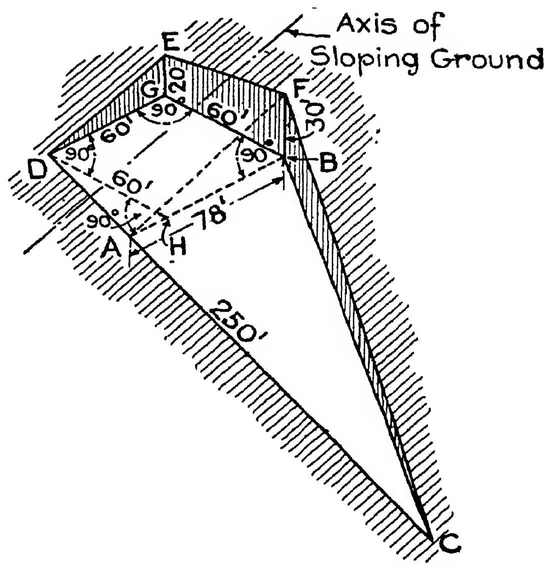


Fig. 173.—Wedge-shaped Excavation.

Then area of triangle ABC $= \frac{1}{2}$ BA . AC sin $\angle$ BAC

$$= \tfrac{1}{2} \times 78 \times 187{\cdot}4 \times \frac{60}{62{\cdot}65} = 7000 \text{ sq. ft.}$$

Height of pyramid $= 30$ ft.

$\therefore$ Volume of pyramid $= \frac{1}{3} \times 30 \times 7000 = 70000$ cu ft.
$= 2592$ cu. yds.

For the volume of the prismoidal solid ADEF, using the general rule—

$$A = \tfrac{1}{2} \times 20 \times 60 = 600 \text{ sq. ft.}$$
$$B = \tfrac{1}{2} \times 30 \times 78 = 1170 \text{ sq ft.}$$
$$M = \tfrac{1}{2} \times 25 \times 69 = 862{\cdot}5 \quad \text{and } L = 60 \text{ ft.}$$

$$\therefore \quad \text{Volume} = \frac{60}{6}\{600 + 1170 + 3450\} = 52200 \text{ cu. ft.}$$
$$= 1935 \text{ cu. yds.}$$

$\therefore$ Total volume removed $= \underline{4527 \text{ cu. yds.}}$

Sections of Cuttings.—It will be convenient at this stage to demonstrate the mode of calculation of the areas of simple sections. In Fig. 174 we have the first case, of a cutting whose sides are sloped and whose natural surface of ground DC is horizontal.

Let AB be the base or "formation width" and let its value be $2a$.

GH = height from centre of base to the natural surface $= h$.
θ = inclination to the horizontal of the sloping sides.
FC = horizontal projection of slope.

Then $\cot\theta$ is usually denoted by s; or, in other words, the slope of the sides is s horizontal to 1 vertical.

$$\frac{FC}{FB} = \cot\theta = s \quad \text{and } FC = FB \times s = hs$$

$$GC = \text{half width of surface} = a+hs.$$

Area ABCD (*i. e.*, the area of the section of the cutting)—

$$= \frac{1}{2}(DC+AB)\times h = \frac{h}{2}(2a+2a+2hs)$$
$$= h(2a+hs)$$

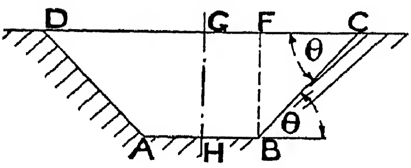


Fig. 174.

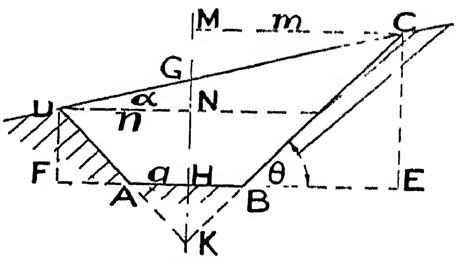


Fig. 175.

Fig. 175 shows the cutting section when the natural surface of the ground takes a slope DC.

Let α = inclination to the horizontal of the natural surface, and let $\cot\alpha = r$.

CM and DN, though not equal, are called the "half-widths" of the section; let these be represented by m and n respectively.

To find m and n—

$$\frac{MG}{m} = \tan\alpha = \frac{1}{r}, \text{ so that } MG = \frac{m}{r} \quad \ldots\ldots \quad (1)$$

Also—

$$\frac{HK}{HB} = \tan\theta = \frac{1}{s}, \text{ so that } HK = \frac{a}{s} \quad \ldots\ldots \quad (2)$$

$$\frac{MK}{m} = \tan\theta = \frac{1}{s}, \text{ whence } MK = \frac{m}{s} \quad \ldots\ldots \quad (3)$$

From (2)—

$$GK = GH+HK = h+\frac{a}{s}$$

From (1) and (3)—

$$GK = MK-MG = \frac{m}{s} - \frac{m}{r}$$

Hence—
$$h+\frac{a}{s} = \frac{m}{s} - \frac{m}{r}$$

and
$$m = \frac{r}{r-s}(a+hs)$$

Similarly—
$$n = \frac{r}{r+s}(a+hs).$$

To find the extreme heights CE and DF—

$$BE = HE-HB = m-a$$

$$\frac{BE}{CE} = \cot\theta = s, \quad \text{whence } CE\times s = BE$$

$$\therefore \quad CE\times s = m-a \quad \text{or} \quad CE = \frac{m-a}{s}$$

and similarly—
$$DF = \frac{n-a}{s}$$

To find the area of the section—

$$\begin{aligned}
\text{Area ABCD} &= CDFE-DFA-CBE \\
&= FE\left(\frac{CE+FD}{2}\right)-\frac{1}{2}AF.FD-\frac{1}{2}BE.CE \\
&= \frac{1}{2}\left\{\frac{(m+n)(m-a+n-a)}{s}-\frac{(n-a)(n-a)}{s}-\frac{(m-a)(m-a)}{s}\right\} \\
&= \frac{1}{2s}\left\{m^2+n^2+2mn-2am-2an-n^2-a^2+2an-m^2-a^2+2am\right\} \\
&= \frac{mn-a^2}{s}
\end{aligned}$$

Example 4.—A cutting is to be made through ground having a transverse slope of 5 horizontal to 1 vertical, and the sides are to slope at $1\frac{1}{4}$ horizontal to 1 vertical. If the formation width is 60 ft. and the height of the cutting (at centre) is 12 ft., find the half-widths, the extreme heights and the area of the section.

Adopting the notation as applied to Fig. 175—

$$2a = 60, \ h = 12, \ s = 1\tfrac{1}{4}, \text{ and } r = 5$$

Then—
$$m = \frac{r}{r-s}(a+hs) = \frac{5}{3{\cdot}75}[30 + (12 \times 1\tfrac{1}{4})] = \underline{60 \text{ ft.}}$$

$$n = \frac{r}{r+s}(a+hs) = \frac{5}{6{\cdot}25}(30 + 15) = \underline{36 \text{ ft.}}$$

$$CE = \frac{m-a}{s} = \frac{60-30}{1{\cdot}25} = \underline{24 \text{ ft.}}$$

$$DF = \frac{n-a}{s} = \frac{36-30}{1{\cdot}25} = \underline{4{\cdot}8 \text{ ft.}}$$

$$\text{Area} = \frac{mn-a^2}{s} = \frac{(60 \times 36) - 30^2}{1{\cdot}25} = \underline{1008 \text{ sq. ft.}}$$

Example 5.—**Volume of a cutting having symmetrical sides,** the dimensions being as in Fig. 176.

Calculate the volume of earth removed, if the cutting enters a hill normally to the slope of the latter and emerges at a vertical wall or cliff.

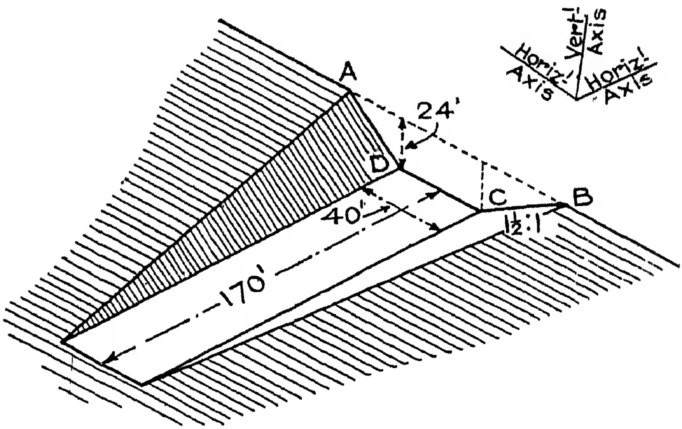


Fig. 176.—Cutting on a Hill.

The volume is found by application of the general rule.

$$\text{Volume} = \frac{L}{6}\{A+B+4M\}$$

To find A— $h = 24, \quad s = 1\tfrac{1}{2}, \quad 2a = 40$

$\therefore$ Area $= h(2a + hs) = 24(40 + 36) = 1824$ sq. ft.

In the case of the other end section, $h = 0$ and thus $B = 0$.

For M— $h = 12, \quad s = 1\tfrac{1}{2}, \quad 2a = 40$

Area $= 12(40 + 18) = 696$ sq. ft.

also $L = 170$ ft.

Hence— Volume $= \frac{170}{6}\{1824 + 0 + 2784\} = 130600$ cu. ft.

or $\underline{4837 \text{ cu. yds.}}$

Example 6.—**To find the volume of a cutting having unequal sides.**

In this case, shown at (*a*), Fig. 177, the cutting enters the hill in an oblique direction, although the outcrop is vertical as before. The

sides of the cutting slope at $1\frac{1}{2}$ horizontal to 1 vertical, while the natural surface of the ground slopes upward at $4\frac{1}{2}$ horizontal to 1 vertical.

The solid can be split up into a prismoidal solid SRFE, together with the two pyramids SE and RF.

To deal first with the prismoidal solid SRFE; its volume can be found from the general rule Volume $= \frac{L}{6}\{A + B + 4M\}$, and in order to find the values of A and B the lengths of BS and AR must first be found.

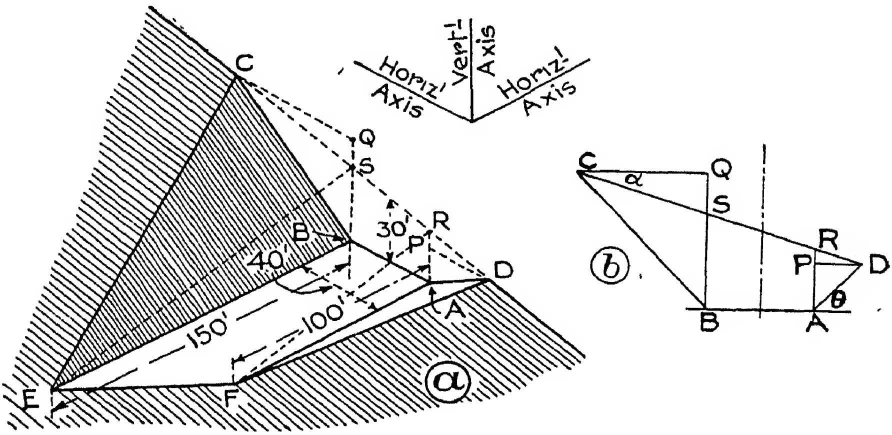


Fig. 177.—Cutting with Unequal Sides.

Referring to (*b*), Fig. 177—

$$CQ = m - a, \quad \text{and } m = \frac{r}{r-s}(a+hs)$$

Also— $r = 4\frac{1}{2}, \quad s = 1\frac{1}{2}, \quad h = 30, \quad \text{and } 2a = 40$

so that $m = \frac{4\frac{1}{2}}{3}(20+45) = 97.5$ ft.

Hence— $CQ = 97.5 - 20 = 77.5$ ft.

Now— $\frac{SQ}{CQ} = \tan \alpha = \frac{1}{4.5}$

$\therefore$ $SQ = \frac{CQ}{4.5} = \frac{77.5}{4.5} = 17.22$ ft.

Also— $BQ = \frac{77.5}{1.5} = 51.66$ ft.

$\therefore$ $BS = BQ - SQ = 51.66 - 17.22 = 34.44$ ft.

Again— $n = \frac{r}{r+s}(a+hs) = \frac{4\frac{1}{2}}{6}(20+45) = 48.75$ ft.

$$PD = 48.75 - 20 = 28.75 \text{ ft.}$$

$$\frac{PR}{PD} = \frac{1}{4.5}, \quad \text{whence } PR = \frac{28.75}{4.5} = 6.39$$

and $AP = PD \tan \theta = \frac{28.75}{1.5} = 19.17$ ft.

Hence— $AR = AP + PR = 19.17 + 6.39 = 25.56$ ft.

We can now proceed to find the volume of the solid SRFE.

$$A = \tfrac{1}{2} \times BS \times BE = \tfrac{1}{2} \times 34 \cdot 44 \times 150 = 2583 \text{ sq. ft.}$$
$$B = \tfrac{1}{2} \times AR \times AF = \tfrac{1}{2} \times 25 \cdot 56 \times 100 = 1278 \text{ sq. ft.}$$
$$M = \tfrac{1}{2} \times 30 \times 125 = 1875 \text{ sq. ft.}$$

and $L = 40$

$$\therefore \quad \text{Volume} = \frac{40}{6}\{2583 + 1278 + 7500\}$$
$$= 75750 \text{ cu. ft.} = 2805 \text{ cu. yds.}$$

To find the volume of the pyramid SE—

The base is the triangle CSB, of which the area $= \tfrac{1}{2} \times BS \times CQ$

$$= \tfrac{1}{2} \times 34 \cdot 44 \times 77 \cdot 5$$
$$= 1335 \text{ sq. ft.}$$

The height = 150 ft., and hence—

$$\text{Volume} = \frac{1}{3} \times \frac{150}{27} \times 1335 \text{ cu. yds.} = 2471 \text{ cu. yds.}$$

To find the volume of the pyramid RF—

The base is the triangle ARD; and its area $= \tfrac{1}{2} \times AR \times PD$

$$= \tfrac{1}{2} \times 25 \cdot 56 \times 28 \cdot 75$$
$$= 367 \cdot 5 \text{ sq. ft.}$$

The height = 100 ft., and hence—

$$\text{Volume} = \frac{1}{3} \times \frac{100}{27} \times 367 \cdot 5 \text{ cu. yds.} = 453 \cdot 6 \text{ cu. yds.}$$

$$\therefore \quad \text{Total volume} = 2805 + 2471 + 453 \cdot 6 = \underline{5730 \text{ cu. yds.}}$$

Cutting and Embankment continuously combined; the Sides being Symmetrical.—If a road or a railway track has to be constructed through undulating ground, both cuttings and embankments may be necessary. The cost of the road-making depends to a large extent on the "net" weight of earth removed, seeing that the earth may be transferred from the cutting to the embankment. The calculation of the net volume removed will be dealt with according to two methods :—

First Method.

Example 7.—A cutting is to be made through the hill AC (Fig. 178) and an embankment in the valley BC so as to give a straight horizontal road from A to B. The formation width is to be 40 ft., and the sides of the cutting and the embankment slope $1\tfrac{1}{2}$ horizontal to 1 vertical. Calculate the net weight of vegetable earth removed (25 cwt. per cu. yd.).

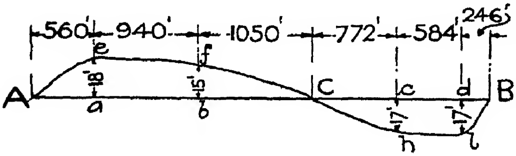


Fig. 178.

The volume of the cutting will be found by considering it made up of three prismoidal solids, and the volume of the embankment will be

found in the same way. Then the net volume is the difference of these separate volumes.

Dealing with the portion between A and C, *i. e.*, with the cutting :—

For the portion A*ae*—

$$A = 0$$
$$B = 18[40 + (18 \times 1\tfrac{3}{4})] = 1287, \quad \text{since } h = 18$$
$$M = 9[40 + (9 \times 1\tfrac{3}{4})] = 501{\cdot}75, \quad \text{since } h = 9$$
$$L = 560$$

$$\therefore \quad \text{Volume} = \frac{560}{6}\{0 + 1287 + (4 \times 501{\cdot}75)\} = 307400 \text{ cu. ft.}$$

For the portion abfe—

$$A = 1287$$
$$B = 15[40 + (15 \times 1\tfrac{3}{4})] = 994, \quad \text{since } h = 15$$
$$M = 16{\cdot}5[40 + (16{\cdot}5 \times 1\tfrac{3}{4})] = 1136, \quad \text{since } h = 16{\cdot}5$$
$$L = 940$$

$$\therefore \quad \text{Volume} = \frac{940}{6}\{1287 + 994 + (4 \times 1136)\}$$
$$= 1{,}069{,}000 \text{ cu. ft.}$$

*For the portion fb*C—

$$A = 994$$
$$B = 0$$
$$M = 7{\cdot}5[40 + (7{\cdot}5 \times 1\tfrac{3}{4})] = 398$$
$$L = 1050$$

$$\therefore \quad \text{Volume} = \frac{1050}{6}\{994 + 0 + (4 \times 398)\} = 453000 \text{ cu. ft.}$$

Thus the total volume removed to make the cutting

$$= 307400 + 1{,}069{,}000 + 453000 = 1{,}829{,}400 \text{ cu. ft.}$$

Dealing with the embankment portion, viz. that from B to C :—

For the solid C*ch*—

$$A = 0$$
$$B = 17[40 + (17 \times 1\tfrac{3}{4})] = 1185$$
$$M = 8{\cdot}5[40 + (8{\cdot}5 \times 1\tfrac{3}{4})] = 466$$
$$L = 772$$

$$\therefore \quad \text{Volume} = \frac{772}{6}\{0 + 1185 + (4 \times 466)\} = 392000 \text{ cu. ft.}$$

For the solid chld—

$$A = 1185 \qquad L = 584$$
$$B = 1185$$
$$M = 1185$$

$$\therefore \quad \text{Volume} = 1185 \times 584 = 692000 \text{ cu. ft.}$$

*For the solid dl*B—

$$A = 1185$$
$$B = 0$$
$$M = 8{\cdot}5[40 + (8\ 5 \times 1\tfrac{3}{4})] = 466$$
$$L = 246$$

$$\therefore \quad \text{Volume} = \frac{246}{6}\{1185 + 0 + (4 \times 466)\} = 125000 \text{ cu. ft.}$$

Hence the total volume required for the embankment

$$= (392 + 692 + 125) \times 10^3 \text{ cu. ft.} = 1{,}209{,}000 \text{ cu. ft.}$$

Then— net volume removed $= (1\cdot829 - 1\cdot209) \times 10^6$

$= 620000$ cu. ft. or 22960 cu. yds.

and the net weight removed $= \dfrac{22960 \times 25}{20}$ tons

$= \underline{28700 \text{ tons}}$

Second Method.

Example 8.—Fig. 179 shows the longitudinal section of some rough ground through which the road AC is to be cut. The sides of the cutting and of the embankment slope at $1\frac{1}{2}$ horizontal to 1 vertical,

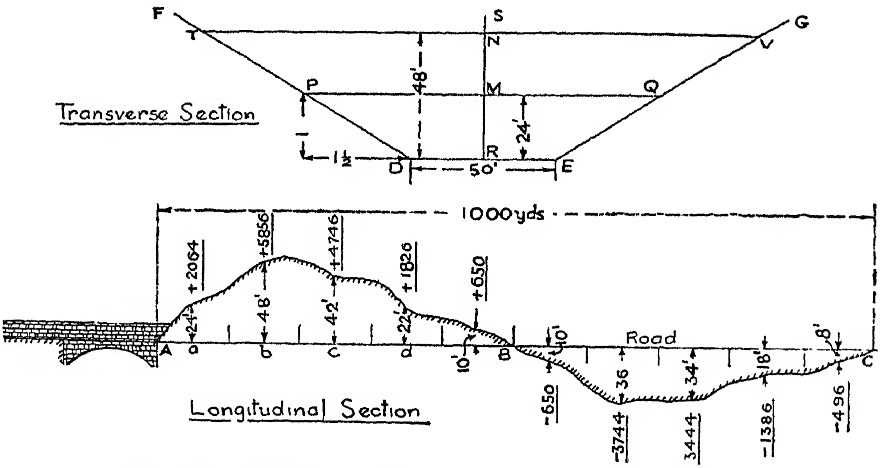


Fig. 179.—Volume of Earth removed in making Road.

and the road is to be 50 ft. wide. Calculate the net volume of earth removed in the making of the road.

Divide the length AC into ten equal distances and erect mid-ordinates as shown. Scale off the lengths of these, which are the heights of the various sections.

The areas of the sections at *a*, *b*, *c*, *d*, etc., can be found by calculation as before, or, if very great accuracy is not desired, the various sections may be drawn to scale and the areas thus determined. To illustrate the latter method: Draw DE = 50 ft., and also the lines DF and EG, having the required slope, viz. $1\frac{1}{2}$ to 1. Through R, the middle point of DE, erect a perpendicular RS, and along it mark distances like RM, RN, etc., to represent the respective heights of the sections: thus RM = 24, and RN = 48. Then to find the area of the section at *a*, which is really the figure DPQE, add the length of PQ to that of DE and multiply half the sum by RM. The area of the section at *b* is $\frac{1}{2}$(TV + DE) × RN, and so on. The areas of the

respective sections are 2064, 5856, 4746, 1826, and 650 sq. ft., these being reckoned as positive; and 650, 3744, 3444, 1386, and 496 sq. ft., these being regarded as negative. The average of all these sections, added according to sign, is 5422 sq. ft. or 602·4 sq. yds. Then the net volume of earth removed = 602·4 × 1000 = 602400 cu. yds.

Cutting with Unequal Sides, in Varying Ground.

First Method.

To find the average cross-section of ground with twisted surface (Fig. 180); an end view being shown in Fig. 181.

The surface slopes downwards to the left at A and to the right at B.

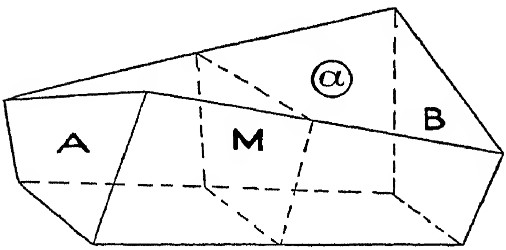


Fig. 180.

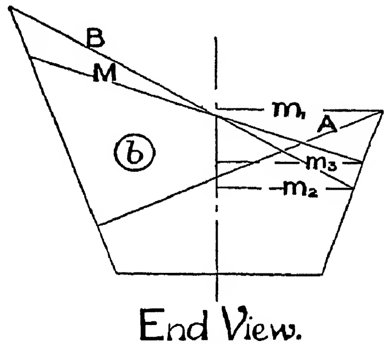


Fig. 181.

Let m_1, n_1, h_1, and r_1 be the half-widths, etc., for A; and m_2, n_2, h_2, and r_2 the corresponding values for B.

Then— Area of A $= \dfrac{m_1 n_1 - a^2}{s}$; area of B $= \dfrac{m_2 n_2 - a^2}{s}$

For the mid-section M— $m_3 = \dfrac{m_1 + m_2}{2}$ and $n_3 = \dfrac{n_1 + n_2}{2}$

and the area of M $= \dfrac{m_3 n_3 - a^2}{s} = \dfrac{(m_1+m_2)(n_1+n_2) - 4a}{4s}$

∴ Average cross-section—

$$= \frac{A + B + 4M}{6}$$

$$= \frac{m_1 n_1 - a^2 + m_2 n_2 - a^2 + (m_1+m_2)(n_1+n_2) - 4a^2}{6s}$$

$$= \frac{2m_1 n_1 + 2m_2 n_2 + m_1 n_2 + m_2 n_1 - 6a^2}{6s}$$

$$= \frac{m_1 n_1 + m_2 n_2 + (m_1+m_2)(n_1+n_2) - 6a^2}{6s}$$

Example 9.—Find the average cross-section of ground with twisted surface, when the formation width is 20 ft. and the side-slopes are $1\frac{1}{2}$ horizontal to 1 vertical. At the one end of the embankment the height is 12 ft. and the natural surface of the ground slopes at 20 horizontal to 1 vertical downwards to the right; while at the other end the height is 6 ft. and the slope of the ground is 10 to 1 downwards to the left.

Adhering to the notation employed in the general description :—

For the section A—

$$m_1 = \frac{20}{20 - 1\cdot5}\{10 + (12 \times 1\cdot5)\} = 30\cdot3 \text{ ft.}$$

$$n_1 = \frac{20}{20 + 1\cdot5}\{10 + 18\} = 26\cdot05 \text{ ft.}$$

For the section B—

$$m_2 = \frac{-10}{-10 - 1\cdot5}\{10 + (6 \times 1\cdot5)\} = 16\cdot5 \text{ ft.}$$

$$n_2 = \frac{-10}{-10 + 1\cdot5}(10 + 9) = 22\cdot4 \text{ ft.}$$

Hence the average cross-section—

$$= \frac{(30\cdot3 \times 26) + (16\cdot5 \times 22\cdot4) + (46\cdot8 \times 48\cdot4) - 600}{9}$$

$$= \underline{\underline{313\cdot6 \text{ sq. yds.}}}$$

Second Method.

Example 10.—Calculate the volume of earth removed in making a cutting of which AE is a longitudinal centre section (Fig. 182). The formation width is 20 ft., the length of the cutting is 4 chains, the

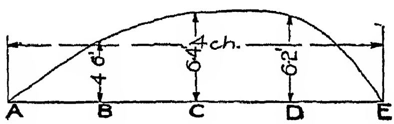
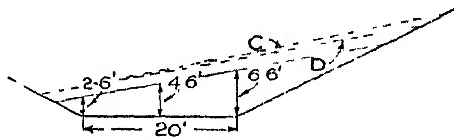


Fig. 182.



Section at B.

Fig. 183.

sections are equally spaced, and the slope of the sides is 2 horizontal to 1 vertical. All the sections slope downwards to the left, as indicated in Fig. 183.

The heights of the sections, in feet above datum level, are :—

Section	Left.	Centre.	Right.
A	0	0	0
B	2·6	4·6	6·6
C	4·1	6·4	8·7
D	4·0	6·2	8·4
E	0	0	0

The areas of the sections may be found by drawing to scale and

then using the planimeter; and Simpson's rule can afterwards be employed, since there are an odd number of sections.

The results in this case are as follows :—

Section.	m	n	Area
A	10	10	0
B	32	13·7	169·5
C	42·2	15·7	280
D	39·9	15·55	260
E	10	10	0

$$\text{Then the volume} = \frac{66}{3}\{0 + 0 + 4(169{\cdot}5 + 260) + 2(280)\}$$
$$= 50100 \text{ cu. ft. or } \underline{1855 \text{ cu. yds.}}$$

Surface Areas for Cuttings and Embankments.

The area of land required for a cutting or an embankment can be determined when the half-widths of the various transverse sections are known; the method of procedure being detailed in the following example :—

Example 11.—Fig. 184 represents the horizontal projection of the cutting dealt with in *Example* 10. Find the area of land required for this cutting if a space of 5 ft. between the outcrops and a fence be allowed.

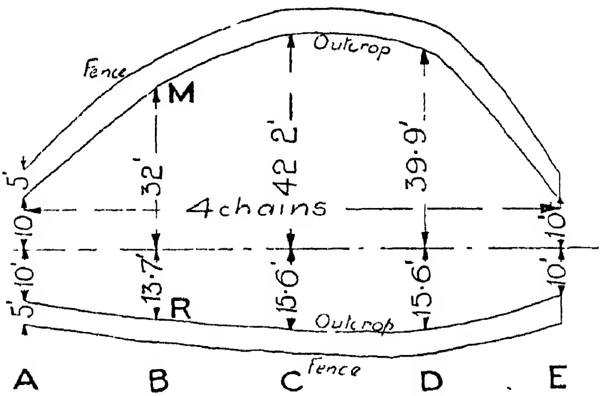


Fig. 184.—Surface Area for Cutting.

The width RM is the extreme width of the section, *i. e.*, its value $= m + n$; accordingly, allowing 5 ft. on each side, the widths to be considered are of the form $m + n + 10$.

Taking the values of m and n as in the previous example, the widths are as in the table :—

Section	A	B	C	D	E
$m + n + 10$	30	55·7	67·8	65·45	30

Applying Simpson's rule—

$$\text{Area of land required} = \frac{66}{3}\{30 + 30 + 4(55{\cdot}7 + 65{\cdot}45) + (2 \times 67{\cdot}8)\}$$
$$= 14960 \text{ sq. ft. or } \underline{1663 \text{ sq. yds.}}$$

Volumes of Reservoirs.

Example 12.—Find the volume of water in the reservoir formed as shown in Fig. 185, when the water stands at a level of 45 ft. above datum level, the bottom of the reservoir being at the level 22 ft.

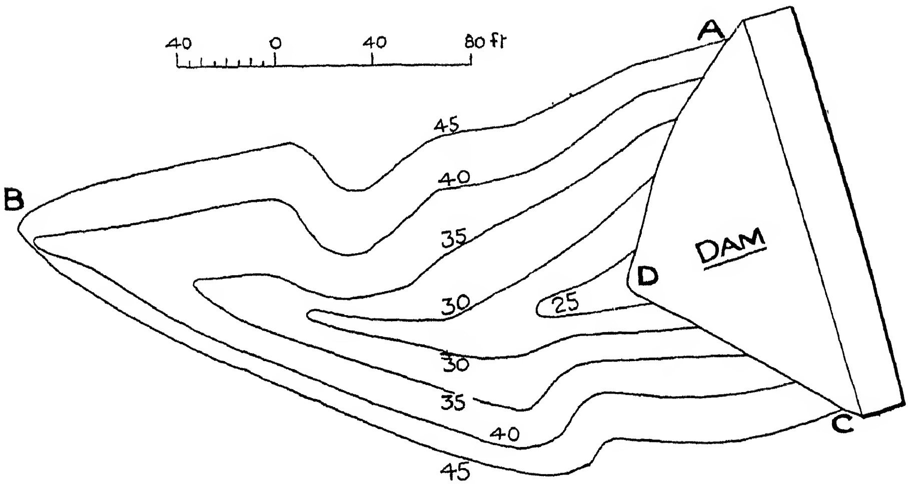


Fig. 185.—Volume of Reservoir.

In the diagram the land is shown contoured, *i. e.*, the line marked 40, for example, joins all points having the level 40 ft. above datum.

The problem, then, is to find the volume of an irregular solid, and this may be done in either of two ways, viz.—

(a) *By taking vertical sections.*—According to this method, we should find the extreme length of the reservoir, which is about 320 ft., and then draw the cross-sections at intervals of, say, 40 ft. The area of each cross-section would then be found, preferably by the planimeter, and the volume calculated by adding the areas according to Simpson's rule.

This process is somewhat tedious, as each section must be plotted separately; and consequently it is better to proceed as in method (*b*).

(b) *By taking horizontal sections,* i. e., sections at heights of 45, 40, 35, etc., ft. respectively.

To find the area of the section at the height 45 ft., determine the area of the figure ABCD by means of the planimeter. This area is found to be 5·083 sq. ins. Now the linear scale is 1″ = 80 ft., and therefore each square inch of area on the paper represents 80 × 80 or

6400 sq. ft. Thus the area of the section at the level of 45 ft. $= 5 \cdot 083 \times 6400 = 32500$ sq. ft.; and in the same way the areas at the levels 40, 35, 30, 25, and 22 ft. are 21550, 10560, 3780, 577, and 0 sq. ft. respectively. The length of the irregular solid is 23 ft., *i. e.*, $45 - 22$, and we may plot the various areas to a base of length,

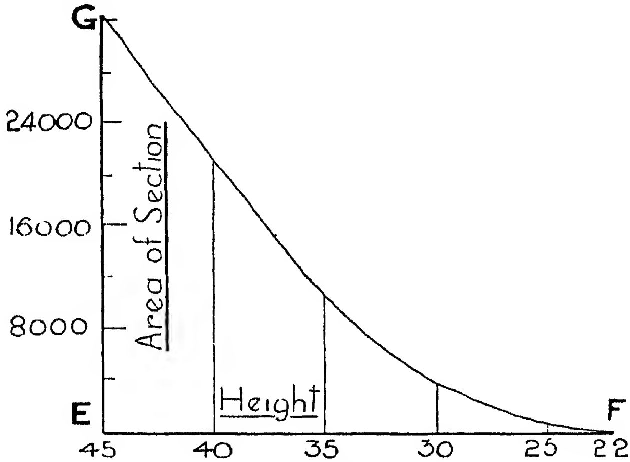


Fig. 186.

as indicated in Fig. 186. The area of the figure EFG, which is found to be 1·633, gives the volume of water in the reservoir, to some scale. In the actual drawing 1″ = 10 ft. (horizontally), and 1″ = 16000 sq. ft. (vertically), so that 1 sq. in. on the paper represents 10×16000 or 160000 cu. ft.

Hence volume of the reservoir $= 160000 \times 1 \cdot 633 = 261300$ cu. ft.

or its capacity = 1630000 gallons.

Exercises 36.—On the Calculation of Volumes and Weights of Earthwork.

1. Calculate the volume of the solid with vertical sides shown in Fig. 187.

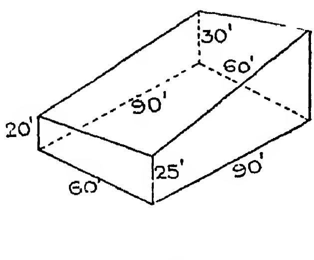


Fig. 187.

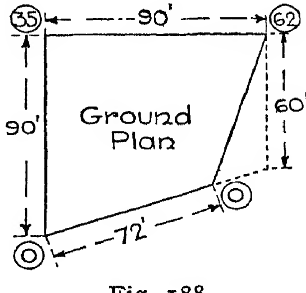


Fig. 188.

2. Fig. 188 shows the plan of a wedge-shaped excavation, where the encircled figures indicate heights. Calculate the weight of clay removed in making the excavation.

3. Fig. 189 is the longitudinal section of some rough ground through which a straight horizontal road is to be cut, the width of the road being 64 ft. The soil is vegetable earth (25 cwts. per cu. yd.), and

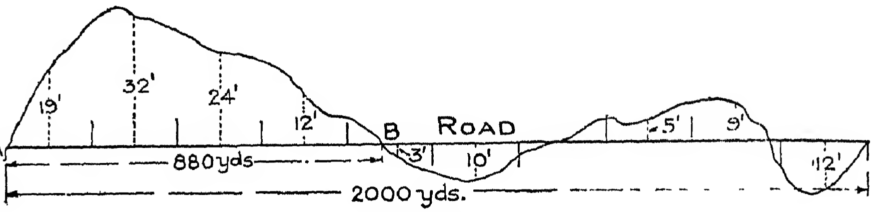


Fig. 189.

the sides of the cutting and embankment slope at 2 horizontal to 1 vertical. Calculate the weight of earth removed in making the road, if the natural surface of the ground is horizontal.

4. Determine the area of land required for making the cutting from A to B in Fig. 189. The side-slopes are 2 horizontal to 1 vertical, the formation width is 64 ft., and a fence is to be built round the working at a distance of 6 ft. from the outcrops.

5. Calculate the capacity of a reservoir for which the horizontal sections at various heights have the values in the following table :—

Height above sea level (ft.)	180	170	160	155	150	147
Area of section in sq. ft.	47200	31000	21700	19000	11300	0

6. The depth of a cutting at a point on the centre line is 20 ft., the width of the base being 30 ft. The slope of the bank is $1\frac{1}{2}$ horizontal to 1 vertical, and the sidelong slope of the ground is 12 horizontal to 1 vertical. Find the horizontal distances from the vertical centre plane to the top of each slope.

7. Find the volume of earth removed from a cutting, if the formation width is 20 ft., the side-slopes are $1\frac{1}{2}$ to 1, and the slope of the surface is 10 to 1. The depth of the cutting at the first point is 25 ft.; at the end of the cutting (200 ft. long) it is 30 ft.; and half-way between it is 26 ft.

8. The base of a railway cutting is 32 ft. in width, the depth of the formation is 34 ft. below the centre line of the railway, the side-slopes are $1\frac{3}{4}$ to 1, and the surface of the ground falls 1 in 8. Calculate the half-breadths for the cutting.

At a distance of 1 chain along the centre line the depth of formation level is 28 ft., and at a distance of 2 chains it is 20 ft. Find the volume of earth to be removed.

9. On the centre line of a railway running due N. the difference in level between the natural ground and the formation level of the embankment is 5·6 ft., 8·4 ft. and 6 ft. at the 23rd, 24th, and 25th chain pegs respectively. The width of the formation level is 20 ft., and the sides of the embankment slope at 2 to 1.

The natural ground slopes down across the railway from E. to W. at 1 in 10. Determine at each chain peg the distances of the toes of the embankment from the centre line and the area of the cross-section; determine also the volume of the embankment between the 23rd and 25th chain pegs.

10. A cutting runs due E. and W. through ground sloping N. and S. The formation level is 15 ft. below the surface centre line and is 20 ft. wide. The ground slopes upwards on the north side of the centre line 1 vertical to 6 horizontal, and on the south side the ground slopes downwards 1 vertical to 10 horizontal. The sides of the cutting slope 1 vertical to $1\frac{1}{2}$ horizontal. Calculate the positions of the outcrops.

CHAPTER IX

THE PLOTTING OF DIFFICULT CURVE EQUATIONS

Plotting of Curves of the Type $y = ax^n$.—The plotting in Chapter IV was of a rather elementary character in that integral powers only of the quantities concerned were introduced. All calculations could there be performed on the ordinary slide rule; *e. g.*, such curves as that representing $y = 5x^2 + 7x - 9$ were possible. If, now, a formula occurs in which one, say, of the quantities is raised to a fractional or negative power, and a curve is required to represent the connection between the two quantities for all values within a given range, the necessary calculations must be made by the aid of logs. Suitable substitutions will in some cases make these calculations simpler, but unless great care is exercised over the arrangement of the calculations and the selection of suitable values for the quantities, a great deal of time will be wasted. In fact, the method of tabulating values is of more importance than is the actual plotting.

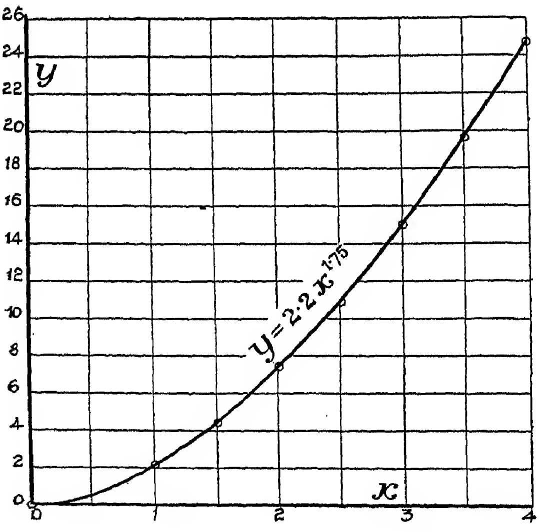


Fig. 190.—Curve of $y = 2{\cdot}2x^{1{\cdot}75}$.

Example 1.—To plot the curve $y = 2{\cdot}2x^{1{\cdot}75}$, values of x ranging from 0 to 4.

$$y = 2{\cdot}2x^{1{\cdot}75}$$

$$\therefore \quad \log y = \log 2{\cdot}2 + 1{\cdot}75 \log x.$$

Arrange a table according to the following plan: In the first

column write the selected values of x; in the second column write the values of log x. With one setting of the slide rule the values of 1·75 log x can be read off; and these must be written in the third column. In the fourth column we must write the values of log y, which are obtained by addition; then the antilogs of the figures in column 4 will be the values of y in column 5.

The advantage of working with columns rather than with lines is seen; thus we write down all the values of log x before any figure is written in the third column, and this saves needless turning over of pages, etc.

Table :—

x	log x	1·75 log x + log 2·2	log y	y
0	$-\infty$	$-\infty$ + ·3424	$-\infty$	0
·5	$\bar{1}$·699 = − ·301	− ·527 + ·3424	$\bar{1}$·8154	·654
1·0	0	0 + ·3424	·3424	2·2
1·5	·1761	·308 + ·3424	·6504	4·47
2·0	·3010	·527 + ·3424	·8694	7·40
2·5	·3979	·696 + ·3424	1 0384	10·92
3·0	·4771	·835 + ·3424	1·1774	15·04
3·5	·5441	·952 + ·3424	1·2944	19·7
4·0	·6021	1·054 + ·3424	1·3964	24·91

The plotting is shown in Fig. 190.

Use of the Log Log Scale on the Slide Rule.—The use of the log log scale now placed on some slide rules would obviate a great amount of the calculation in this and similar examples.

e. g., taking $y = 2{\cdot}2x^{1\cdot75}$ and disregarding the factor 2·2 until the end—

$$\log y = 1{\cdot}75 \log x$$

$$\therefore \quad \log(\log y) = \log 1\ 75 + \log(\log x)$$

$$i.\,e., \quad \log Y = \log 1{\cdot}75 + \log X$$

$$\text{where} \quad Y = \log y, \text{ and } X = \log x.$$

Therefore if a length on the ordinary log scale, say the C scale, be added to a length on the log log scale, which is usually the extreme scale, the result on the log log scale will be that required.

If $y = x^{1\cdot75}$; and supposing the value of y is required when $x = 2{\cdot}5$.

Set the index of the C scale level with 2·5 on the log log scale, move the cursor until over the power, 1·75, on the C scale: then the reading on the log log scale (4·96) is the value of $2{\cdot}5^{1\cdot75}$. Multiplication by 2·2, for $y = 2{\cdot}2x^{1\cdot75}$, can be done with one setting of the rule after

all the powers have been found. The tabulation would in this case reduce to—

x	$x^{1.75}$	$y = 2.2x^{1.75}$
2·5	4·96	10·92

as an example.

The log log scale is most useful for finding roots.

E. g., to find $\sqrt[5]{432}$. Set 5 on the C scale level with 432 on the log log scale; then the reading on the log log scale opposite the index of the C scale is 3·37, *i. e.*, the 5th root of 432.

Expansion Curves for Gases.—The formula $pv^n = C$, for the expansion or compression curves of gases, is of the same type as that in the last example.

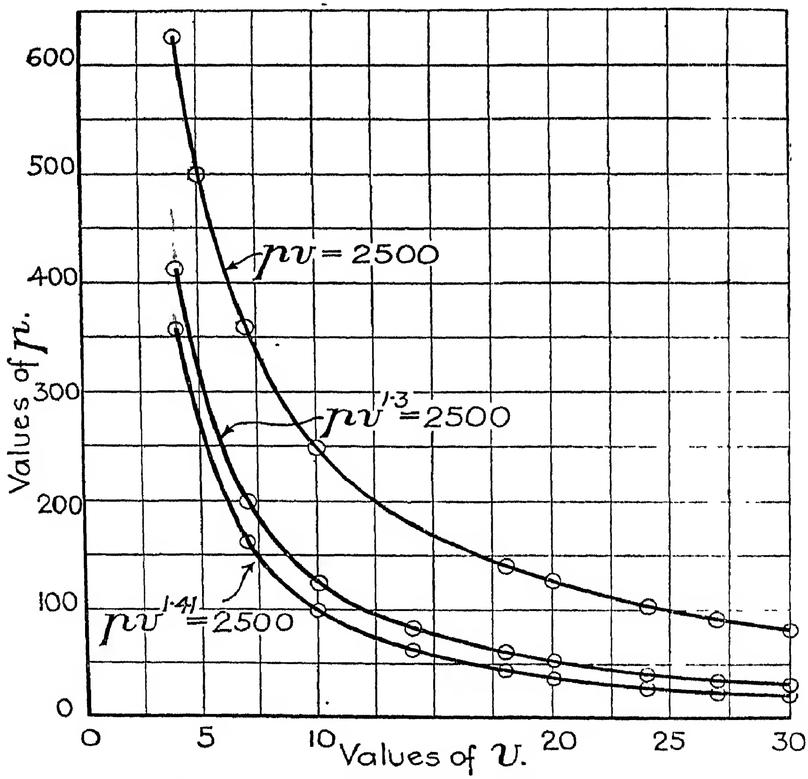


Fig. 191.—Expansion Curves for Gases.

In this formula p is the pressure in lbs. per sq. in. or per sq. ft. and v is the "specific volume," *i. e.*, volume in cu. ft. of 1 lb. whilst n and C are constants varying with the conditions

Thus for air expanding adiabatically, *i. e.*, without loss or gain of heat, $n = 1{\cdot}41$: for the gas in the cylinder of a gas engine $n = 1{\cdot}37$: for isothermal expansion, *i. e.*, expansion at constant temperature, $n = 1$. It is instructive to plot two or three expansion curves on the same diagram, n alone varying, and thus to note the effect of this change.

Example 2.—Plot, on the same diagram and to the same scales, from $v = 4$ to $v = 30$, the curves representing the equations: (*a*) $pv^{1\cdot41} = 2500$, (*b*) $pv^{1\cdot3} = 2500$, (*c*) $pv = 2500$. The plotting is shown in Fig. 191.

Each equation is of the form $pv^n = C$

$$\therefore \quad \log p + n \log v = \log C$$
$$\text{or} \quad \log p = \log C - n \log v.$$

Dealing with the separate cases—

(a) *Adiabatic expansion of air*; $n = 1{\cdot}41$

$$\log p = \log 2500 - 1{\cdot}41 \log v.$$

The arrangement of the table is as follows :—

v	$\log v$	$\log 2500 - 1{\cdot}41 \log v$	$\log p$	p
4	·602	3·398 — ·847	2 551	356
7	·845	1·191	2·207	161
10	1·0	1·41	1·988	97 3
14	1·146	1·615	1·783	60·7
18	1·255	1·770	1·628	42·5
20	1·301	1·834	1·564	36·6
24	1·380	1·945	1·453	28·4
27	1·431	2·02	1·378	23·9
30	1·477	2·082	1·316	20·7

(b) *Expansion of superheated steam*; $n = 1{\cdot}3$.

Values of v and $\log v$ are as above; and the table is completed as shown :—

$\log 2500 - 1{\cdot}3 \log v$	$\log p$	p
3·398 — ·783	2·615	412
1·097	2·301	200
1·3	2·098	125
1·49	1·908	80 9
1·632	1·766	58·3
1·691	1·707	50 9
1·794	1·604	40·2
1·860	1·538	34·5
1·920	1·478	30·1

(c) *Isothermal expansion*; $n = 1$.

v	4	7	10	14	18	20	24	27	30
$p = \frac{2500}{v}$	625	357	250	179	139	125	104	92·6	83·3

It will be seen that the bigger the value of n, the steeper is the curve, or, in other words, the slope of the curve depends on n. All these curves are hyperbolas, that for (*c*) being the special case of the *rectangular* hyperbola.

A Construction for drawing Curves of the Type $pv^n = C$. —In this construction the position of one point on the curve must be known.

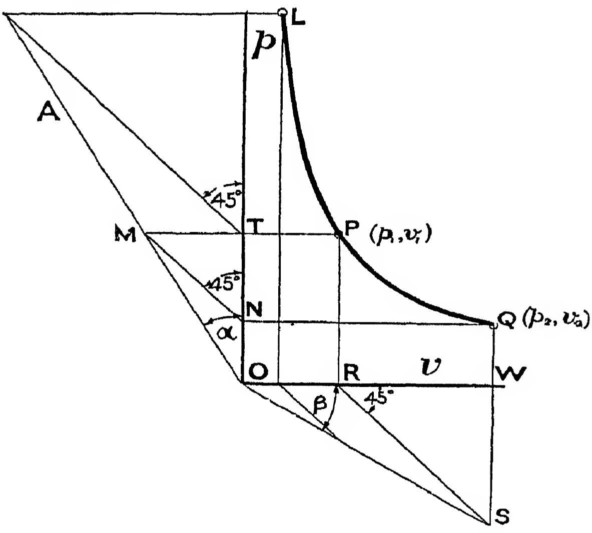


Fig. 192.—Construction for Curves of the $pv^n = C$ type.

Let P, $(p_1 v_1)$, be the given point. Fig. 192.

Choose any angle α, say 30°: set it off as shown, and also an angle β, calculated from the equation—

$$\tan\beta = 1 - (1 - \tan\alpha)^{\frac{1}{n}}$$

Draw the horizontal PM to meet OA in M, and the vertical PR to meet OR in R. Draw MN making 45° with ON and RS making 45° with OR.

A horizontal through N meets a vertical through S in Q; then Q is another point on the curve: also the construction for the point L on the other side of P is indicated.

Proof of the Method—

$$\tan\alpha = \frac{\text{MT}}{\text{OT}} = \frac{\text{TN}}{\text{OT}} = \frac{p_1 - p_2}{p_1} = 1 - \frac{p_2}{p_1}$$

$$\tan\beta = \frac{\text{SW}}{\text{OW}} = \frac{\text{RW}}{\text{OW}} = \frac{v_2 - v_1}{v_2} = 1 - \frac{v_1}{v_2}$$

but $\quad \tan\beta = 1 - (1 - \tan\alpha)^{\frac{1}{n}}$

$$\therefore \quad 1 - \frac{v_1}{v_2} = 1 - \left(1 - 1 + \frac{p_2}{p_1}\right)^{\frac{1}{n}}$$

$$\therefore \qquad \frac{v_1}{v_2} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{n}}$$

$$i.\ e., \qquad \frac{v_1^{\ n}}{v_2^{\ n}} = \frac{p_2}{p_1}$$

$$\text{or} \qquad p_1 v_1^{\ n} = p_2 v_2^{\ n}, \qquad i.\ e., \qquad pv^n = \text{Constant.}$$

Example 3.—If $n = \cdot 9$ and $\alpha = 30°$, calculate the value of β.

$$\tan\beta = 1 - (1 - \tan 30°)^{\frac{1}{\cdot 9}} = 1 - (1 - \cdot 5774)^{1 \cdot 111}$$
$$= 1 - (\cdot 4226)^{1 \cdot 111}$$

Let— $x = (\cdot 4226)^{1 \cdot 111}$

then $\log x = 1 \cdot 111 \times \log \cdot 4226 = 1 \cdot 111 \times \bar{1} \cdot 6259$

$$= -1 \cdot 111 + \cdot 696$$
$$= \bar{1} \cdot 585$$

whence $x = \cdot 3846$

Then $\tan\beta = 1 - \cdot 3846 = \cdot 6154 = \tan 31° 36'$

or $\beta = \underline{31° 36'}.$

If $n = 1$, then $\tan\beta = \tan\alpha$

$i.\ e., \qquad \beta = \alpha.$

Note.—30° is rather a large angle for α if the range to be covered is small. Accordingly, the value of β is stated here, for $\alpha = 10°$ and $n = 1 \cdot 37$.

$$\tan\beta = 1 - (1 - \tan 10°)^{\frac{1}{1 \cdot 37}} = 1 - (1 - \cdot 1763)^{\cdot 73}$$
$$= 1 - \cdot 871 = \cdot 129$$
$$\therefore \qquad \beta = 7° 21'.$$

Example 4.—A tube 3″ internal and 8″ external diameter is subjected to a collapsing pressure of 5 tons per sq. in.: show by curves the radial and circular stresses everywhere, it being given that at a point r ins. from the axis of the cylinder—

The radial stress $p = A + \frac{B}{r^2}$ and the circular stress $q = A - \frac{B}{r^2}$

Note that $p = 5$ tons per sq. in. when $r = 4''$; and $p = 0$ when $r = 1 \cdot 5''$, and the object is to first find the values of the constants A and B from the data given.

From the given conditions—

$$5 = A + \frac{B}{16}$$

$$0 = A + \frac{B}{2 \cdot 25}$$

Subtracting— $$5 = B\left(\frac{1}{16} - \frac{1}{2 \cdot 25}\right)$$
$$= B\,(\cdot 0625 - \cdot 4444)$$

$$\therefore \qquad 5 = -\cdot 382B$$

$$\text{or} \qquad B = \frac{5}{-\cdot 382} = -13 \cdot 1.$$

Also—
$$5 = A + (\cdot 0625 \times -13\cdot 1) = A - \cdot 818$$
$$\therefore \quad A = 5\cdot 818.$$
Hence—
$$p = 5\cdot 818 - \frac{13\cdot 1}{r^2}, \quad q = 5\cdot 818 + \frac{13\cdot 1}{r^2}$$

[Note that $(p + q) = 11\cdot 636 =$ constant. The material is subjected to crushing stresses p and q in two directions at right angles to one another and in the plane of the paper: therefore dimensions at

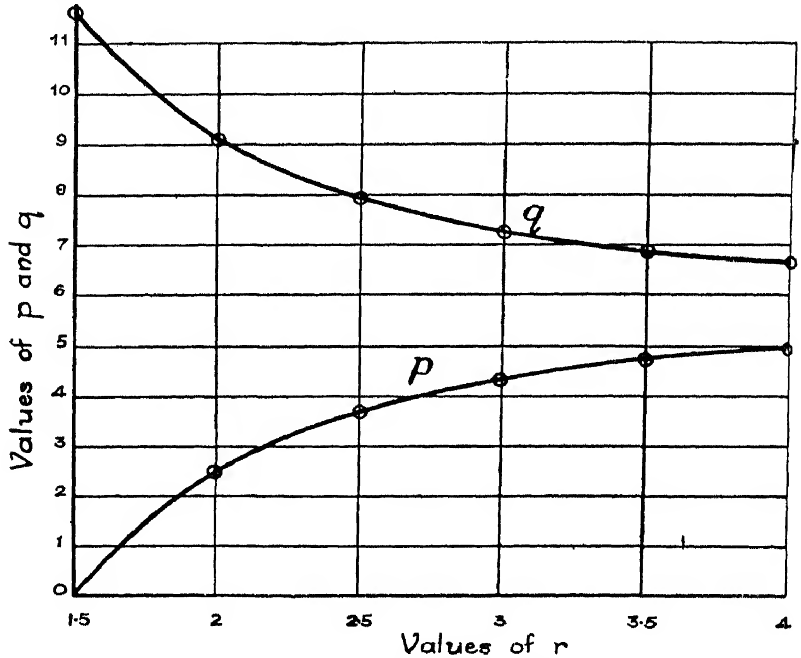


Fig. 193.—Curves of Radial and Hoop Stresses.

right angles to the paper must elongate by an amount proportional to $(p + q)$. If the cross-section is to remain plane this elongation must be constant; hence $(p + q)$ must also be constant.]

To calculate values of p and q the table would be set out as follows:—

r	r^2	$\frac{13\cdot 1}{r^2}$	$5\cdot 818 + \frac{13\cdot 1}{r^2} = q$	$5\cdot 818 - \frac{13\cdot 1}{r^2} = p$
1·5	2·25	5·818	11·636	0
2·0	4	3·275	9·093	2·543
2·5	6·25	2·095	7·913	3·723
3·0	9	1·455	7·273	4·363
3·5	12·25	1·068	6·886	4·750
4·0	16	·818	6·636	5

The curves are shown plotted in Fig. 193. It is customary, however, to plot the curves of radial and hoop stress in the manner shown in Fig. 194, where curve (1) gives the radial stress at any point between a and b, and curve (2) gives the circular or hoop stress at any point between a_1 and b_1.

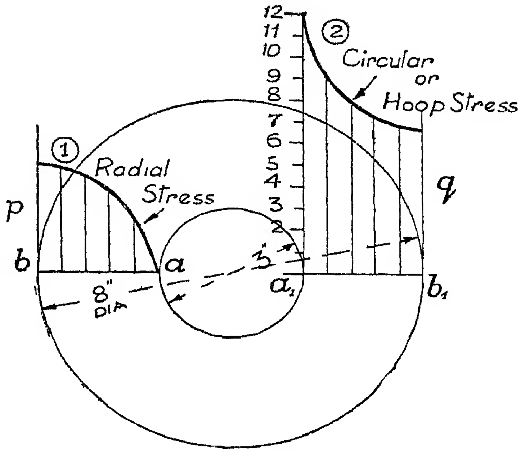


Fig. 194.—Curves of Radial and Hoop Stresses.

Example 5.—According to a certain scheme (refer to p. 212), the depreciation fund in connection with a machine can be expressed by—

$$A = \frac{D}{r}\{(1 + r)^n - 1\}$$

where—

D = amount contributed yearly to the sinking fund, and—

$100r$ = percentage rate of interest allowed on same.

For a machine whose initial value is £500 and scrap value is £80, D is found to be £14 14*s*., if 3% interest per annum be allowed. If the life of the machine is 21 years, plot a curve to show the state of the sinking fund at any time, *i. e.*, plot the curve—

$$A = \frac{14 \cdot 7}{\cdot 03}\{1 \cdot 03^n - 1\}, \quad n \text{ varying from 0 to 21.}$$

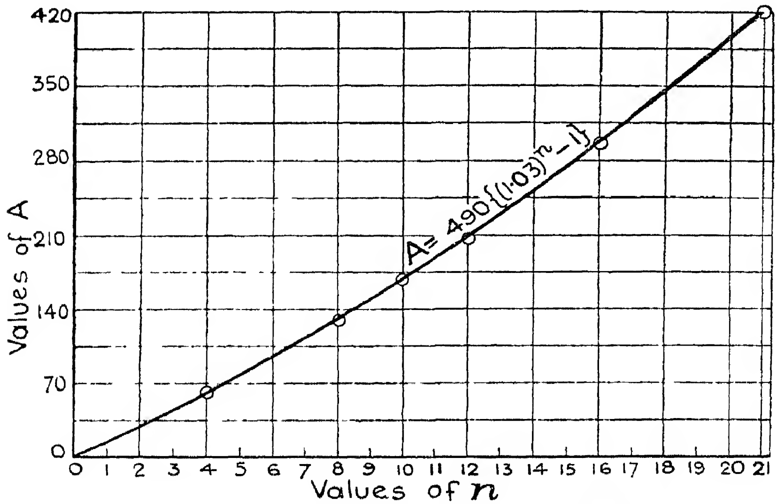


Fig. 195.—Curve of Depreciation Fund for Machine.

It will be advisable to work out $1{\cdot}03^n$ separately.

Let— $1{\cdot}03^n = x$; then $\log x = n \log 1{\cdot}03 = {\cdot}0128n$

also $\dfrac{14{\cdot}7}{{\cdot}03} = 490.$

Taking a few values only for n, between 0 and 21, the tabulation will be as follows:—

n	${\cdot}0128n = \log x$	x	$x - 1$	$\frac{14{\cdot}7}{{\cdot}03}(x-1) = A$
0	0	1	0	0
4	·0512	1·126	·126	61·7
8	·1024	1·266	·266	130
10	·128	1·343	·343	168
12	·1536	1·424	·424	207·5
16	·2048	1·603	·603	296
21	·2688	1·857	·857	420

and the plotting is shown in Fig. 195.

Equations to the Conic Sections.—A knowledge of the form of the curve that represents some particular type of equation may ensure a great saving of time and thought. Values for the variables need not then be chosen at random and beyond the range of the curves

The equations to the conic sections are here given because many of the curves occurring in practice are of one of these forms.

The Ellipse.—If the origin be taken at the centre of the ellipse, the equation is—

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are the half-major and half-minor axes respectively, or the maximum values of x and y.

If the equation is given in a slightly different form, it should be put into the standard form before any values are selected.

Example 6.—Plot the curve representing the equation $3x^2 + 5y^2 = 60$

$3x^2 + 5y^2 = 60$ is the equation of an ellipse, and can be written—

$$\frac{3x^2}{60} + \frac{5y^2}{60} = 1$$

i. e., the equation is divided throughout by 60, so that the right-hand side becomes unity.

Thus— $$\frac{x^2}{20} + \frac{y^2}{12} = 1$$

so that $a^2 = 20$, and $a = \pm 4{\cdot}472$
$b^2 = 12$, and $b = \pm 3{\cdot}464$.

Hence the range of x is from $-4{\cdot}472$ to $+4{\cdot}472$, and no lower or higher values respectively should be taken.

If the values of y are to be calculated, we have—

$$5y^2 = 60 - 3x^2$$
$$y^2 = 12 - \cdot 6x^2$$
$$y = \pm\sqrt{12 - \cdot 6x^2}$$

Dealing only with one-half of the ellipse, the table of values reads—

x	x^2	$12 - \cdot 6x^2$	y^2	y
0	0	12 − 0	12	± 3·464
1	1	12 − ·6	11·4	± 3·38
2	4	12 − 2·4	9·6	± 3·10
3	9	12 − 5·4	6·6	± 2·55
4	16	12 − 9·6	2·4	± 1·547
4·472	20	12 − 12	0	0

The other half can be obtained by projection, and Fig. 196 is plotted. If the graphic method of drawing an ellipse is known this

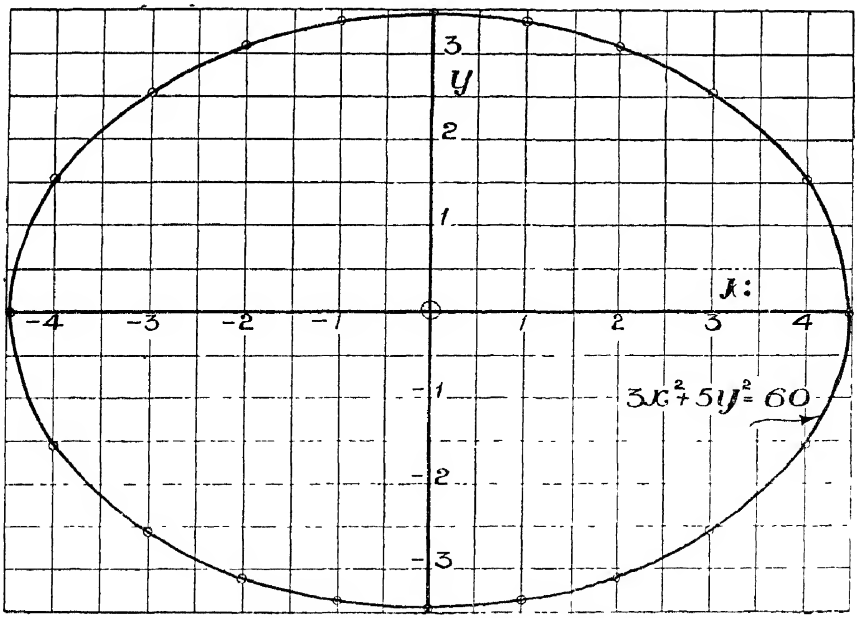


Fig. 196.—Curve of Equation to Ellipse.

calculation is unnecessary: all that is required from the equation being the lengths of the axes.

An application of the ellipse is found in the *Ellipse of Stress*, in the subject of Strengths of Materials. It is required to determine the magnitude and direction of the resultant stress on a

plane BD, due to the stresses f_1 and f_2 acting as indicated in Fig. 197.

It is found that the resultant stress $f = \sqrt{f_1^2 \cos^2 \theta + f_2^2 \sin^2 \theta}$, and if α is the angle made with f_1

$$\tan \alpha = \frac{f_2}{f_1} \tan \theta.$$

If an ellipse be constructed with axes to represent the original stresses, the resultant stress can very easily be read from it.

Along OQ in Fig. 198 and perpendicular to BD, mark off a length OQ to represent f_1, and a length OR to represent f_2. Draw

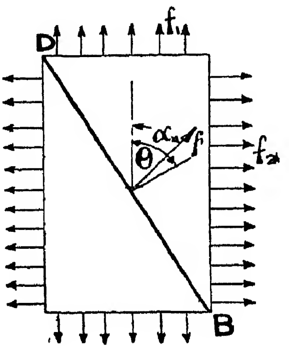


Fig. 197.

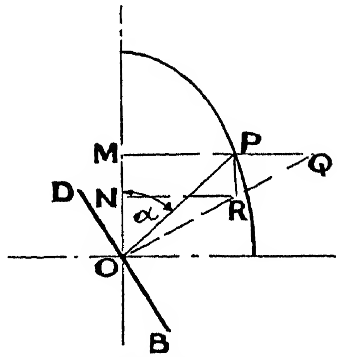


Fig. 198.

Ellipse of Stress.

a horizontal QM to meet a vertical PR in P; then OP represents f and $\angle$ MOP $= \alpha$.

To show that P lies on an ellipse, we must prove that the equation governing P's position is of the nature $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\text{OM} = \text{OQ} \cos \theta = f_1 \cos \theta$$
$$\text{MP} = \text{RN} = \text{OR} \sin \theta = f_2 \sin \theta$$
$$\therefore \quad (\text{PO})^2 = (\text{OM})^2 + (\text{MP})^2 = f_1^2 \cos^2 \theta + f_2 \sin^2 \theta = f^2$$
$$\textit{i. e.,} \quad \text{OP} = f$$

If the origin is at O, and x and y are the co-ordinates of P—

then $\quad x = \text{MP} = f_2 \sin \theta, \quad$ and $\quad y = \text{OM} = f_1 \cos \theta$

$$\therefore \quad \frac{x}{f_2} = \sin \theta, \ \frac{y}{f_1} = \cos \theta.$$

but $\quad \sin^2 \theta + \cos^2 \theta = 1$ for all values of θ

$$\therefore \quad \frac{x^2}{f_2^2} + \frac{y^2}{f_1^2} = 1$$

or P lies on an ellipse the lengths of whose axes are $2f_2$ and $2f_1$

The circle may be regarded as a special case of the ellipse, where $a = b$, i.e., $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ or $x^2 + y^2 = a^2$, a being the radius of the circle.

e.g., $$5x^2 + 5y^2 = 45$$

can be written— $$x^2 + y^2 = 9,$$

which represents a circle of radius 3 units.

The Parabola.—If the axis is horizontal, and the vertex at the origin, then the equation is $y^2 = 4ax$.

If the axis is vertical, the equation is $x^2 = 4ay$, where $4a =$ length of the "latus rectum," the chord through the focus perpendicular to the axis.

To make the investigation more general, let x be changed to $x + c =$ say, $x + \cdot 7$: and y to $y + c_1 =$ say, $y + 11\cdot 45$; also let $4a = \cdot 2$.

Then the case will be that of the parabola having a latus rectum of $\cdot 2$, and the axis will be vertical, with the vertex at the point $-\cdot 7$, $-11\cdot 45$.

The equation is—

$$\begin{aligned} (x+\cdot 7)^2 &= \cdot 2(y+11\cdot 45) \\ x^2+1\cdot 4x+\cdot 49 &= \tfrac{1}{5}(y+11\cdot 45) \\ 5x^2+7x+2\cdot 45-11\cdot 45 &= y \\ \text{or} \qquad y &= 5x^2+7x-9. \end{aligned}$$

(This curve is shown plotted in Fig. 88.)

Conversely, the equation $y = 5x^2 + 7x - 9$ might be put into the standard form, thus—

$$\begin{aligned} y &= 5(x^2+1\cdot 4x-1\cdot 8) \\ &= 5(x^2+1\cdot 4x+(\cdot 7)^2-(\cdot 7)^2-1\cdot 8) \\ &= 5(x+\cdot 7)^2-11\cdot 45 \\ \therefore \qquad \frac{y+11\cdot 45}{5} &= (x+\cdot 7)^2 \end{aligned}$$

which equation is of the form $4aY = X^2$

where $4a = \frac{1}{5}$, $Y = y+11\cdot 45$, and $X = x+\cdot 7$.

This analysis is useful if the position of the vertex, say, is desired and the curve itself is not needed. (Compare maximum and minimum values.)

For the parabolas occurring in practical problems the simpler forms are sufficient.

The Hyperbola.—If the centre of the hyperbola is at the origin, the equation is—

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $2a$ = the length of the transverse axis (along the x axis)
$2b$ = the length of the conjugate axis (along the y axis).

No values should be taken for x between $-a$ and $+a$, for there is no part of the curve there.

Example 7.—Plot the curve representing the equation—

$$2x^2 - 5y^2 = 48. \qquad \text{(Fig. 199.)}$$

By dividing throughout by 48 the equation may be written—

$$\frac{x^2}{24} - \frac{y^2}{9 \cdot 6} = 1$$

so that $a = \pm\sqrt{24} = \pm 4 \cdot 9$

and $b = \pm\sqrt{9 \cdot 6} = \pm 3 \cdot 1$

If a rectangle be constructed by verticals through $x = -4 \cdot 9$ and $+4 \cdot 9$, and horizontals through $y = -3 \cdot 1$ and $+3 \cdot 1$, the diagonals of this rectangle will be the "asymptotes" of the hyperbola, *i. e.*, the boundaries of the curves are known.

To calculate values—

$$-5y^2 = 48 - 2x^2$$
$$5y^2 = 2x^2 - 48$$
$$y^2 = \cdot 4x^2 - 9 \cdot 6$$
$$\therefore \qquad y = \pm\sqrt{\cdot 4x^2 - 9 \cdot 6}$$

i. e., an expression is found for y in terms of x.

The table of values reads ;—

x	x^2	$\cdot 4x^2 - 9 \cdot 6$	y^2	y
4·9	24	9·6 − 9·6	0	0
5·0	25	10 − 9·6	·4	± ·632
5·5	30·3	12·1 − 9·6	2·5	± 1·58
6·0	36	14·4 − 9·6	4·8	± 2·19

This is the calculation for one branch of the curve only; and the other branch may be obtained by writing $-x$ for $+x$ throughout, *e. g.*, when $x = -5$, $y = \pm \cdot 632$; therefore project across.

If $a = b$, then—

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$$

or

$$x^2 - y^2 = a^2.$$

For this case the asymptotes are at right angles, and the hyperbola is rectangular.

To find the equation of the hyperbola when referred to the asymptotes as axes (see Fig. 199.)—From P a point on the curve, draw PN parallel to OF and PM parallel to OE. Let PM $= p$,

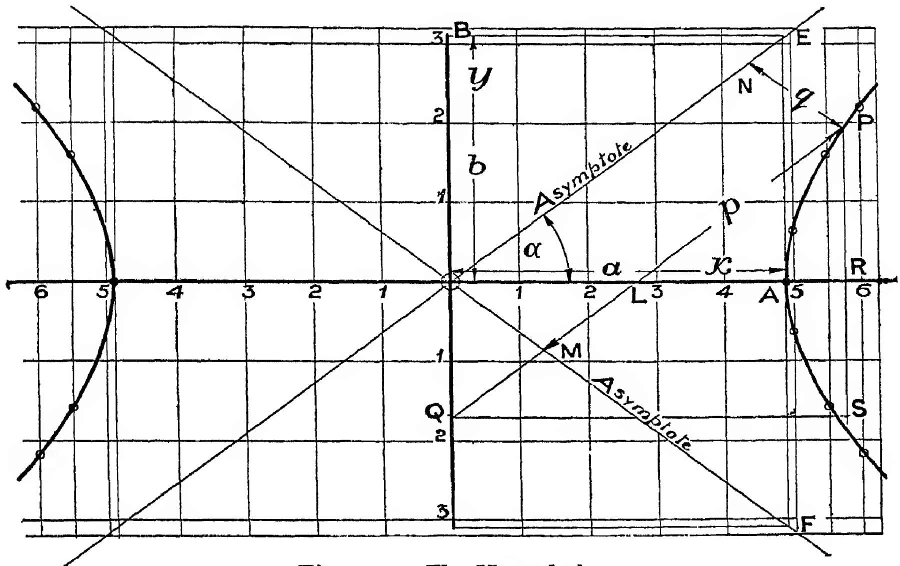


Fig. 199.—The Hyperbola.

and PN $= q$; then the co-ordinates, when the asymptotes are axes, are (p, q). Note that PN and PM are parallel to the asymptotes, and not perpendicular to them.

Let— $\angle \text{EOA} = \alpha$; then $\tan \alpha = \frac{b}{a}$

i. e., $\cos \alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin \alpha = \frac{b}{\sqrt{a^2+b^2}}$

$$\text{OM} = \text{NP} = q.$$

also $\text{QM} = \text{OM} = \text{ML} = q$ (From equality of angles)

$$\text{PL} = \text{PM}-\text{ML} = p-q$$

$$\text{PQ} = \text{PM}+\text{MQ} = p+q$$

$$\frac{\text{PR}}{\text{PL}} = \sin \alpha = \frac{b}{\sqrt{a^2+b^2}} \quad \text{i.e.,} \quad \frac{y}{p-q} = \frac{b}{\sqrt{a^2+b^2}}$$

$$p-q = \frac{y}{b}\sqrt{a^2+b^2}$$

or $$p^2+q^2-2pq = \frac{y^2}{b^2}(a^2+b^2) \quad \ldots\ldots\ldots \quad (1)$$

also $$\frac{\text{QS}}{\text{PQ}} = \cos \alpha = \frac{a}{\sqrt{a^2+b^2}} \quad \text{i. e.,} \quad \frac{x}{p+q} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\therefore \quad p^2+q^2+2pq = \frac{x^2}{a^2}(a^2+b^2) \quad \ldots\ldots\ldots \quad (2)$$

By subtracting (1) from (2)—

$$4pq = \left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)(a^2+b^2)$$

$$= a^2+b^2 \;.\;.\;.\;.\;.\; \text{since } \frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$$

$$\therefore \quad pq = \frac{a^2+b^2}{4}$$

But a and b are constants, therefore the product of the co-ordinates p and q is constant: *this is a most important relation.*

If the hyperbola is rectangular, $b = a$ (the asymptotes being at right angles)

$$\text{and} \qquad pq = \frac{a^2}{2}$$

Compare the equation $pv = C$, for the isothermal expansion of a gas.

Example 8.—Find the equation of the hyperbola $x^2-3y^2 = 3$, referred to its asymptotes. Answer: $\underline{pq = 1}$.

Exercises 37.—On the plotting of Equations of the Type $y = ax^n + b$.

1. Plot, for values of x ranging from 1 to 9, the curve $y = 5{\cdot}76x^{1{\cdot}29}$.
2. Plot the curve $2y = {\cdot}064x^{-{\cdot}27}$ from $x = 0$ to $x = 2$.
3. Plot on the same axes the curves $y_1 = 4{\cdot}2x^{1{\cdot}63}$ and $y_2 = {\cdot}31x^{3{\cdot}47}$ and by adding corresponding ordinates obtain the curve
$y = 4{\cdot}2x^{1{\cdot}63} + {\cdot}31x^{3{\cdot}47}$. {$x$ to range from ${\cdot}2$ to $3{\cdot}5$ }
4. Plot, from $y = -{\cdot}5$ to $y = +{\cdot}5$, a curve to give values of C,

$$\text{when } C = 1{\cdot}69\left(\log_e 3 - \frac{1}{y+1}\right)$$

5. Formulæ given for High Dams are as follows :—

where x = depth in feet of a given point from the top
y = horizontal distance in feet from such point to flank of dam
z = horizontal distance in feet from such a point to face of dam
P = safe pressure in tons per sq. ft. on the masonry

$$y = \sqrt{\frac{{\cdot}05x^3}{P+{\cdot}03x}}; \quad z = \left(\frac{{\cdot}09x}{P}\right)^4$$

Draw the section of a dam 30 ft. deep, allowing $P = 4{\cdot}5$.

6 For a steam engine, if x = mean pressure (absolute) expressed as a percentage of the initial pressure (absolute), and y = cut-off expressed as a percentage of the stroke, then—

$$x = y(5{\cdot}605 - \log_e y).$$

Plot a curve giving values of x for values of y between 0 and 70.

7. If a number of observations have been made, say, for a length

of a chain line in a survey, then the probable error e of the mean of the observations can be calculated from—

$$e = \cdot 6745 \sqrt{\frac{\Sigma r^2}{n(n-1)}}$$

where r = difference between any observation and the mean observation and n = number of observations. If $\Sigma r^2 = 7\cdot 2$, plot a curve to give values of e for values of n between 2 and 30.

8. Plot on the same axes the curves—

(a) $pv^{1\cdot 13} = 4000$ and (b) $pv^{\cdot 8} = 2540$, v ranging from 4 to 32.

9. In Fig. 200,

D = the outside diameter of a worm wheel
$= 2A\left(1-\cos\frac{a}{2}\right) + d.$

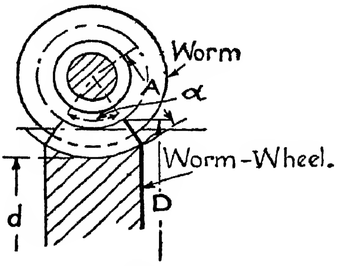


Fig. 200.

If $d = 4$ and $A = \cdot 75$, show by a graph the variation in D due to a variation in a from 20° to 60°.

10. The calculated efficiency η of worm gearing is found from—

$$\eta = \frac{\tan a(1 - \mu \tan a)}{\mu + \tan a}$$

where μ = coefficient of friction and a = angle of the worm.

If $\mu = \cdot 15$, plot a curve to show efficiencies for angles from 0° to 50°.

11. The ideal efficiency η of a gas engine is given by $\eta = 1 - \left(\frac{1}{r}\right)^{n-1}$

If $n = 1\cdot 41$, and r = compression ratio, plot a curve giving the efficiency for any compression ratio between 3 and 18.

12. A machine costs £500; its value as scrap is £80.

Plot curves to show the state of the depreciation fund as reckoned by the two methods—

(a) Equal amounts put away each year.
(b) A constant percentage of the value of the preceding year set aside each year.

The fund at the end of n years $= 500\,[1-(1-\cdot 0836)^n]$, and the life of the machine is 21 years.

13. The capacity K per foot of a single telegraph wire far removed from the earth is $K = \dfrac{33\cdot 9}{2\log_e\frac{l}{r} - \cdot 618}$ microfarads. Plot a curve to give the capacity for wires for which the ratio $\frac{l}{r}$ increases from 500 to 20000.

14. Hutton's formula for wind pressure on a plane inclined to the actual direction of the wind is—

$$p = P(\sin\theta)^{1\cdot 84\cos\theta - 1}$$

where P = pressure on a plane at right angles to the direction of the wind,

p = pressure on a surface inclined at θ to the direction of the wind.

If P = 20 lbs. per sq. ft., plot a curve giving values of p for any angle up to 90°.

15. Plot a curve showing the H.P. transmitted by a belt lapping 180° round a pulley for values of the velocity v from 0 to 140, the coefficient of friction μ being ·2.

$$\text{H.P.} = \frac{v}{1100}\left(\text{T}-\frac{wv^2}{g}\right)\left(1-\frac{1}{e^{\mu\theta}}\right)$$

$\text{T} = 350$, $w = \cdot 4$, $g = 32\cdot 2$, $\theta =$ angle of lap in radians.

16. Aspinall gives as a rule for determining the resistance to motion of trains—

$$\text{R} = 2\cdot 5 + \frac{\text{V}^{\frac{5}{3}}}{65\cdot 82}$$

where R = resistance in lbs. per ton, V = velocity in miles per hour.

Plot a curve to give values of R for all velocities up to 55 m.p.h.

17. Find the value of r (the ratio of expansion), which makes W (brake energy per lb. of steam) a maximum.

$$\text{W} = \frac{120\,\frac{1+\log_e r}{r} - 27}{\frac{\cdot 00833}{r} + \cdot 000903}$$

18. The efficiency η of a three-stage air compressor with spray injection is given by—

$$\eta = \frac{\log_e r}{\frac{3n}{n-1}\left(r^{\frac{n-1}{3n}} - 1\right)}$$

where $n = 1\cdot 2$ and $r =$ ratio of compression.

Plot a curve giving the efficiency for any compression ratio between 2 and 12.

19. Determine the length of the latus rectum and also the co-ordinates of the vertex of the parabola $5y = 2x^2 - 11x - 27$.

20. A rectangular block is subjected to a tensile stress of 5 tons per sq. in. and a compressive stress of 3 tons per sq. in. Draw the ellipse of stress and read off the magnitude and direction of the resultant stress on the plane whose normal is inclined at 40° to the first stress. [*Hint.*—Refer to p. 346.]

Curves representing Exponential Functions.—To plot the curve $y = e^x$, where e has its usual value, one may work directly from the tables, or a preliminary transformation of the formulæ may be necessary. If tables of powers of e are to hand, the values of y corresponding to certain values of x are read off at a glance; and in such a case the values of x selected are those appearing in these tables.

Example 9.—Plot the curves $y = e^x$ and $y = e^{-x}$ from $x = -4$ to $+4$.

From Table XI at the end of the book the figures are found thus :—

x	-4	-3	-2	-1	0	1	2	3	4
$y = e^x$	·0183	·0498	·1353	·3679	1	2·7183	7·3891	20·08	54·6

When $x = -4$, e^{-4} is required, and this is found in the 3rd column.
" $x = 3$, e^3 is required, and this is found in the 2nd column.

The plotting for these figures is shown in Fig. 201, by the curve (1).
If tables of powers of e are not available, proceed as follows:—

$$y = e^x, \text{ and therefore } \log y = x \log e = \cdot 4343x$$

and the table is arranged thus—

x	$\cdot 4343x = \log x$	y
2	·8686	7·389

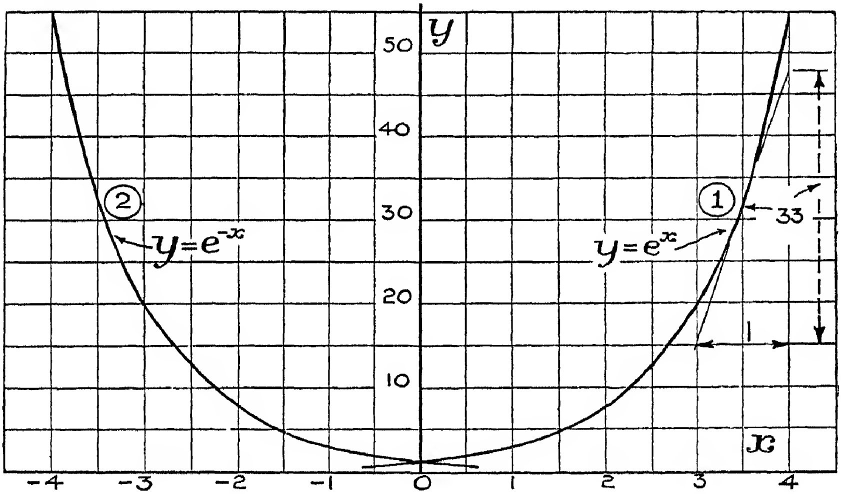


Fig. 201.—Curves of $y = e^x$ and $y = e^{-x}$

Having drawn the curve $y = e^x$, draw the tangent to it at some point and measure its slope; and it will be found that the value of the slope is also the value of the ordinate to the point of contact of the tangent and the curve. Thus, the tangent is drawn to touch the curve at the point for which $x = 3 \cdot 5$: its slope is measured and found to be 33, and this is seen to be the value of y when $x = 3 \cdot 5$.

If for x we write $-x$, i. e., we plot the curve $y = e^{-x}$, we find that this gives a curve exactly similar to the last, but on the other side of the y axis: such would be expected, since x must now be measured as positive towards the left instead of to the right. The curve $y = e^{-x}$ is shown plotted in Fig. 201, and is curve (2).

All equations of this type will be represented by exactly the same form of curve, drawn to different scales.

Example 10.—To plot the curve $y = e^{3x}$.

Write this as $Y = e^X$, where $Y = y$ and $X = 3x$.

Plot the curve $Y = e^X$ exactly as before, and then alter the horizontal scale in such a way that 1 on it now reads $\frac{1}{3}$, and so on.

$$\text{For} \qquad X = 3x$$
$$i.e., \quad \text{construction scale} = 3 \times \text{required scale}$$
$$\text{or} \quad \text{required scale} = \frac{\text{construction scale}}{3}$$

Example 11.—Plot the curve $5y = 4e^{\frac{1}{7}x}$.

This can be written— $\qquad \frac{5}{4}y = e^{\frac{1}{7}x}$

$$i.e., \qquad Y = e^X$$
$$\text{where} \qquad Y = \frac{5}{4}y,\ X = \frac{1}{7}x.$$

Hence, plot $Y = e^X$ from the tables, and alter both scales in such a way that the—

$$\text{New scale for } y = \frac{4}{5} \times \text{construction scale}$$
$$\text{,,} \quad \text{,,} \quad \text{,,} \quad x = 7 \times \quad \text{,,} \quad \text{,,}$$

so that where the vertical construction scale reads 5, 4 must be written; and 7 must be written in place of 1 along the horizontal.

Example 12.—If the E.M.F. is suddenly removed from a circuit containing resistance R, and self-induction (coefficient of self-inductance L), the current C at any time t after removal of the E.M.F. is given by the equation—

$$C = C_0 e^{-\frac{Rt}{L}}$$

Plot a curve to show the dying away of the current for the case when $C_0 = 50$ amps, $R = \cdot 32$ ohm, and $L = \cdot 004$ henry.

Substituting the numerical values—

$$C = 50e^{-\frac{\cdot 32t}{\cdot 004}}$$
$$= 50e^{-80t}$$

It will be sufficient to plot values of C for values of t between $t = 0$ and $\cdot 05$ sec.

$$C = 50e^{-80t}$$
$$\therefore \qquad \bar{C} = e^{-T} \qquad [\ \bar{C} \text{ is spoken of as C bar}]$$
$$\text{where} \qquad \bar{C} = \frac{C}{50} \quad \text{and } T = 80t$$

If the maximum value of t is $\cdot 05$, the maximum value of T must be $80 \times \cdot 05$, *i. e.*, 4.

Hence from the tables :—

T . . .	0	·5	1	2	3	4
$\bar{C}$ i.e., e^{-T}	1	·6065	·3679	·1353	·0498	·0183

These values are shown plotted in Fig. 202, and then the scales are altered so that 1 on the vertical becomes 50, and 1 on the horizontal becomes $\frac{1}{80}$.

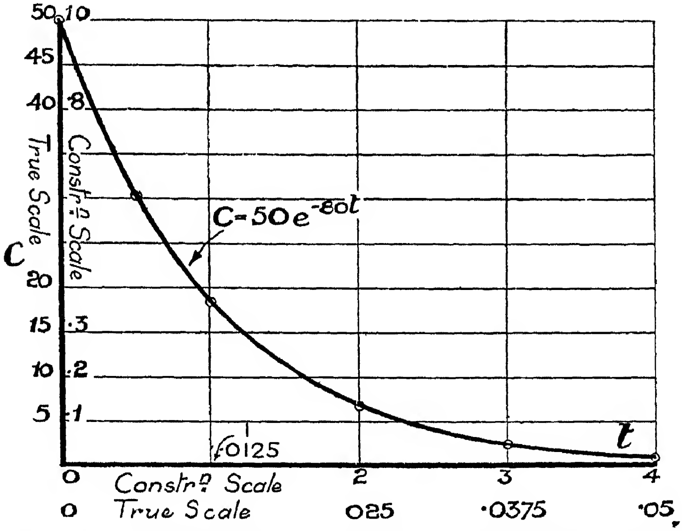


Fig. 202.—"Dying away" of Current in an Electric Circuit.

The saving of time and thought in the calculation of values more than compensates for the somewhat awkward scales that may result, and even this difficulty may be avoided by choosing the original or construction scales suitably.

If it is found that the necessary values of the x or t cannot readily be used, *i. e.*, if values are necessary for x for which no values of e^x, etc., are given in the table, recourse must be made to calculation.

In this case the work would be arranged thus :—

$$\begin{aligned}\log C &= \log 50 - 80t \log e \\ &= 1{\cdot}699 - 80 \times {\cdot}4343t \\ &= 1{\cdot}699 - 34{\cdot}74t\end{aligned}$$

t	$1{\cdot}699 - 34{\cdot}74t$	$\log C$	C
0	1·699 − 0	1·699	50
·0025			
005			
etc.			
·05	1·699 − 1·737	$\bar{1}{\cdot}962$	·9162

Example 13.—If a pull t is applied at one end of a belt passing over a pulley and lapping an angle θ (radians), the pull T at the other end is greatly increased owing to the friction between the belt and the pulley.

If μ = coefficient of friction between belt and pulley

$T = te^{\mu\theta}$

Plot a curve to show values of T as θ increases from 0 to 180°, taking $t = 40$, and $\mu = \cdot 3$.

The angle θ ranges from 0 to 3·14. (π radians = 180°.)

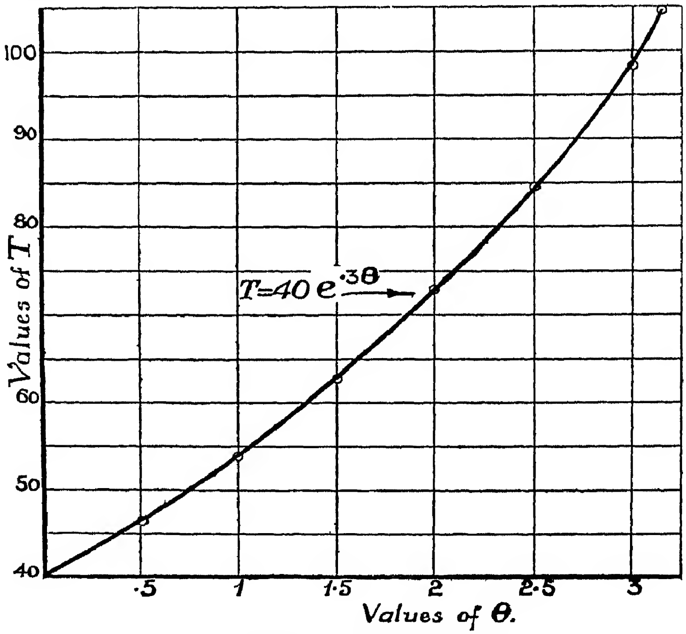


Fig. 203.—Pull on a Belt.

It will be rather more convenient in this case to calculate :—

Substituting values— $T = 40e^{\cdot 3\theta}$

Then— $\log T = \log 40 + \cdot 3\theta \log e$

$= 1\cdot 6021 + \cdot 3 \times \cdot 4343\theta$

$= 1\cdot 6021 + \cdot 1303\theta$

θ	$1\cdot 6021 + \cdot 1303\theta$	log T	T
0	1·6021 + 0	1·6021	40
·5	+ ·0652	1·6673	46·5
1·0	+ ·1303	1·7324	54
1·5	+ ·1955	1·7976	62·8
2·0	+ ·2606	1·8627	72·9
2·5	+ ·3258	1·9279	84·7
3·0	+ ·3909	1·9930	98·4
3·14	+ ·4180	2·0201	104·7

i. e., when the belt is in contact for half the circumference of the pulley the tension is increased in the proportion of 2·6 to 1. In practice a ratio of 2 to 1 is very often adopted. The plotting for this example is shown in Fig. 203.

Example 14.—If an electric condenser of capacity K has its coats connected by a wire of resistance R, the relation between the charge q at any time t secs. and the initial charge q_0 at zero sec. is given by—

$$\frac{q}{q_0} = e^{-\frac{t}{RK}}$$

Find the time that elapses before the charge falls to a value $= \frac{1}{e} \times$ initial charge, and indicate the form of the curve which represents the discharge.

$$\text{If } t = \text{RK}, \quad \text{then } q = q_0 e^{-1} = \frac{q_0}{e} = \frac{1}{2{\cdot}718} q_0$$

i. e., the charge falls to $\frac{1}{2{\cdot}718}$ of its initial value in time RK secs. This time is termed the "time constant" of the condenser circuit.

The curve representing this discharge would be similar to that plotted for *Example* 12, viz. in Fig. 202.

The Catenary.—Referring to the curves $y = e^x$ and $y = e^{-x}$ if the "mean" curve of these is drawn it will represent the equation—

$$y = \frac{e^x + e^{-x}}{2} \quad \text{i. e., } y = \cosh x.$$

This curve is known as the "catenary"; and it is the curve taken by a cable or wire hanging freely under its own weight. The catenary when inverted is the theoretically correct shape for an arch carrying a uniform load per foot curve of the arch.

If the cable is strained to a horizontal tension of H lbs., and the weight per foot run of the cable is w lbs., then the equation becomes—

$$\frac{y}{c} = \frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2}$$

$$\text{where} \qquad c = \frac{\text{H}}{w}$$

The proof of this rule is rather difficult, and is given in Volume II of *Mathematics for Engineers*.

From what has already been mentioned it should be seen that the catenary is the curve $y = \cosh x$ with the scales in both directions multiplied by c, since its equation can be written—

$$\text{Y} = \frac{e^{\text{X}} + e^{-\text{X}}}{2} = \cosh \text{X}$$

$$\text{Provided} \quad \begin{cases} Y = \dfrac{y}{c} & \text{and} \\ X = \dfrac{x}{c} & \text{then} \end{cases} \begin{cases} y = cY \\ x = cX \end{cases}$$

Therefore, to plot any catenary one can select values of x, read off corresponding values of cosh x from the tables and plot one against the other, afterwards multiplying both scales by c.

If a definite span is suggested, the range of values for X must be selected in the manner indicated in the following example.

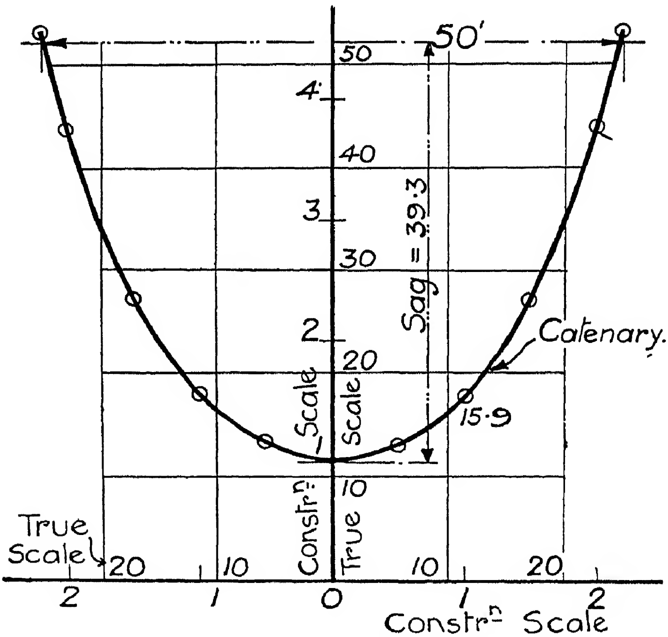


Fig. 204.—The Catenary.

Example 15.—A cable weighing 3·5 lbs. per ft. has a span of 50 ft., and is strained to a tension of 40 lbs. Draw the curve representing the form of the cable. Find the sag, and the tension at 10 ft. from the centre.

Here—
$$c = \frac{40}{3 \cdot 5} = 11 \cdot 42.$$

Also the span is to be 50 ft., *i. e.*, on the "new" or "final" scale 25 ft. must be represented on either side of the centre line.

But, new scale = $c \times$ construction scale

$\therefore$ 25 ft. on new scale = $11 \cdot 42 \times$ X on construction scale

$$\text{or} \qquad X = \frac{25}{11 \cdot 42} = 2 \cdot 19$$

so that no values of X need be taken beyond, say, 2·2.

Taking values of X from 0 to 2·2, the values of cosh X are found from Table XI, thus :—

X	0	·5	1·0	1·5	2·0	2·2
cosh X = Y	1	1·128	1·543	2·352	3·762	4·568

The curve is now plotted, as in Fig. 204, and then for unity on the construction scales 11·42 must be written, and the 25 ft. is marked off on either side of the centre line. The sag is read off as 39·3 ft., using the final scale.

The tension in the cable at any point is measured by the ordinate to the curve multiplied by w; *e. g.*, the tension at 10 ft. from the centre = 3·5 × 15·9 = 55·6 lbs.

Exercises 38.—On the plotting of Curves representing Exponential Functions.

1. Plot, for values of x from $-\cdot 8$ to 2·9, the curve $y = 2e^{-x}$. Find its slope when $x = 1{\cdot}6$.

2. Plot the curve $y = \cdot 25e^{-\frac{1}{3}x}$ from $x = 0$ to $x = 15$.

3. Plot, from $x = -5$ to $x = +3$, the curve $y = \cdot 021 \times 1{\cdot}62^x$.

4. If $C = C_0e^{-at}$, $C_0 = 14{\cdot}6$, $a = 410$, and t ranges from 0 to ·023, represent by a graph the change in C (the dying-away of a current).

5. Plot a curve to give the tension T at one end of a belt for various coefficients of friction μ; the angle of lap (θ radians) being constant. Given that—

$$T = te^{\mu\theta},\ \theta = 165°,\ \text{and}\ t = 50;\ \mu \text{ ranges from } \cdot 1 \text{ to } \cdot 35.$$

6. A cable weighing 2·18 lbs. per ft. and strained to a tension of 56 lbs. hangs freely. Depict the form taken by the cable when the span is 30 ft., and find the tension in it 12 ft. from the centre.

7. $C = 48{\cdot}7(1 - e^{-\frac{Rt}{L}})$. If $R = \cdot 56$, $L = \cdot 008$, plot a current-time (C—t) curve for values of t from 0 to ·062.

8. Trace a graph to show the drop in electric potential down a uniform conductor, if the potential at the receiving end is 200 volts, the resistance per kilometre r of the conductor is 10 ohms, the leakage g of the insulation is $\cdot 5 \times 10^{-6}$ megohms per kilometre, and the distance from the "home" end to the receiving end is 500 kilometres.

If e = the potential at distance x from the receiving end—

$$e = 200 \cosh \sqrt{gr} \, . \, x.$$

Graphs of Sine Functions.

Consider the equation $y = \sin x$. We have already seen in Chapter VI that as the angle x increases from 0° to 90°, the sine y increases from 0 to 1; and as x increases from 90° to 180°, y decreases from 1 to 0. Continuing into the 3rd and 4th quadrants : for x increasing from 180° to 270°, y decreases from 0 to -1; and for x increasing further to 360°, y increases from -1 to 0.

After 360° has been reached the cycle of changes is repeated, *i. e.*, 360° is what is called the *period* for the function $y = \sin x$.

Because y and x are connected by a law, we conclude that the changes will not be abrupt or disjointed, or in other words, the curve representing $y = \sin x$ will be a smooth one.

The sine curve is perhaps the most familiar of all curves, there being so many instances of periodic variation in nature.

Thus, if a curve be plotted showing the variation in the magnetic declination of a place over a number of years, its form will be that of a sine curve: so also for a curve showing the mean temperature, considered over a number of years, for each week of the year.

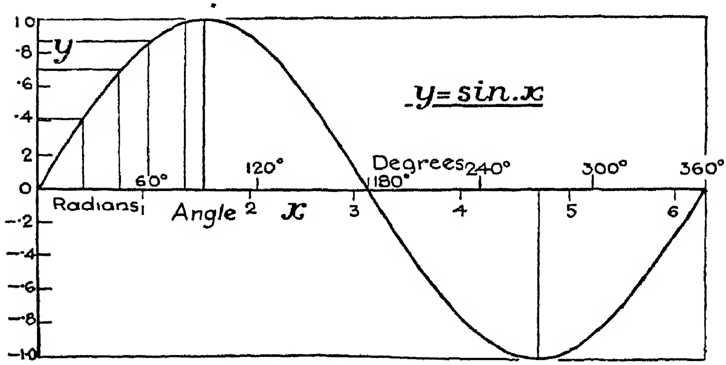


Fig. 205.—Sine Curve.

Sine curves occur frequently in engineering theory and practice; in fact, a sine curve results whenever uniform circular motion is represented to a straight line base.

All sine curves are of the same nature, and therefore it is necessary to carefully study one case, and that the simplest, to serve as a basis.

To plot $y = \sin x$: select values of x between 0° and 90°, thus:—

x degs.	0	25	45	60	80	90
y	0	·423	·707	·866	·985	1

Choose suitable scales so as to admit the full period to be plotted and plot for these values, as in Fig. 205.

No further recourse to the tables is necessary, this portion of the curve being simply drawn out three times.

For— $\sin 100 = \sin (180 - 100) = \sin 80$

and therefore for 10° to the right of 90° the value of y is the same as that for 10° to the left of it, *i. e.*, the curve already drawn can

be traced and pricked through to give the portion of the curve between $x = 90°$ and $x = 180°$.

Again, $\sin 205° = \sin (180° + 25°) = - \sin 25°$
and $\sin 240° = \sin (180° + 60°) = - \sin 60°$

i. e., the 3rd portion of the curve is the 1st portion "folded over" the horizontal axis. Similarly, the 4th will correspond to the 2nd "folded over"; and accordingly we need only concern ourselves with calculations for the 1st quarter of the curve.

The maximum value of y, viz. 1, is spoken of as the *amplitude* of the function. Thus in the case of a swinging pendulum, the greatest distance on either side of its centre position is the amplitude of its motion.

If $y = 5 \sin x$, then the amplitude is 5, and the curve could be obtained from $y = \sin x$ by multiplying the vertical scale by 5.

Example 16.—Plot the curve $y = \cdot 5 \sin 4x$.

Writing this as— $$\frac{y}{\cdot 5} = \sin 4x$$

or $$Y = \sin X$$

$$\left[\text{where} \quad Y = \frac{y}{\cdot 5} = 2y, \quad \text{and} \quad X = 4x\right]$$

we see that the simple sine function is obtained.

Accordingly we plot the curve $Y = \sin X$ (making use of the table on p. 360), and then *alter both scales* so that $x = \dfrac{X}{4}$ and $y = \dfrac{Y}{2}$

Dealing with the last example, we see that the period is $\dfrac{360°}{4}$ or $90°$; *i. e.*, if x is multiplied by 4, the period must be divided by 4.

Similarly for the curve representing $y = \sin \frac{1}{5}x$, the period would be $360 \div \frac{1}{5} = 1800°$. We thus obtain the important rule: **"To obtain the period for a 'sine' function, divide 360° by the coefficient of x or t** (**whichever letter is adopted for the base or 'independent variable'**") or briefly—

$$\textbf{Period in degrees} = \frac{360°}{\textbf{coefficient of } x \textbf{ or } t}$$

Since 2π radians $= 360°$, wherever we have written $360°$ above we should write 2π, if the angle is to be expressed in radians, *i. e.*, the period in radians or seconds (of time)—

$$= \frac{2\pi}{\text{coefficient of the } x \text{ or } t}$$

Thus if— $$y = 4 \sin 6x$$

$$\text{Period} = \frac{2\pi}{6} = \frac{\pi}{3} \quad \text{or} \quad 60°$$

and amplitude $= 4$.

Example 17.—The current in an electric circuit at any time t secs. is given by the expression $C = 4 \cdot 5 \sin 100\pi t$.

Plot a curve to show the change in the current for a complete period.

The general formula is $C = C_0 \sin 2\pi ft$, where f = number of cycles per second = frequency. In this case $2\pi f = 100\pi$, $\therefore f = 50$.

If $f = 50$, the time for one cycle, or the period, must $= \frac{1}{50} = \cdot 02$ sec. Thus the periodic time $= \cdot 02$ sec.

Notice that the period is given in terms of *seconds* (of time) in this case, and not in degrees.

The same periodic time would have been obtained if our previous rule had been applied, for—

$$\text{Period} = \frac{2\pi}{\text{coeff. of } t} = \frac{2\pi}{100\pi} = \cdot 02 \text{ sec.}$$

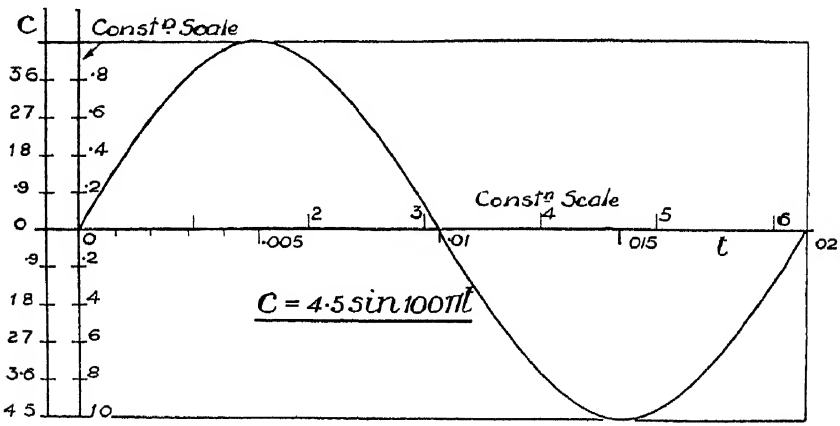


Fig. 206.—Change in Current in Circuit.

Either of two methods can be used for the calculation of values—

(*a*) Plotting from the simple sine function.

According to this scheme write the equation in the form—

$$\frac{C}{4 \cdot 5} = \sin 100\pi t$$

$$\text{or} \qquad \bar{\underline{C}} = \sin T$$

$$\text{where} \qquad \bar{\underline{C}} = \frac{C}{4 \cdot 5}, \quad \text{and } T = 100\pi t$$

Hence to plot the curve (Fig. 206) $\bar{\underline{C}} = \sin T$, select values of T between 0 and $\frac{\pi}{2}$ (0 and 1·571), and thus read off values for $\bar{\underline{C}}$ so that the first quarter of the curve can be plotted, remembering always that the base must be numbered in radians.

Values for this portion would be of this character :—

T	0	·2	·4	·7854	·96	1·1	1·4	1·571
$\underline{\text{C}}$	0	·198	·385	·707	·819	·891	·985	1

To obtain the scales so that the given equation is represented, multiply the vertical scale by 4·5, *i. e.*, 1 on the original scale now reads 4·5; and divide the horizontal scale by 100π, so that $\frac{\pi}{2}$ now reads ·005 $\left(\frac{\pi}{200\pi}\right)$. The curve is shown in Fig. 206.

(*b*) According to the second method, the simple sine curve is not used and no alterations are necessary. Having found the period, ·02 sec., it is known that values of t need only be taken for one quarter of this, *i. e.*, between 0 and ·005.

The tabulation would be arranged as follows :—

t	$100\pi t$ or (radians)	$18{,}000t$ (degrees)	sin $100\pi t$	$C = 4{\cdot}5 \sin 100\pi t$
0	0	0	0	0
·001	·314	18°	·309	1·39
·002	·628	36°	·588	2·645
·003	·942	54°	·809	3·64
·004	1·256	72°	·951	4·275
·005	1·571	90°	1	4·5

Note that the 2nd column is not really necessary, it is only inserted here to make clear the reason for the 3rd column.

Example 18.—A crank 1′-6″ long rotates uniformly in a right-hand direction, starting from the inner dead centre position, and making 30 revs. per minute. Construct a curve to show the height of the end of the crank above the line of stroke at any time, assuming pure harmonic motion.

$$\text{Time for 1 revolution} = \frac{60}{30} = 2 \text{ secs.}$$

or, in 2 secs , 2π radians is the angular distance travelled.

In 1 sec. π radians is the angular distance travelled, or the "angular velocity," usually denoted by ω, $= \pi$ radians per sec.

Construction —Draw, to some scale, a circle of radius 1′-6″, to the left of the paper. (Fig. 207)

Divide its circumference into a number of equal parts, say 10 or 12; in this case 10 is chosen, lines making 36° with one another being drawn. These division lines will correspond to the positions of the crank at time divisions of a tenth of a period, *i. e.*, ·2 sec. apart.

Number these divisions of the circumference 0, 1, 2, 3 . . .

Produce the horizontal through O and along it mark off to some scale a distance to represent 2 secs., and divide this into 10 equal parts.

When the crank is in the position O1, *i.e.*, at time ·2 sec. after start, its projection on the vertical axis is OA: hence produce 1A to meet

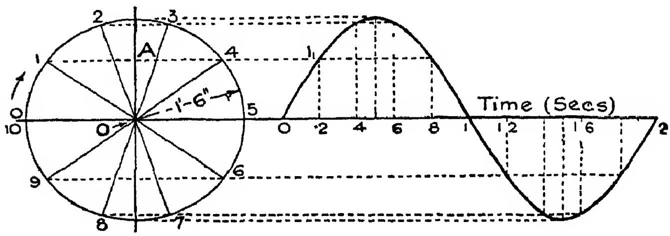


Fig. 207.

the vertical through ·2 at 1_1; and this will be a point on the curve required. Proceeding similarly for the other positions of the crank, the full curve is obtained, and from its form we conclude that it is a sine curve.

To prove that it is a sine curve—

Suppose that in time t secs. the crank moves to the position OC (Fig. 208).

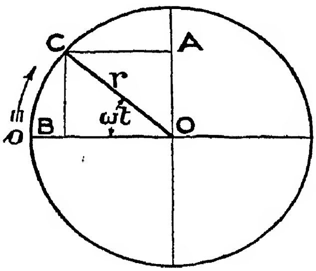


Fig. 208.

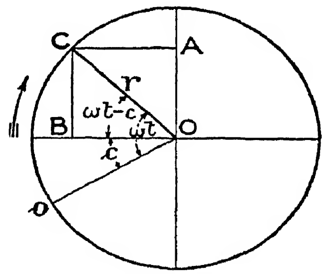


Fig. 209.

In 1 sec. the angle moved $= \pi$ (in this case) or ω (in general)

$\therefore$ In t sec. the angle moved $= \pi t$ (in this case) or ωt (in general)

where ωt is the angle in radians.

$$\therefore \quad \angle COB = \omega t$$
$$OA = CB = CO \sin \angle COB = r \sin \omega t$$

where $r =$ radius of crank circle.

Therefore the curve obtained by the construction is that representing the equation $y = r \sin \omega t$.

Hence a graphic means of drawing sine curves can be employed in place of that by calculation. Great care must, however, be taken in connection with the magnitudes involved.

e. g., to plot $C = 4{\cdot}5 \sin 100\pi t$ by this means.

Radius of circle $= 4{\cdot}5$, the amplitude of the function
and $\omega t = 100\pi t$ or $\omega = 100\pi$
i. e., 100π radians must be swept out per sec.
$\therefore$ 2π radians are swept out in ·02 sec.

Therefore, if the circle were divided into 10 equal parts, the distances along the time base corresponding to the angular displacements would be ·002 sec. each.

Simple Harmonic Motion.—If the crank in Fig. 209, which is supposed to revolve uniformly, were viewed from the right or left, it would appear to oscillate up and down the line OA. Such motion is known as simple harmonic motion, or more shortly S.H.M.

Looking, also, from the top, the motion as observed would be an oscillation along OB, and this again would be S.H.M.; therefore, if the connecting-rod were extremely long compared with the crank the motion of the piston would be approximately S.H. In the case of the valve rod it would be more nearly true that the movement of the valve was S.H., for the valve rod would be very long compared with the valve travel.

At a later stage of the work it will be shown that the acceleration along OB, say, is proportional to the displacement from O; and this is often taken as a basis for a definition of S.H.M.

S.H.M., then, is the simplest form of oscillatory motion, and can be illustrated by a sine curve.

Suppose that the crank does not start from the inner dead centre position, but from some position below the horizontal, what modification of the equation and of the curve results?

If at time t secs. after starting, the crank is at OC (Fig. 209) {Oo is the initial position of crank}—

then $\angle COo = \omega t$
and $\angle COB = \omega t - c$
where $c = \angle BOo$
$\therefore$ $y = r \sin \angle COB = r \sin(\omega t - c)$.

Similarly, if the crank is inclined at an angle c above the horizontal at the start, $y = r \sin(\omega t + c)$.

A moment's thought will show that the curve will be shifted along the horizontal axis one way or the other, but that its shape will be unaltered.

Example 19.—Plot a curve to represent the equation—

$C = 4{\cdot}5 \sin(100\pi t - 1{\cdot}1)$ for a complete period.

Let us reduce the equation to a form with which we have already dealt; thus—

$$C = 4\cdot5 \sin (100\pi t - 1\cdot1)$$

$$\frac{C}{4\cdot5} = \sin 100\pi\left(t - \frac{1\cdot1}{100\pi}\right)$$

$$= \sin 100\pi(t - \cdot0035)$$

i. e., $\overline{C} = \sin 100\pi T_1 = \sin T$

where $T_1 = t - \cdot0035,\quad T = 100\pi T_1 \quad \text{and } \overline{C} = \dfrac{C}{4\cdot5}$

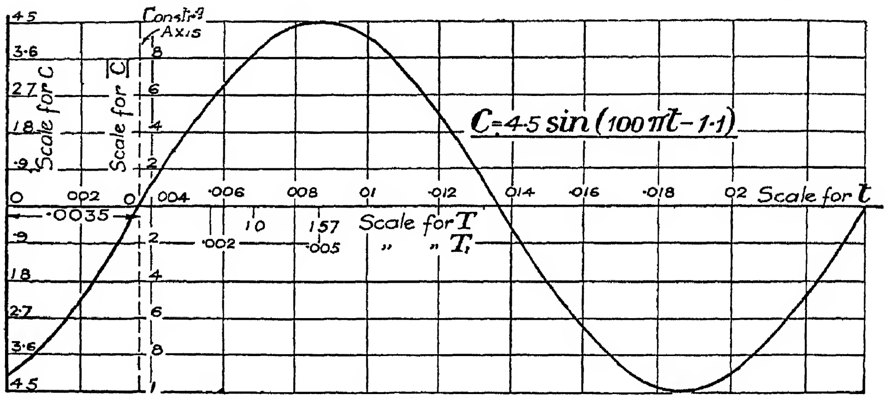


Fig. 210.

We have already seen, viz. in *Example* 17, how to plot this curve, the period being ·02 sec. Then, having altered the two scales according to previous instructions, the vertical axis must be shifted a distance of ·0035 sec. to the left (see Fig. 210), because $t = T_1 + \cdot0035$. Hence the scale for t, which is the final scale, must be measured from an axis ·0035 unit to the left of that used in the construction, *i. e.*, the horizontal scale must again be altered, not in magnitude but in position

The changes in the scales may appear rather confusing, but on the other hand calculations have only to be made for the one fundamental curve (the table for this being given on p. 360), and all the others are derived from it. Therefore, when once the table of values for the simple sine curve has been set out, it will serve for all sine and cosine curves, *i. e.*, it is a "template."

It serves for the cosine curve because this curve is merely the sine curve shifted along the axis a distance equal to one quarter of the period.

Thus— $y = \sin t$

and $y = \cos t = \sin (90 - t)$

will be the same curve measured from different vertical axes.

Graph of tan x.—The graph representing $y = \tan x$ is not of the same type as the sine and cosine curves. As x increases from 0 to 45°, y increases from 0 to 1, but after x has the value 45°

y increases very much more rapidly; while at 90° the value of y is infinitely large. After 90° the tangent is negative, for the angle is in the 2nd quadrant. Supposing some form of continuity in

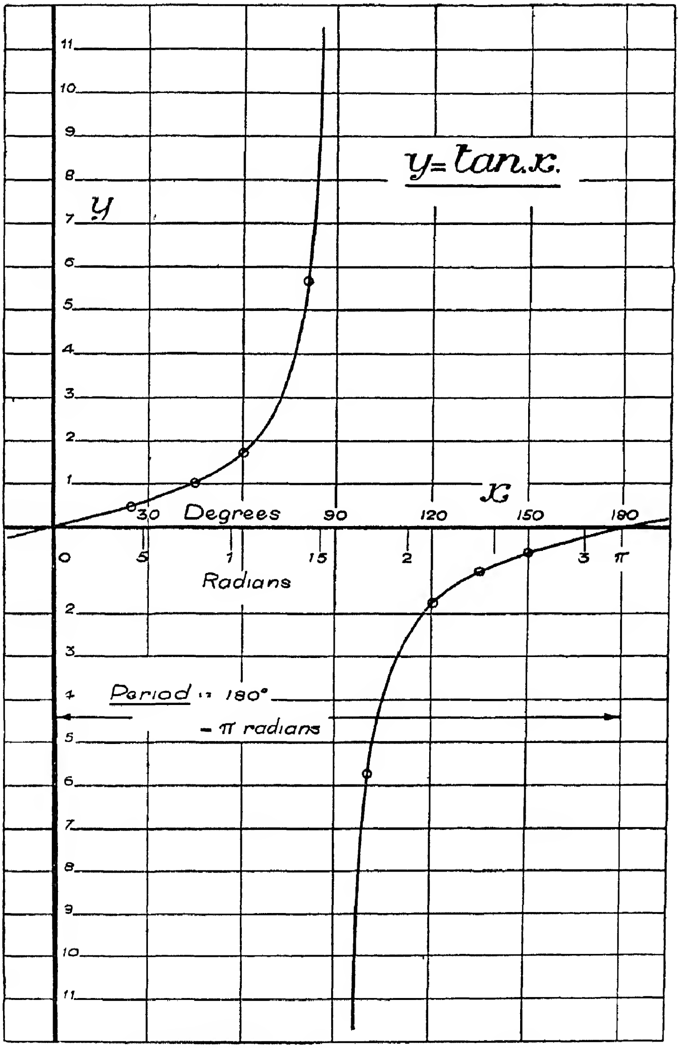


Fig. 211.—Graph of tan x.

the curve, it must now approach from infinity from the negative side and come up to cross the axis at 180°. After this the curve is repeated, so that the period for the simple tangent function is 180° or π.

Selecting values for x, those for y can be read off from the tables :—

x	$y = \tan x$
0	0
25°	·466
45°	1·0
60°	1·732
80°	5·671
85°	11·43
90°—	$+\infty$

x	$y = \tan x$
90° +	$-\infty$
95°	— 11·43
100°	— 5·671
120°	— 1·732
135°	— 1
155°	— ·466
180°	0

Note that 90°— indicates that a value of x is supposed to be taken just *less* than 90°, but practically differing nothing from 90°; thus 90°— would be of the nature 89·99°. Similarly 90°+ would indicate 90·01°, say.

The curve on either side is asymptotic to the vertical through 90°, as will be seen from the curve plotted in Fig. 211.

All other simple tangent curves can be obtained from this fundamental curve by suitable change of scales.

Example 20.—Plot the curve representing the equation—

$$y = 8 \tan 400t.$$

Rewrite the equation as— $\quad Y = \tan T$

where $\quad Y = \frac{y}{8}$ and $T = 400t$.

Then plot $Y = \tan T$ from 0° to 180°, *i. e.*, for a complete period, and afterwards alter the scales so that 1 on the vertical scale becomes 8, and 1 on the horizontal scale becomes $\frac{1}{400}$

$\operatorname{Tan} x = \frac{\sin x}{\cos x}$, and therefore, if we had drawn the curves $y_1 = \sin x$ . . . . . . (1) and $y_2 = \cos x$ . . . . . . (2), we should obtain the value of the ordinate of the curve $y = \tan x$ by dividing any ordinate of curve (1) by the corresponding ordinate of curve (2).

Example 21.—The efficiency of a screw-jack is given by

$$\eta = \frac{\tan \theta}{\tan (\theta + \phi)}$$

where θ is the angle of the developed screw, and ϕ is the angle of friction. If θ varies from 0° to 12°, plot a curve to give the value of the efficiency; μ, the coefficient of friction, being ·1465.

The angle of friction is such that its tangent is equal to the coefficient of friction, *i. e.*, $\phi = \tan^{-1} \mu$.

Thus $\tan \phi = \cdot 1465$ and $\phi = 8°20'$; also $\eta = \dfrac{\tan \theta}{\tan (\theta + 8°20')}$

The tabulation of values is as follows :—

θ	$\theta + \phi$	$\tan \theta$	$\tan (\theta + \phi)$	η
0	8° 20′	0	·1465	0
2	10° 20′	·0349	·1823	·191
4	12° 20′	·0699	·2186	·320
6	14° 20′	·1051	·2555	·413
8	16° 20′	·1405	·2931	·480
10	18° 20′	·1763	·3314	·533
12	20° 20′	·2126	·3706	·574

and the plotting is shown in Fig. 212.

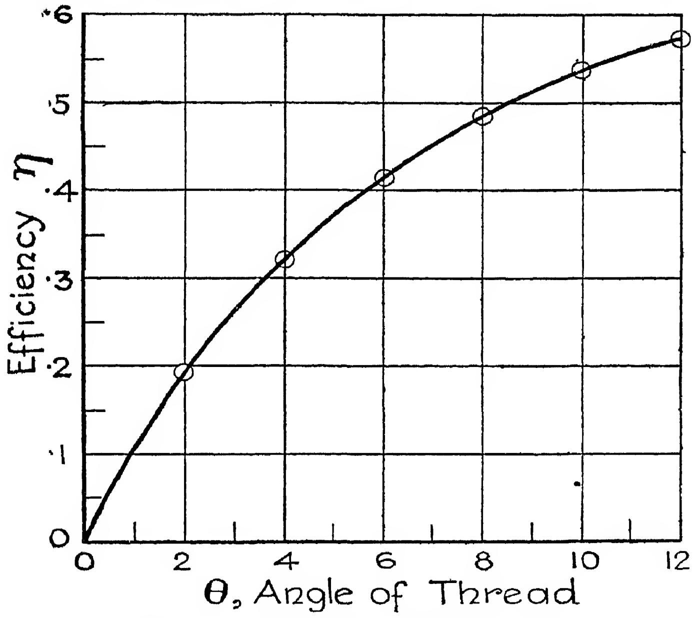


Fig 212.—Efficiency of Screw-Jack.

Compound Periodic Oscillations.—In engineering practice one often meets with curves which are quite periodic, but are not of the sine or tangent type. Many of these can be broken up or "analysed" into a number of sine curves. The process is spoken of as *harmonic analysis*, and reference to this is made in Volume II of *Mathematics for Engineers*. At this stage, however, it is well to consider the work from the reverse or the synthetic point of view, in which the resultant curve is constructed from its components by the addition of ordinates.

An example of importance to surveyors concerns the "equation of time," which is the difference between the "apparent" and the "mean" time of day. The apparent time is the actual time as recorded by a sun-dial, whilst the mean time is calculated from its average over a year. Two causes contribute to the difference between the two times, viz.—

(*a*) The earth in its journey round the sun moves in an ellipse having an eccentricity $\left(\frac{\text{distance between foci}}{\text{diameter}}\right)$ of $\frac{1}{60}$, and in consequence of the laws of gravity its speed is greater when nearer to the sun than when more remote.

(*b*) The earth's orbit is inclined to the plane of the equator.

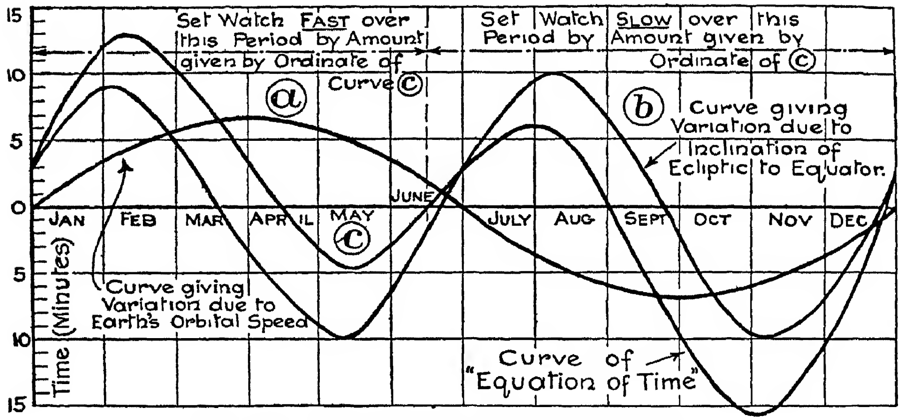


Fig. 213.—Curves for "Equation of Time."

The corrections due to these two causes are found separately, and are represented by the respective curves (*a*) and (*b*) in Fig. 213. For curve (*a*) the period is one year, and the period of (*b*) is half a year.

These, when combined by adding corresponding ordinates, due attention being paid to the algebraic sign, give curve (*c*), for which the period is one year. By the use of this curve the correction to be added to or subtracted from the observed "sun time" can be obtained. Thus to determine the longitude, *i. e.*, the distance in degrees east or west of Greenwich, of, say, a village in Ireland, it would be first necessary to find the meridian of the place by observation of the pole star. Next the time of the crossing of the meridian by the sun *i. e.*, the local time, would be noted, and this would be corrected by adding or subtracting the equation of time for the particular day. Then the difference between the corrected

local time and Greenwich mean time as given by a chronometer would give the longitude, since one hour corresponds to fifteen degrees.

Example 22.—Plot the curve $y = 4 \sin t + \cdot 5 \sin 2t$ sufficiently far to show a complete period.

Let $y_1 = 4 \sin t$ . . . . . . (1), and $y_2 = \cdot 5 \sin 2t$ . . . . . . (2); then the curve required is $y = y_1 + y_2$, *i. e.*, it is the sum of two curves of different periods.

The period of $y = 4 \sin t$ is 2π, while the period of $y = \cdot 5 \sin 2t$ is $\frac{2\pi}{2}$ or π.

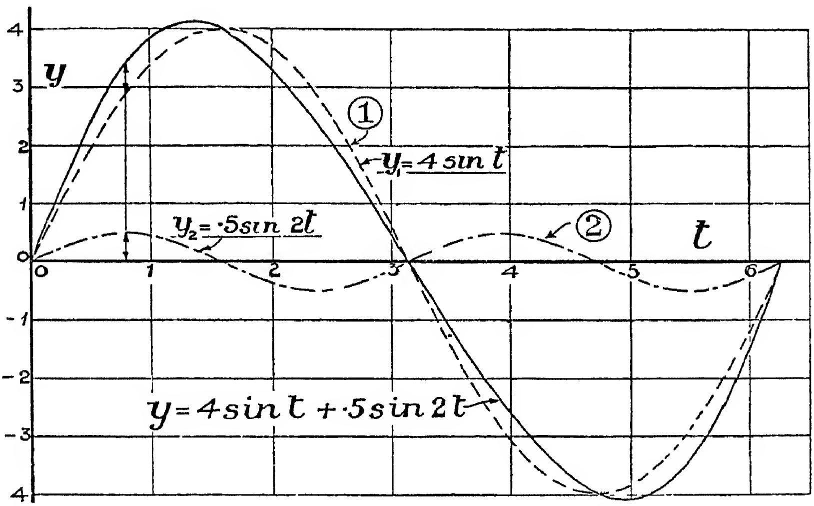


Fig. 214.—Complete period of curve $y = 4 \sin t + 5 \sin 2 t$.

Therefore the curves must be plotted between $t = 0$ and $t = 2\pi$ to give the full period of the resultant curve, so that there will be one period of curve (1) and two of curve (2).

The curves are now dealt with separately, because, being of different periods, values suitable for the one would not be so for the other.

For curve (1) period $= 2\pi$, and amplitude $= 4$.

The two curves must be plotted to the same scales. The simple sine curve "template" already mentioned would serve for curve (1), but curve (2) must be previously adjusted in scale to make it possible to apply the "template."

It may be sometimes easier to set out the work as follows instead of using a template ;—

Curve (1).—Values of t need only be taken between 0 and $\frac{\pi}{2}$

Curve (2).—Values of t need only be taken between 0 and $\frac{\pi}{4}$; therefore take values one-half of those in the previous case, so that the calculation is simplified.

Curve (1)

t	$\sin t$	$y_1 = 4 \sin t$
0	0	0
·2	·198	·792
·4	·385	1·54
·7854	·707	2·828
·960	·819	3·276
1·1	·891	3·564
1·4	·985	3·94
1·571	1·0	4·0

Curve (2)

t	$2t$	$\sin 2t$	$y_2 = \cdot 5 \sin 2t$
0	0	0	0
·1	·2	·198	·099
·2	·4	·385	·193
·3927	·7854	·707	·354
·48	·96	·819	·41
·55	1·1	·891	·446
·7	1·4	·985	·493
·7854	1·571	1·0	·5

These curves are plotted as shown in Fig. 214, and the resultant curve is obtained by adding corresponding ordinates, paying careful attention to the signs.

One further example of this compounding of curves will be given.

Example 23.—The current in an electric circuit is given by—
$C = 50 \sin 628t$, whilst the voltage is given by $V = 148 \sin (628t + \cdot 559)$.
Plot curves to represent the variations in the current, voltage and power at any time.

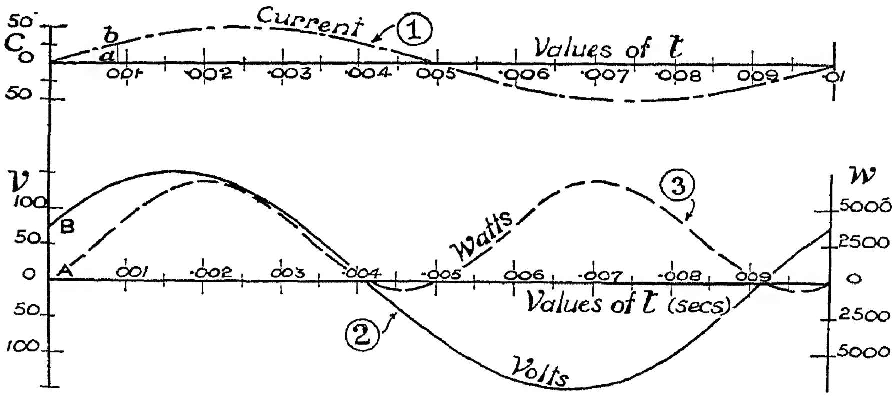


Fig. 215.—Variations in Current, Voltage and Power in Electric Circuit.

Dealing with the three curves in turn :— (See Fig. 215.)
Curve (1).—This is the curve of current.

$$C = 50 \sin 628t, \text{ and the periodic time } = \frac{2\pi}{628} = \cdot 01 \text{ sec.}$$

Plot the curve $\underline{\mathrm{C}} = \sin \mathrm{T}$ from 0 to 2π, and then multiply the vertical scale by 50 and divide the horizontal by 628.

Curve (2), the curve of voltage—

$$\mathrm{V} = 148 \sin 628\left(t + \frac{\cdot 559}{628}\right)$$

$$= 148 \sin 628\mathrm{T}_1 = 148 \sin \mathrm{T}$$

provided that— $\mathrm{T} = 628\mathrm{T}_1$ and $\mathrm{T}_1 = t + \cdot 00089$.

This is the first curve with its axis moved to the right a distance of ·00089 sec. and with all ordinates multiplied by $\frac{148}{50}$ or 2·96. Thus $\mathrm{AB} = 2\cdot 96 \times ab$.

Curve (3), the curve of power, is obtained by multiplying corresponding ordinates of curves (1) and (2).

Confusion is avoided by plotting curve (2) along a different horizontal axis from that used for (1).

The reader will find it convenient to draw out the simple sine curve on tracing paper to a scale convenient for his book or paper, and to use that as a template; much time and labour being saved by this means.

Curves for Equations of the Type $y = e^{-ax} \sin(bx+c)$.—In plotting such a curve it is not wise to select values of x and then calculate values of y directly: it is easier to split the function up into $y_1 = e^{-ax}$ and $y_2 = \sin(bx+c)$, and plot the curves representing these equations separately, obtaining the final curve $y = y_1 \times y_2$ by multiplication of ordinates.

The forms of the two component curves are already known. They must, however, be plotted to the same horizontal scale, which should always be a scale of radians (if an angle is measured along the horizontal) or one of seconds (if time is measured along the horizontal).

Example 24.—Plot the curve $y = e^{-\frac{1}{2}x} \sin(5x + 2\cdot 4)$, showing two complete waves.

Let $y = y_1 \times y_2$ where $y_1 = e^{-\frac{1}{2}x}$ and $y_2 = \sin(5x + 2\cdot 4)$.

To avoid any trouble with the scales, this example is worked in full, *i. e.*, templates are not used.

It will be slightly more convenient to deal first with curve (2).

Curve (2)— $y_2 = \sin(5x + 2\cdot 4) = \sin 5(x + \cdot 48) = \sin 5\mathrm{X}$

where $\mathrm{X} = x + \cdot 48$

Hence the vertical axis through the zero of x in Fig. 216 will be ·48 unit to the right of that for X; hence, since the second scale has to be used again in the plotting, the construction vertical axis must be chosen ·48 unit to the left of some convenient starting-line.

$$y_2 = \sin 5\mathrm{X}, \text{ the period being } \frac{2\pi}{5} = 1\cdot 256 \text{ radians}$$

Hence values of x need only be taken between 0 and $\frac{1 \cdot 256}{4}$ or to $\cdot 314$.

X . .	·0	·04	·08	·1571	·192	·22	·28	·314
5X . .	0	·2	·4	·7854	·96	1·1	1·4	1·571
sin 5X .	0	·198	·385	·707	·819	·891	·985	1

This curve can now be plotted, reckoning the horizontal scale from the construction vertical axis; and then the zero is shifted to its correct position, ·48 unit to the right, as shown in Fig. 216.

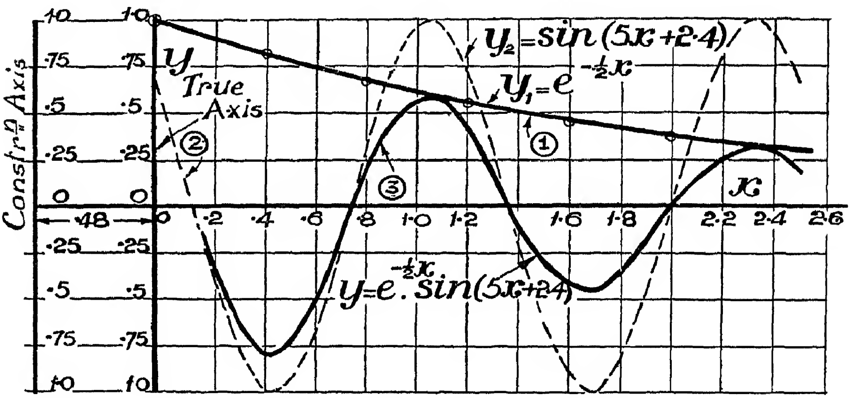


Fig. 216.—Curve of $y = e^{-\frac{1}{2}x} \sin(5x + 2 \cdot 4)$.

Curve (1)— $y_1 = e^{-\frac{1}{2}x}$. For one complete wave of curve (2) $x = 1 \cdot 256$, and therefore for two waves it will be more than sufficient if values of x are taken up to 3.

The table of values is as follows :—

x . . .	0	·4	·8	1·2	1·6	2·0	2·4	2·8	3·0
$\frac{1}{2}x$ or X	0	·2	·4	·6	·8	1·0	1·2	1·4	1·5
$e^{-X} = y_1$	1	·819	·67	·549	·449	·368	·301	·247	·223

A word of explanation regarding this table is necessary. Consider the value $x = 1 \cdot 2$; then to find y, the value of $e^{-\frac{1}{2} \times 1 \cdot 2}$ or $e^{-\cdot 6}$ must be found. Therefore $X = \cdot 6$ is read in the first column of Table XI at the end of the book, and $e^{-\cdot 6}$ is read off in the third column.

This curve can now be plotted, always to the same horizontal scale as that chosen for curve (2), but not necessarily to the same vertical scale. In this example, however, the same scale is convenient for both.

Curve (3), or $y = y_1 \times y_2$ can next be obtained by selecting corresponding ordinates of the two curves and multiplying them together.

When $x = 1{\cdot}06$, $y_1 = {\cdot}58$ and $y_2 = 1$; hence in this case the particular product of y_1 and y_2 has the same value as y_1, and accordingly the vertical scale chosen for curve (3) is advisedly that for (1), so that the curve when plotted touches the curve (1) at its highest points.

Glancing at the curve (3) we observe that the amplitude is now diminished in a constant ratio, although the period remains the same, *i. e.*, there is some damping action represented.

If a condenser discharges through a ballistic galvanometer and deflections left and right are taken, then by plotting the readings a curve is obtained (naturally of a very small period) of the character of curve (3). The logarithm of the ratio of the amplitudes of successive swings is called the **logarithmic decrement** of the galvanometer.

For the case considered, the ratio of consecutive amplitudes is—

$$\frac{e^{-\frac{1}{2}x+1\cdot256}}{e^{-\frac{1}{2}x}} = \frac{e^{-\frac{1}{2}x} \times e^{1\cdot256}}{e^{-\frac{1}{2}x}} = e^{1\cdot256} = 3{\cdot}5 \text{ (approx.)}$$

$$\therefore \quad \text{logarithmic decrement} = \log_e 3{\cdot}5 = 1{\cdot}253.$$

Again, imagine a horizontal metal disc within a fluid, hung by a vertical wire. If the wire is twisted and then released, the disc oscillates from the one side to the other. Measurements of the amplitudes of the respective swings demonstrate the facts that (*a*) the ratio of the amplitude of one swing to the amplitude of the preceding swing is constant for any fluid, and (*b*) this ratio is less for the more viscous fluids.

Thus if the disc oscillated in air, the successive swings would be very nearly alike as regards amplitude; or, in other words, the motion is practically simple harmonic, and its representation in the usual manner gives a sine curve. If the medium is water or thick oil, the motion is represented by a curve like No. (3) in Fig. 216, but the damping effect would be much more marked in the case of the oil.

Exercises 39.—On the Plotting of Graphs Representing Trigonometric Functions.

1. Write down the amplitudes and periods of the following functions: $8 \cos 4x$; ${\cdot}2 \sin (3x - 4)$; $51{\cdot}8 \sin 314t$ (t is in seconds); ${\cdot}116 \sin (615t - {\cdot}214)$; ${\cdot}91 \cos (5 - {\cdot}17x)$.

2. Plot the curve $y = {\cdot}4 \sin 2\theta$ for 2 complete periods. Write down the amplitude and also the period of this function.

3. The range of a projectile fired with velocity V at elevation A is given by $\frac{V^2 \sin 2A}{g}$. Plot a curve to show the range for angles of elevation up to 45°, the velocity of projection being 1410 ft. per sec.

4. On the same diagram and to the same scales plot the curves $y_1 = 2 \sin x$ and $y_2 = 5 \sin \frac{1}{2}x$, and also, by addition of ordinates, the curve $y = 2 \sin x + 5 \sin \frac{1}{2}x$.

5. A crank rotates in a right-hand direction with angular velocity 10, starting from the inner dead centre position. To a time base draw a curve whose ordinates give the displacement of a valve, the connecting-rod (or valve-rod) being many times as long as the crank. The travel of the valve is to be $1\frac{1}{2}''$.

6. Plot the curve $s = 2{\cdot}83 \sin (4t - {\cdot}016)$ for one complete period, the angle being in radians.

7. Plot the curve $y = {\cdot}81 \cos 3\theta$ for a complete period.

8. Plot the curve $5y = 4{\cdot}72 \tan 4\theta$ for a complete period.

9. The current from an alternator is given by $C = 15 \sin 2\pi ft$, and the voltage by $E = 100 \sin (2\pi ft - n)$. If the frequency f is 40 and n (the lag) $= {\cdot}611$, draw curves of current and E.M.F., and by multiplication of corresponding ordinates plot the curve of power.

10. The acceleration A of the piston of a reciprocating engine is given by—

$$A = 4\pi^2 n^2 r \left\{\cos \theta + \frac{\cos 2\theta}{m}\right\}$$

Plot a curve to give values of the acceleration for one complete revolution when r = crank radius = 1 ft., $m = \frac{\text{connecting-rod length}}{\text{crank length}} = 10$, n = R.P.S. = 2.

11. The displacement y of a certain slide valve is given by—

$$y = 2{\cdot}6 \sin (\theta + 32°) + {\cdot}2 \sin (2\theta + 105°).$$

Plot a curve to give the displacement for any angle between 0 and 360°.

12. Plot the curve $y = e^{-{\cdot}2x} \sin 5x$, showing two complete waves.

13. Plot a curve to give the displacement x of a valve from its centre position when $x = -1{\cdot}2 \cos pt - 1{\cdot}8 \sin pt$ and p = angular velocity of the crank, which revolves at 300 R.P.M.

14. Plot the curve $y = 5 \operatorname{cosec} \theta$, showing a complete period.

15. What is the period of the curve $7y = 2{\cdot}8 \sec 3\theta$? Plot this curve.

16. An E.M.F. wave is given by the equation—

$$E = 150 \sin 314t + 50 \sin 942t.$$

Draw a curve to show the variation in the E.M.F. for a complete period.

17. The "range" of an object from a point of observation is found by multiplying the tangent of the observed angle by the length of the base. Draw a curve to give ranges for angles varying from 45° to 70°, the measured base being two chains long.

Graphic Solution of Equations.—The application of purely algebraic rules will enable us to solve simple or quadratic equations. Equations of higher degree, or those **not** entirely algebraic,

can best be solved by graphs; and in some cases no other method is possible.

The general plan is to first obtain some approximate idea of the expected result, either by rough plotting or by calculation, and to then narrow the range, finally plotting to a large scale the portion of the curve in the neighbourhood of the result.

Occasionally the work is simplified by plotting two easy curves instead of the more complex one.

Example 25.—Solve the equation— $e^{3x} - 5x^2 - 17 = 0$.

The equation may be written— $e^{3x} = 5x^2 + 17$.

Then if the two curves $y_1 = e^{3x}$ and $y_2 = 5x^2 + 17$ are plotted, their point or points of intersection will give the value or values required.

Tabulating :—

For Curve (1) $y_1 = e^{3x}$.

x	$3x$ or X	$e^X = y_1$
0	0	1
·5	1·5	4·48
1·0	3	20·09
1·5	4·5	90
2	6	404

For Curve (2) $y_2 = 5x^2 + 17$.

x	$5x^2 + 17$	y_2
0	0 + 17	17
·5	1·25 + 17	18·25
1	5 + 17	22
1·5	11·25 + 17	28·25
2	20 + 17	37
3	45 + 17	62

We conclude from an examination of these tables that y_1 and y_2 are alike when x has some value between 1·0 and 1·5.

Values of x are next taken between 1·0 and 1·5; thus :—

x	$3x$ = X	y_1	$5x^2 + 17$	y_2
1·0	3·0	20·1	5 + 17	22
1·1	3·3	27 1	6·05 + 17	23·05

Therefore, the solution is evidently between 1·0 and 1·1; hence plot the two curves for these values and note the point of intersection (see Fig. 217).

This is found to be at the point for which $x = 1{\cdot}032$, and therefore $\underline{x = 1{\cdot}032}$ is one solution, and as the curves intersect at one point only, it is the only solution.

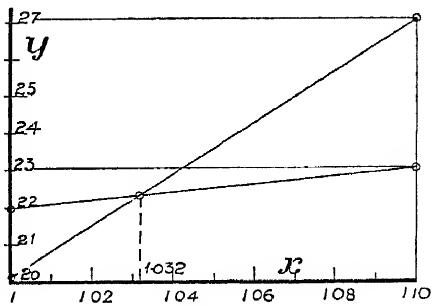


Fig. 217.
Solution of $e^{3x} - 5x^2 - 17 = 0$.

Numerous examples of this method of solution of equations

occur in connection with problems in hydraulics. As an example take the following :—

Example 26.—Water flows at 7·45 cu. ft. per sec. through a pipe of diameter d ft., and the loss of head in 10 miles is 350 ft. The coefficient of resistance is $f = \cdot 005\left(1 + \frac{1}{12d}\right)$. Find the diameter of the pipe, given that—

$$\text{Head lost} = \frac{flv^2}{2gm} \quad \text{where} \quad m = \frac{d}{4}$$

$$\text{Area of pipe} = \frac{\pi}{4}d^2$$

$$\text{Then the velocity} = \frac{7{\cdot}45}{\text{area}} = \frac{7{\cdot}45}{\pi d^2} \times 4 = \frac{9{\cdot}48}{d^2}$$

$$\therefore \qquad 350 = \frac{f \times 4 \times 10 \times 5280 \times 9{\cdot}48^2}{d \times 64{\cdot}4 \times d^4}$$

$$\text{and} \qquad d^5 = \frac{40 \times 5280 \times 9{\cdot}48 \times 9{\cdot}48}{350 \times 64{\cdot}4}f$$

$$= 838f.$$

$$\text{Substituting for } f\text{—} \qquad = 838\left(1 + \frac{1}{12d}\right) \times \cdot 005$$

$$= 4{\cdot}190\left(1 + \frac{1}{12d}\right)$$

$$= 4{\cdot}19 + \frac{4{\cdot}19}{12d}$$

$$\text{from which} \quad d^6 - 4{\cdot}19d - \cdot 35 = 0.$$

To solve this equation, we know that no negative values need be taken; hence as a first approximation—

$$\text{Let } y = d^6 - 4{\cdot}19d - \cdot 35$$

Then—

d	$d^6 - 4{\cdot}19d - \cdot 35$	y
0	0 − 0 − ·35	− ·35
1	1 − 4·19 − ·35	− 4·54
2	64 − 8·38 − ·35	55·27

Since— $d = 1$ makes y negative
and $d = 2$ makes y positive

the value of d that makes $y = 0$ must lie between 1 and 2 and nearer to 1.

For $d = 1{\cdot}5$, $y = (1{\cdot}5)^6 - (4{\cdot}19 \times 1{\cdot}5) - \cdot 35 = 4{\cdot}76$.

Thus the required value of d is between 1 and 1·5.

If $d = 1{\cdot}3$, $y = -\cdot 98$, and we see that the required value is between 1·3 and 1·5. Plot the values of y for the values of d 1·3 and 1·5, as

in Fig. 218, allowing a fairly open scale for d, and join the two points by a straight line. The intersection of this line with the axis of d gives the value of d required, which is seen to be 1·334.

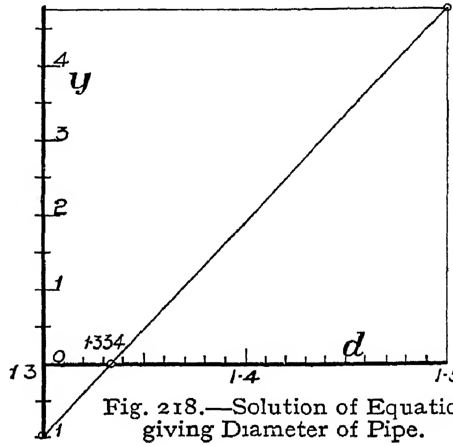


Fig. 218.—Solution of Equation giving Diameter of Pipe.

Example 27.—The length of an arc is 2·67″, and the length of the chord on which it stands is 2·5″. Find the angle subtended at the centre of the circle. [This question has reference to the length of sheet metal in a corrugated sheet.]

Arc = radius × θ where θ is in radians.

Now— $r\theta = 2{\cdot}67$ and $r = \dfrac{2{\cdot}67}{\theta}$

Also— $\sin\dfrac{\theta}{2} = \dfrac{1{\cdot}25}{r}$, *i. e.*, $r = \dfrac{1{\cdot}25}{\sin\frac{\theta}{2}}$

$$\therefore \quad \frac{2{\cdot}67}{\theta} = \frac{1{\cdot}25}{\sin\frac{\theta}{2}}$$

or $\sin\dfrac{\theta}{2} = \dfrac{1\ 25}{2\ 67}\theta = {\cdot}468\theta.$

Making our first approximation, taking θ from 0 to 3·14 :—

θ	$\frac{\theta}{2}$	$\sin\frac{\theta}{2}$	468θ
0	0	0	0
·5	·25	·247	·234
1·0	·5	·479	·468
1·5	·75	·682	·702
2·0	1·0	·842	·936
2·5	1·25	·945	1·17
3·0	1·5	·998	1·403
3·14	1·57	1·0	1·47

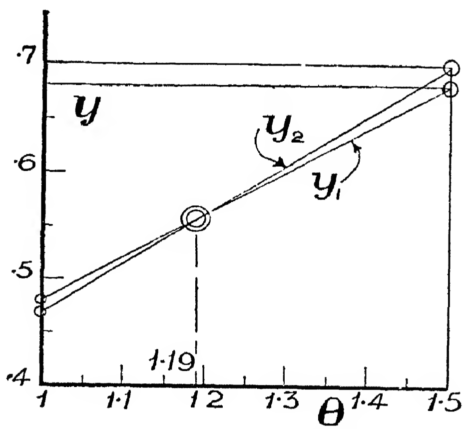


Fig 219.—Length of Metal in Corrugated Sheet.

we see that the solution must lie between $\theta = 1{\cdot}0$ and $\theta = 1{\cdot}5$.

When $\theta = 1{\cdot}2$, $\sin\dfrac{\theta}{2} = {\cdot}565$, and ${\cdot}468\theta = {\cdot}562$.

Plotting the two curves in Fig. 219, $y_1 = \sin\frac{\theta}{2}$, and $y_2 = \cdot 468\theta$ for values of θ from $\theta = 1\cdot 0$ to $1\cdot 5$; we note the point of intersection to be at $\theta = 1\cdot 19$. $\therefore$ $\underline{\theta = 1\cdot 19 \text{ radians or } 68\cdot 2^\circ.}$

Exercises 40.—On the Graphic Solution of Equations.

1. Find a value of x in terms of l to satisfy the equation—

$$3x^3 - 3l^2x + l^3 = 0$$

x being a distance from one end of a beam of length l.

2. Solve for z the equation $l^3 - 3lz^2 - z^3 = 0$ when $l = 10$.

3. In order that a hollow shaft may have the same strength as a solid one the following equation must be satisfied—

$$\frac{\pi f}{16} \cdot \frac{D^4 - d^4}{D} = \frac{\pi f}{16} \cdot d^3$$

Writing x for $\frac{D}{d}$ this equation reduces to $x^4 - x - 1 = 0$. Find the ratio of the diameters so that the given condition may be satisfied.

4. Find a value of d (a diameter) to satisfy the equation—

$$d^3 - \frac{4Pd}{\pi f} - \frac{32Pr}{\pi f} = 0$$

where $r = 3\cdot 2$, $f = 6$, $P = 15$.

5. Solve the equation $e^x = 4x$.

6. Find values of x between -4 and $+3$ to satisfy the equation—

$$10^{\frac{2x}{3}} = 16 + 4x - x^2$$

7. Find a value of x between 1 and 5 to satisfy the equation—

$$x^2 \log_e x = 8$$

8. Solve for positive values of x the equation $5e^{-3x}\sin 4x = 1\cdot 8$. (Note that the value of x must be in radians.)

9. Determine a value of x between 0 and π to satisfy the equation—

$$x^{1\cdot 5} - 3\sin x = 3$$

10. To find the height of the water in a cylindrical pipe so that the flow shall be a maximum it is necessary to solve the equation—

$$\theta(2 - 3\cos\theta) + \sin\theta = 0$$

Find the value of θ (radians) to satisfy this equation.

11. Solve for l in terms of L the equation—

$$56l^3 - 111Ll^2 + 72lL^2 - 14L^3 = 0$$

which occurs when finding the most economical arrangement of the three spans of a continuous beam; l being the length of each of the end spans and L being the total span.

12. In finding the ratio of expansion r for a direct acting single cylinder steam engine of 14″ diameter and 22″ stroke, the equation $1 + \log_e r - \cdot 389r = 0$ was obtained.

Find the value of r to satisfy this equation.

13. The maximum velocity of flow through a circular pipe is reached when the angle θ at the centre of the circular section subtended by the wetted perimeter has the value given by the equation—

$$\frac{\sin\theta}{\theta} - \cos\theta = 0.$$

Find this value of θ.

14. Solve, for positive values of f (the length of a link of a certain mechanism), the equation—

$$f^3 - 19{\cdot}5f^2 + 42{\cdot}5f + 546 = 0.$$

15. Forty cu. ft. per sec. are to pass through a pipe laid at a slope of 1 in 1500, the pipe to run half full. The velocity is given by—

$$\frac{157}{1 + \frac{\cdot 5}{\sqrt{m}}}, \text{ where } m = \cdot 153\text{D and the quantity} = \frac{\pi}{4}\text{D}^2 v$$

Simplifying and collecting these equations we arrive at the simpler form—

$$\text{D}^3 - 50{\cdot}3(\sqrt{\text{D}} + 1) = 0$$

Find the value of D to satisfy this equation.

16. The bottom of a trapezoidal channel (the slope of the sides being 2 vertical to 1 horizontal) is 4 ft. wide. Find the depth of flow d, if the discharge is 12000 gallons per min., the slope is 1 in 500, and the coefficient of resistance is ·006.

$$\left\{\text{Equations reduce to } \frac{2{\cdot}32(8d + d^2)^{\frac{3}{2}}}{\sqrt{8 + 4{\cdot}47d}} = 32\right\}$$

17. Find a value of r, the ratio of expansion, to satisfy the equation—

$$\frac{1}{r} - \cdot 1083 \log_e r - \cdot 225 = 0$$

18. A hollow steel shaft has its inside diameter 3″. What must be the outside diameter so that the shaft may safely stand a torque of 200 tons ins., the allowable stress f being 5 tons per sq. in.? Given that—

$$\frac{\text{Torque}}{\frac{\pi}{32}(\text{D}^4 - 3^4)} = \frac{2f}{\text{D}}$$

19. Find a value of θ (the angle of the crank from line of stroke) to satisfy the equation—

$$\sin^6\theta - n^2\sin^4\theta - n^4\sin^2\theta + n^4 = 0 \quad \text{when } n = 5.$$

[*Hint.*—Let $\text{X} = \sin^2\theta$ and then solve for X]

20. An equation occurring in connection with the whirling of shafts is—

$$\cosh x + \frac{1}{\cos x} = 0$$

Find a value of x between 0 and π to satisfy this equation.

[Note that the values of $\cosh x$ should be taken from Table XI at the end of the book.]

21. Find the height above the bottom of a cylindrical tank of diameter 10 ft. at which a pipe must be placed so that the water will overflow when the tank is two-thirds full.

Construction of PV (pressure-volume) and $\tau\phi$ (temperature-entropy) Diagrams.—It is impossible to proceed far in the study of thermodynamics without a sound working knowledge of the indicator and entropy diagrams of heat engines; and to assist in the acquisition of this knowledge these paragraphs are addressed mainly to students of the theory of heat engines. Although we are not concerned in this volume with the full meaning of these

curves, we can deal with them as practical examples of graph-plotting. More can be learned about the advantages and usefulness of an entropy diagram by actual construction and use than by absorbing the remarks of some one else, and taking for granted all that he says. Careful attention should, therefore, be directed to the following exercises, which should be worked out step by step by the reader.

Example 28.—Draw a PV diagram (Fig. 220) and also a $\tau\phi$ diagram (Fig. 221) for 1 lb. of steam expanding from a pressure of 100 lbs. per sq. in. absolute, to atmospheric pressure, the steam being dry and

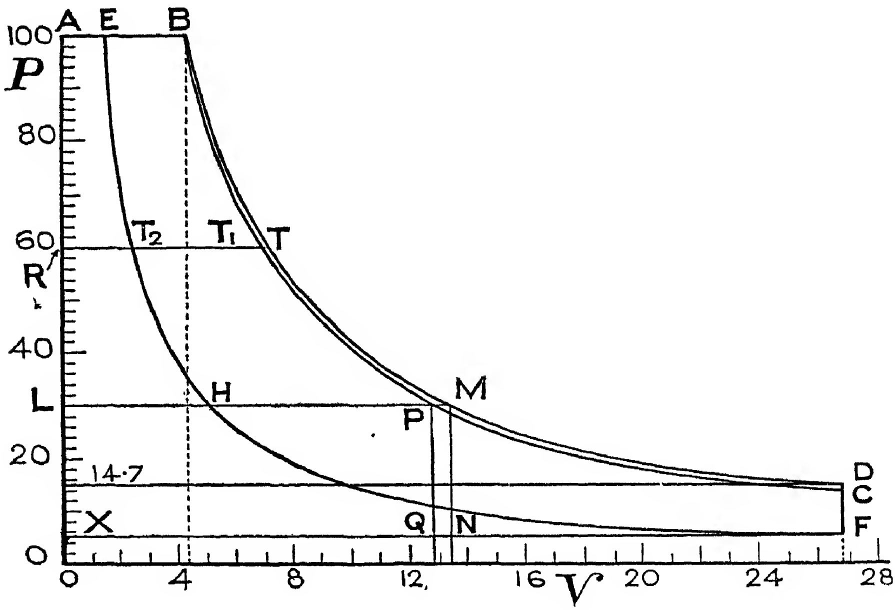


Fig. 220.—Pressure-Volume or PV Diagram.

saturated throughout. [*Note.*—Since these diagrams are to be used for subsequent examples, they must be so constructed that the lowest pressure indicated is 5 lbs. per sq. in. absolute.]

To calculate for points on the expansion line BD in Fig. 220 steam tables must be used; the volumes (V) of 1 lb. weight of steam for various pressures (P) between 100 lbs. per sq. in. and 14·7 lbs. per sq. in. absolute being read off from the tables and tabulated thus :—

P	100	80	60	40	20	14·7
V	4·4	5·48	7·16	10·50	20	26·8

Horizontals through 100 and 14·7 on the pressure scale complete the diagram in Fig. 220. BD is the saturation or 100 % dryness curve.

For the $\tau\phi$ diagram (Fig. 221) rather more calculation is necessary.

The entropy of water at any absolute temperature $\tau°$ Fahrenheit $= \log_e \frac{\tau}{493}$, if the entropy is considered zero at 32° F., *i. e.*, at 461 + 32 or 493° F. absolute.

For our example we require the "water" line from about 160° F. to 320° F., since these temperatures correspond approximately to pressures 5 and 100. Hence the range of $\tau = 621°$ to 781° F. absolute, or, say, 620° to 780°. The tabulation is next arranged as follows, it being noticed that—

$$\log_e \frac{\tau}{493} = \log_e \tau - \log_e 493 = 2{\cdot}303(\log_{10}\tau - \log_{10} 493)$$

τ	$\log_{10}\tau - \log_{10} 493$	$2{\cdot}303 \times$ column (2) $= \log_e \frac{\tau}{493}$
620	2·7924 − 2·6928	·0996 × 2·303 = ·229
660	2·8195 − 2·6928	·1267 × 2·303 = ·292
700	2·8451 − 2·6928	·1523 × 2·303 = ·351
750	2·8751 − 2·6928	·1823 × 2·303 = ·42
780	2·8921 − 2·6928	·1993 × 2·303 = ·459

It is unwise to plot this line until the calculations for the "steam" line have been made.

The width of the diagram, *i. e.*, from the water line to the steam line (*a* to *b*, *r* to *t*, etc., in Fig. 221), is always $\frac{L}{\tau}$, where L is the latent heat at the temperature τ considered. The values of the latent heat are read from the steam tables and are set down thus :—

[Taking 460 instead of 461.]

$t°$ F.	$\tau°$ F. absol.	L	$\frac{L}{\tau}$
160°	620°	1002	1·615
200°	660°	974	1·475
240°	700°	947	1·353
290°	750°	912	1·215
320°	780°	891	1·142

Hence the scale for entropy must be chosen so that the largest value may be shown, viz. 1·844, which is obtained by adding 1·615 to ·229.

Plotting the values of τ, taken from the last two tables, to a horizontal base of ϕ, we obtain the water and steam lines, which are straight lines over short distances.

The vertical scale may also be numbered to read pressures, which

may be obtained for the temperatures required from steam tables. Thus :—

τ	788	753	710	672	622
P	100	60	30	14·7	5

A horizontal through 100 on the scale of P gives the line *ab* (corresponding to AB on the PV diagram), and the intersection of the horizontal through 14·7 lbs. per sq. in. with the steam or saturation line gives the point *d*.

Example 29.—On the $\tau\phi$ diagram (Fig. 221) draw the adiabatic line *bc*, and also the constant volume line *dcf*, the latter on the assumption that qV is constant throughout the curve; q being the dryness fraction and V the volume of 1 lb. of dry steam. Draw also the corresponding line BC in Fig. 220, and the constant volume line DCF.

The line DCF (Fig. 220) is a vertical through D, which meets the horizontal through 5 on the pressure scale in F; but certain calculations are necessary before the line *dcf* (Fig. 221) can be drawn.

As the pressure decreases, the volume increases. Thus at 14·7 lbs. pressure the volume of 1 lb. weight of steam = 26·8 cu. ft., while at 10 lbs. pressure the volume of 1 lb. weight is 38·4 cu. ft. Consequently if only 26·8 cu. ft. of steam are present at the lower pressure instead of the 38·4 cu. ft., the dryness of the steam must be $\frac{26·8}{38·4}$, *i. e.*, ·698; and accordingly the latent heat is only ·698 of its true value. Hence if we make, on the horizontal through 10 lbs. pressure, $kx_1 = ·698kx$ (Fig. 221), the point x_1 lies on the line of constant volume, viz. 26·8 cu. ft.

At 5 lbs. pressure, volume of 1 lb. weight = 72·4 cu. ft.; hence—

$$wf = \frac{26·8}{72·4}ws = ·371ws$$

A number of points can be found in this manner, and the smooth curve through them, viz. *dcf*, is obtained.

All adiabatics on the $\tau\phi$ chart are vertical lines, so that *bc* may be drawn. To draw the line BC in Fig. 220, proceed as follows: Select any convenient pressure, say 60, and calculate the value of the ratio $\frac{rt_1}{rt}$ in Fig. 221. Referring to Fig. 220, determine the position of T_1 on the horizontal through 60,

$$\text{so that} \quad \frac{RT_1}{RT} = \frac{rt_1}{rt}$$

and other points on the curve BC may be found in like manner.

It should be noted that the adiabatic BC lies under the saturation curve BD, since the steam is not dry throughout the expansion; and the dryness fraction at any pressure is the value of a ratio like $\frac{RT_1}{RT}$.

Example 30.—Draw the adiabatics through f and F, the final pressure being 5 lbs. per sq. in. absolute. (Figs. 221 and 220.)

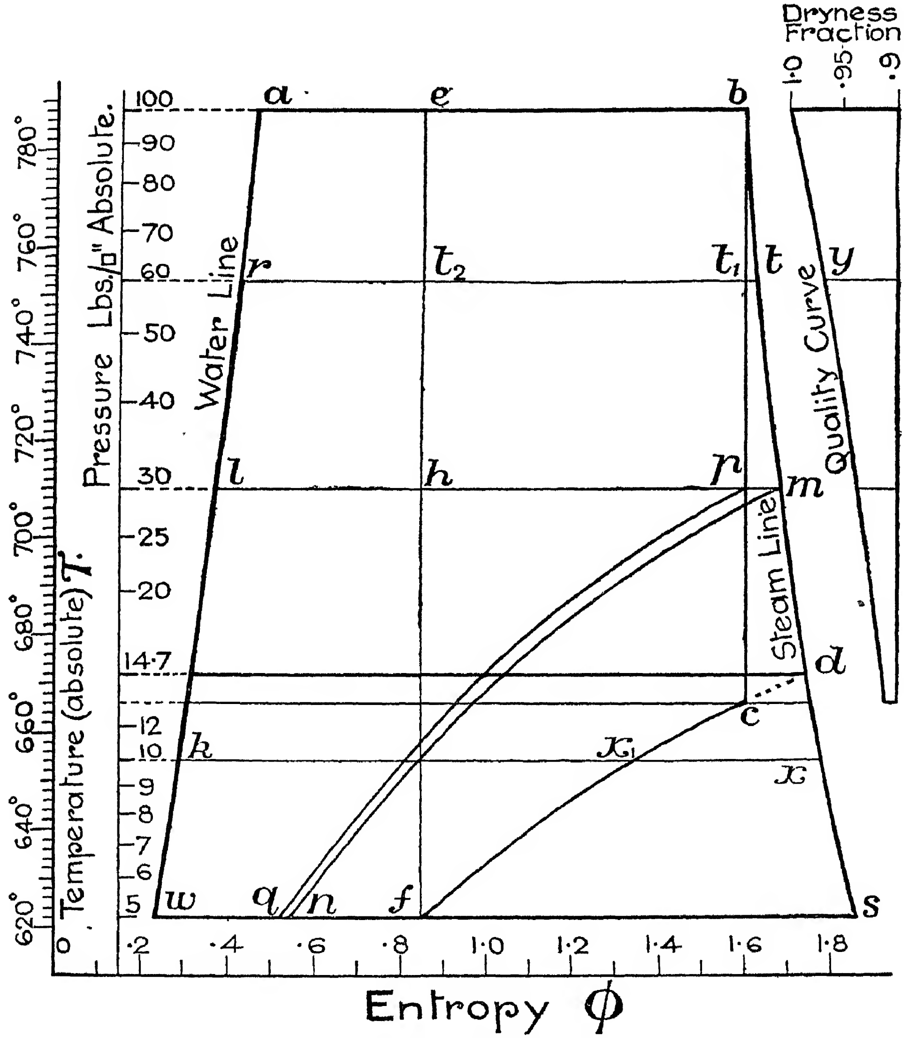


Fig. 221.—Temperature-entropy or $\tau\phi$ Diagram.

The point f, on the constant volume line dcf, has already been fixed; and a vertical through f gives the adiabatic ef.

EF is obtained from BD in just the same way as BC was derived;

$$i.\ e.,\ \frac{\mathrm{RT}_2}{\mathrm{RT}} = \frac{rt_2}{rt} \text{ etc.}$$

Example 31.—Draw the Rankine cycle for the case in which the steam is initially dry; and also for the case in which the steam at the commencement of the expansion has its dryness fraction $= ae \div ab$. The initial and back pressures are 100 and 30 lbs. per sq. in. absolute respectively.

The Rankine cycle is made up of (i) expansion at constant pressure, (ii) adiabatic expansion, (iii) exhaust at constant pressure, and (iv) compression at constant volume.

Thus the horizontals PL and pl (Figs. 220 and 221) must be drawn, and the Rankine cycle is given by the figures ABPL and $abpl$ for the one dryness, and AEHL and $aehl$ for the other.

Example 32.—Draw the common steam engine diagram with a toe drop from 30 lbs. to 5 lbs. per sq. in. absolute; showing the case when the engine is jacketed and also that when there is no jacket. (See Figs. 220 and 221.)

If the engine is jacketed, the steam expansion line lies along the saturation curve, so that the diagram is ABMNX on the PV diagram (Fig. 220) and $abmnw$ on the $\tau\phi$ chart (Fig. 221); the line mn being a line of constant volume obtained in the same way as cf.

If there is no jacket, the diagram is ABPQX in Fig. 220, and $abpqw$ in Fig. 221; pq being a line of constant volume.

Example 33.—Calculate the dryness fraction from the entropy diagram for various temperatures, and thence plot on this diagram the "quality" curve for the adiabatic bc (Fig. 221).

At 100 lbs. pressure the dryness fraction is 1, whilst at 60 lbs. pressure the dryness fraction $= \frac{rt_1}{rt}$; and at 30 lbs. pressure the dryness fraction $= \frac{lp}{lm}$. Selecting some vertical line as the base set off horizontals to represent these various dryness fractions, taking ·9 as the base of the curve: thus the position of y represents the dryness at 60 lbs. pressure. A curve through the points so obtained is the quality curve.

Example 34.—Calculate the values of the exponent in $pv^n = C$ for the expansions represented by BC and EF, Fig. 220.

For the line BC— $p = 100$ when $v = 4{\cdot}44$
$p = 13$ when $v = 26{\cdot}8$

also
$$\log p + n \log v = \log C$$
Thus—
$$\log 100 + n \log 4{\cdot}44 = \log C$$
$$\log 13 + n \log 26{\cdot}8 = \log C$$
or
$$2 + {\cdot}6474n = \log C$$
and
$$1{\cdot}1139 + 1{\cdot}4281n = \log C$$
whence by subtraction—
$$\cdot 8861 = \cdot 7807n$$
$$n = \frac{\cdot 8861}{\cdot 7807} = \underline{1{\cdot}135}$$

In like manner the exponent for the expansion EF is $\underline{1{\cdot}06}$.

We may compare these values with those given by Zeuner's rule; viz.—

$$n = 1{\cdot}035 + {\cdot}1q \text{ where } q \text{ is the initial dryness.}$$

For BC $\quad q = 1 \quad$ and therefore $n = 1{\cdot}035 + {\cdot}1 \quad = 1{\cdot}135$
For EF $\quad q = {\cdot}332$ and therefore $n = 1{\cdot}035 + {\cdot}0332 = 1{\cdot}068.$

Constant heat lines may be plotted on the $\tau\phi$ diagram; but before showing how this may be done, we must indicate what is meant by the term "constant heat line." If steam is throttled by being passed through an orifice its dryness is greater than it would be if the expansion were free. Thus in Fig. 223, at the temperature τ_2 the dryness fraction $= \frac{DC_1}{DE}$ and not $\frac{DC}{DE}$ as for adiabatic expansion; and the line BC_1 is known as a line of constant heat.

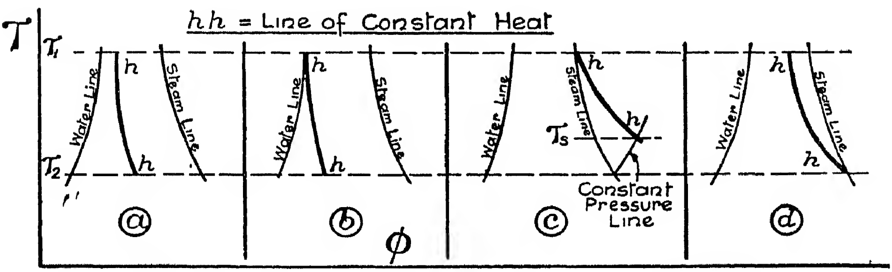


Fig. 222 —Constant Heat Lines.

Four cases of the drying effect of expansion without doing external work, known as "throttling," are possible, these being represented by (*a*), (*b*), (*c*) and (*d*) in Fig. 222.

Case (*a*) illustrates the expansion of a mixture of water and steam from temperature τ_1 to temperature τ_2. At the commencement of the expansion the dryness fraction of the mixture is q_1, its latent heat is L_1 and its sensible heat h_1, while q_2, L_2 and h_2 are the corresponding quantities at the temperature τ_2. Then, since the heat content is unchanged—

$$q_1L_1 + h_1 = q_2L_2 + h_2$$

in which equation q_1, L_1, h_1, L_2 and h_2 would be known, and thus q_2 could be calculated.

Case (*b*) is that of water being dried, thus becoming a mixture of steam and water. The equation here is—

$$h_1 = q_2L_2 + h_2.$$

Case (*c*) is that of dry saturated steam becoming superheated, and for this change

$$h_1 + L_1 = h_2 + L_2 + \cdot 5(\tau_s - \tau_2)$$

τ_s being the temperature to which the steam is raised by the throttling; $\tau_s - \tau_2$ thus being the degrees of superheat (only obtained internally).

In *Case* (*d*) steam of a certain wetness is completely dried by expansion under constant heat. (Any further throttling would naturally superheat.)

For the change shown in the diagram—

$$\begin{aligned} q_1 L_1 + h_1 &= L_2 + h_2 \\ &= 1115 - \cdot 7t_2 + t_2 - 60 \\ &= 1055 + \cdot 3t_2 \end{aligned}$$

from which equation t_2, the temperature at which the steam is just dry, can be found.

From a consideration of the foregoing cases it will be seen that lines of constant heat appear in either the "saturated area," viz. the area between the water and steam lines, or the "superheated area," viz. the area beyond the steam line; and these two cases will be dealt with in the following examples :—

Example 35.—Steam $\cdot 3$ dry at 400° F. expands to 150° F., being dried by throttling. Draw the constant heat line representing this expansion.

If τ_1 and τ_2 are the absolute temperatures, and h_1 and h_2 are the sensible heats—

$$T_1 - T_2 = h_1 - h_2 = t_1 - t_2$$

To draw the line of constant heat it is necessary to calculate the dryness fraction at various temperatures. From the equation $q\,L_2 + h_2 = q_1 L_1 + h_1$

$$q_2 = \frac{q_1 L_1 + t_1 - t_2}{L_2}$$

In this equation q_1, L_1, and t_1 are known, whilst values of t_2 may be assumed and values of L_2 calculated therefrom, or taken from steam tables.

Now— $t_1 = 400$, $L_1 = 835$, and $q_1 = \cdot 3$

Then, taking convenient drops of temperature, say 50° or 100°, a table may be arranged as follows :—

t_2	L_2	$t_1 - t_2 + q_1 L_1$	q_2
400	835	0 + 251	·3
300	905	100 + 251	·388
200	975	200 + 251	·462
150	1010	250 + 251	·495

$\Big[L_1 = 835$ and $q_1 L_1 = \cdot 3 \times 835 = 251$; also $\frac{351}{905} = \cdot 388$, $\frac{451}{975} = \cdot 462$ and $\frac{501}{1010} = \cdot 495. \Big]$

The line of constant heat (Fig. 223) may be drawn after points such as K have been determined; K being so placed that $\frac{LK}{LK_1} = \cdot 388$.

Example 36.—On the $\tau\phi$ chart (Fig. 223) plot the line of constant heat for superheated steam, which is dry at 350° F.

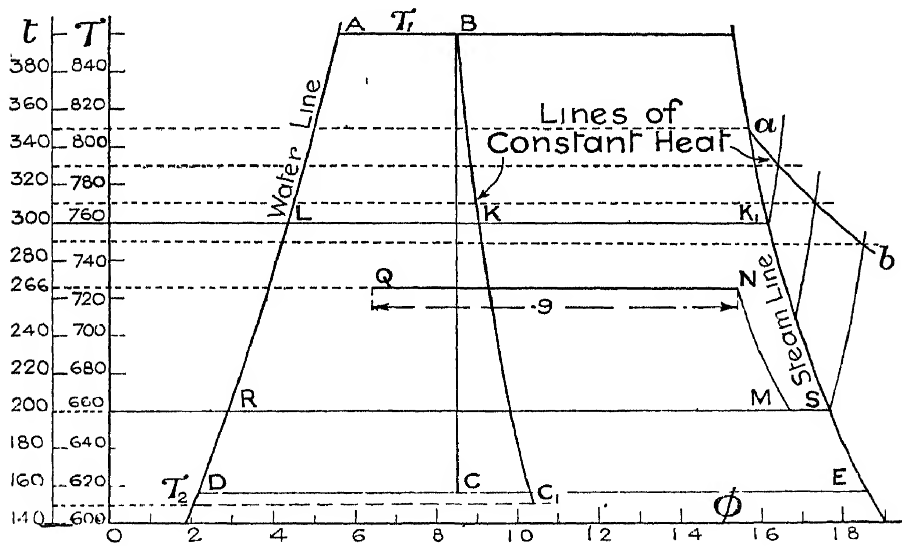


Fig 223 —$\tau\phi$ Diagram showing Constant Heat Lines.

This example is a numerical illustration of Case (*c*), Fig. 222, and hence we must use the equation—

$$h_1 + L_1 = h_2 + L_2 + \cdot 5(\tau_s - \tau_2)$$

By transposition—

$$\tau_s = \frac{h_1 + L_1 - h_2 - L_2}{\cdot 5} + \tau_2$$

$$\tau_s = 2(h_1 - h_2 + L_1 - L_2) + \tau_2 \quad \text{(absolute temp.)}$$

or $$t_s = 2(t_1 - t_2 + L_1 - L_2) + t_2 \quad \text{(F.° temp.)}$$

We know that $t_1 = 350$, $L_1 = 870$; and it is convenient to take drops of temperature of 50° F.

Then the table for the calculation is arranged in the following manner :—

t_2	L_2	L_1	$t_1 - t_2 + L_1 - L_2$	2 × column (4)	t
350	870	870	0 + 0	0	350
300	905	870	50 − 35	30	330
250	940	870	100 − 70	60	310
200	975	870	150 − 105	90	290

Horizontals through these temperatures meet the constant pressure lines (drawn on all charts, the equation being $\phi = K_p \log_e \frac{\tau}{\tau_0}$, *i. e.*, the curve is of the same character as the "water" line) through 350°, 300°, etc. (on the "steam" line), at points on the line required; join these and the line *ab* is obtained (Fig. 223).

Example 37.—Steam of ·2 dryness at 266° F. is dried further by the addition of heat and then allowed to expand through an orifice down to 200° F., where it is 6·9 % wet. Find the number of heat units added at 266° F.

This may be worked by calculation, or by use of the chart.

(a) *By calculation.*—L at 266° F. = 929. Let x heat units be added, and then the dryness at the end of the addition of heat

$$= \frac{x}{929} + \cdot 2$$

Let this dryness $= q_1$

Then— q_2 at 200° F., $= 93\cdot 1\% = \cdot 931$

Also— $L_2 = 975$

But— $q_1 L_1 + t_1 = q_2 L_2 + t_2$

$$\therefore \quad \left(\frac{x}{929} + \cdot 2\right)929 + 266 = (\cdot 931 \times 975) + 200$$

i. e., $x = 907 - 66 - 186 = \underline{655}.$

(b) *By use of chart.*—Draw the constant heat line MN (Fig 223), starting from M. $\left\{\frac{RM}{RS} = \cdot 931\right\}$

Then QN = ·9 rank, or the heat units added = ·9 × (460 + 266),

i. e., $x = \underline{655 \text{ heat units.}}$

Construction of PV and $\tau\phi$ Charts for Engines other than Steam ; *e. g.*, The Stirling, Joule and Ericsson Engines.

Example 38.—Trace the PV and $\tau\phi$ diagrams for the Stirling engine working between 62° F. and 1000° F., the ratio of expansion being 3 to 1. (Work with 1 lb. weight of gas.)

The **PV diagram** consists of two constant volume lines together with two isothermals. See Fig. 224.

Starting from the point A, the pressure $= 14 \cdot 7$, $\tau = 461 + 62 = 523°$, and the volume (read off from the steam tables) $= 13 \cdot 14$ cu. ft. To find the position of the point B :—It is true for all values of p, v and τ that $\frac{pv}{\tau} =$ constant. At B the temperature is 1000° F., or 1461° F. absolute : also the volume is 13·14, hence—

$$\frac{p_B v_B}{\tau_B} = \frac{p_A v_A}{\tau_A}$$

$$i.e., \quad p_B = \frac{14 \cdot 7 \times 13 \cdot 14 \times 1461}{13 \cdot 14 \times 523} = 41 \cdot 1$$

so that the point B is fixed.

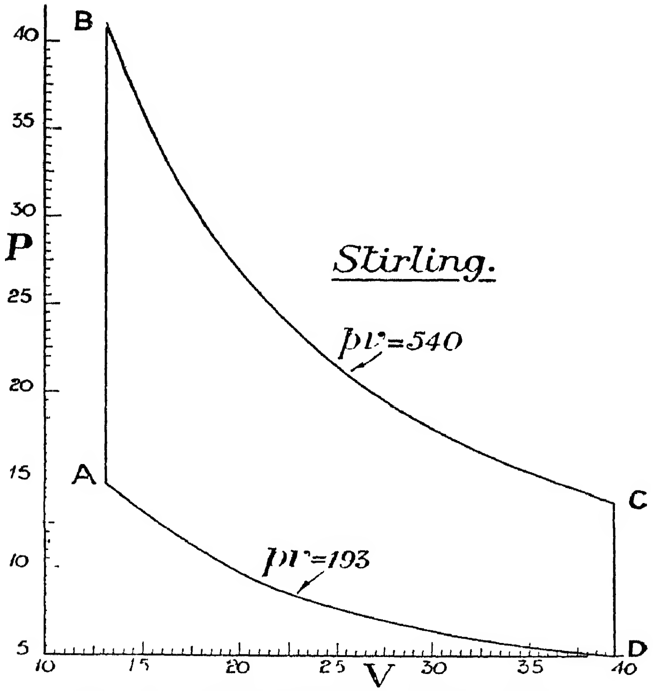


Fig. 224.—PV Diagram for Stirling Engine.

For the isothermal BC, $pv =$ constant, and since $p_B = 41 \cdot 1$ and $v_B = 13 \cdot 14$, the value of the constant is $41 \cdot 1 \times 13 \cdot 14 = 540$.

Using the equation $pv = 540$, points on BC may be found thus :— If $p = 30$, $v = 18$; $p = 20$, $v = 27$, etc.; and the isothermal must be continued until C is reached, the volume at C being three times that at B, *i. e.*, $v_C = 3 \times 13 \cdot 14 = 39 \cdot 42$.

CD is vertical; and also $\frac{p_D v_D}{\tau_D} = \frac{p_C v_C}{\tau_C}$

$$p_D = \frac{540 \times 523}{1461 \times 39 \cdot 42} = 4 \cdot 91$$

so that the position of D is fixed.

The constant for the isothermal DA is $4 \cdot 91 \times 39 \cdot 42 = 193$; and accordingly the points on the line may be obtained.

To draw the **$\tau\phi$ diagram** (Fig. 225) Suppose the entropy is zero at the start. Then points on the line *ab* are calculated from the equation $\phi = K_v \log_e \frac{\tau}{523}$, where K_v = specific heat at constant volume = $\cdot 1691$.

$$\phi = \cdot 1691 \log_e \frac{\tau}{523} = \cdot 1691 \times 2 \cdot 303 (\log_{10}\tau - \log_{10} 523)$$
$$= \cdot 39(\log_{10}\tau - \log_{10} 523)$$

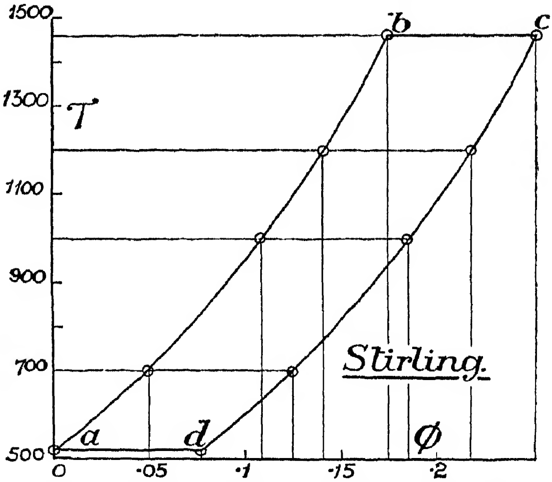


Fig. 225.—$\tau\phi$ Diagram for Stirling Engine.

and the table of values reads as follows :—

τ	$\log_{10}\tau - \log_{10} 523$	$\cdot 39 \times$ column (2) $= \phi$
700	$2 \cdot 8451 - 2 \cdot 7185$	$\cdot 049$
1000	$3 \cdot 0 \quad - 2 \cdot 7185$	$\cdot 1096$
1200	$3 \cdot 0792 - 2 \cdot 7185$	$\cdot 1405$
1461	$3 \cdot 1647 - 2 \cdot 7185$	$\cdot 174$

The position of *c* is fixed, since the work done $= \frac{c\tau \log_e r}{J}$

and thus the distance $bc = \frac{53 \cdot 2}{774} \log_e r$

$$= \frac{53 \cdot 2 \times \log_e 3}{774} = \cdot 0755.$$

The lines *bc* and *ad* are parallel, and *cd* is the curve *ab* shifted to the right a horizontal distance *bc*; and thus the diagram can be completed.

Example 39.—Plot PV and $\tau\phi$ diagrams for the Joule engine, when the compression pressure is 60 lbs. per sq. in. and the lower temperature is 62° F. Work with 1 lb. weight of the gas, and take for the adiabatics $pv^{1\cdot41} = C$.

Dealing with **the PV diagram** (Fig. 226) :—At C the pressure = 14·7, the volume = 13·14 cu. ft., and $\tau = 523$: hence $p_0v_0 = 14\cdot7 \times 13\cdot14 = 193$.

The point A, at pressure 60, is on the isothermal through C; then—

$$p_Av_A = p_0v_0 = 193$$

whence $$v_A = \frac{193}{60} = 3\cdot22$$

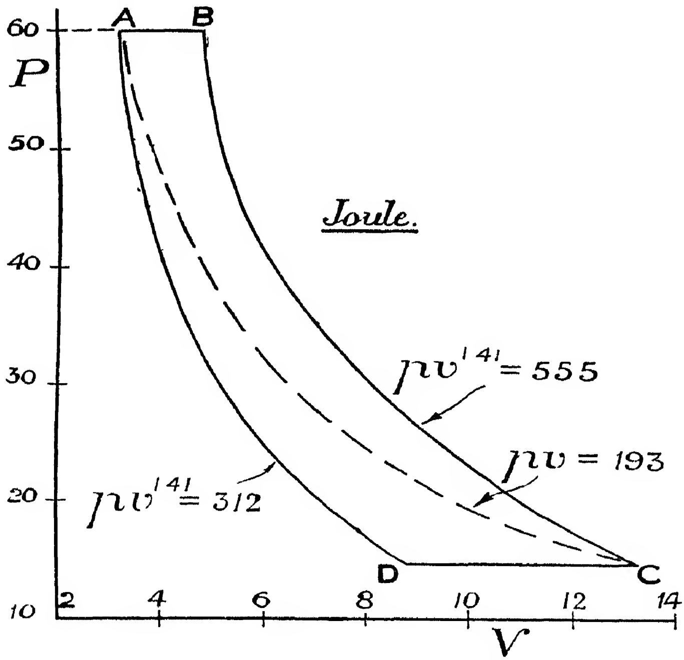


Fig. 226 —PV Diagram for Joule Engine

For the adiabatic AD $\qquad pv^{1\cdot41} = K \quad$ (say)

so that $\qquad K = 60 \times 3\cdot22^{1\cdot41}$

$$\log K = \log 60 + 1\cdot41 \log 3\cdot22 = 1\cdot7782 + (1\cdot41 \times \cdot5079)$$
$$= 2\cdot4943$$
$$\therefore \qquad K = 312\cdot1.$$

Hence points on the line AD may be found from $pv^{1\cdot41} = 312\cdot1$

The pressure at D = 14·7, and the volume = $\sqrt[1\cdot41]{\dfrac{312\cdot1}{14\cdot7}} = 8\cdot732$.

Also—
$$\frac{p_Dv_D}{\tau_D} = \frac{p_Av_A}{\tau_A}$$

$$\therefore \qquad \tau_D = \frac{8\cdot732 \times 14\cdot7 \times 523}{3\cdot22 \times 60} = 347\cdot7^\circ \text{ F. absolute.}$$

For the adiabatic CB, the constant $= p_0 v_0^{1 \cdot 41} = 14 \cdot 7 \times 13 \cdot 14^{1 \cdot 41} = 555 \cdot 1$; and thus this line may be drawn.

Substituting the values of p_A, v_A, τ_A, p_B, $v_B(4 \cdot 845)$ in the equation—

$$\frac{p_B v_B}{\tau_B} = \frac{p_A v_A}{\tau_A},$$ τ_B is found to be $787 \cdot 7°$ F. absolute.

For the $\tau\phi$ diagram (Fig. 227): — Starting from the point c, draw the horizontal through it; this being the isothermal for 523° F. absolute.

The distance

$$ca = \cdot 2375 \log_e \frac{523}{347 \cdot 7} \quad \text{or} \quad \cdot 2375 \log_e \frac{787 \cdot 7}{523};$$ the ratios of the temperatures being the same.

Points on the line ab are obtained from the equation—

$$\phi = \cdot 2375 \log_e \frac{\tau}{523},$$

as also are those on cd; the latter values of ϕ being measured backwards, *i.e.*, towards the left of the diagram. The tabulation for this calculation would be arranged as in the previous example, so that there is no need for a detailed list of values here: and the diagram is completed by the verticals cb and ad.

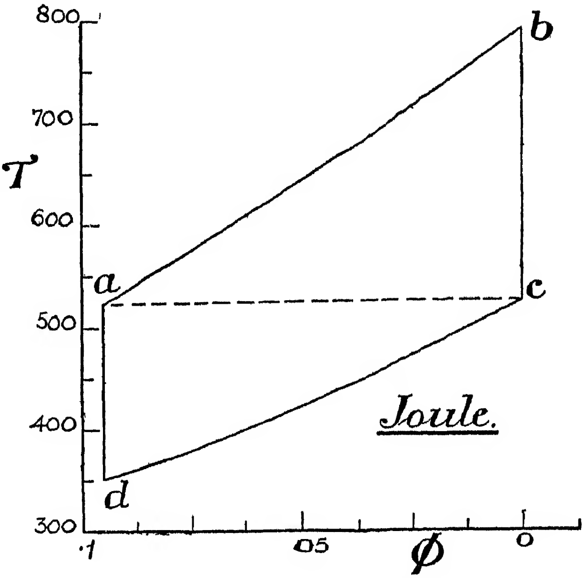


Fig. 227.—$\tau\phi$ Diagram for Joule Engine.

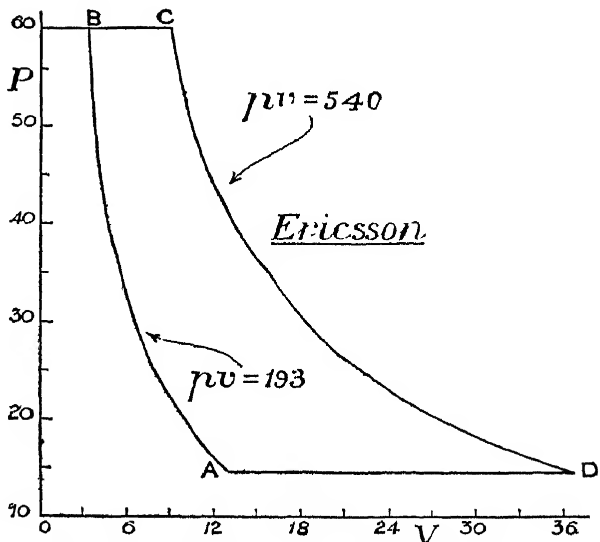


Fig. 228.—PV Diagram for Ericsson Engine.

Example 40.—Plot PV and $\tau\phi$ diagrams for the Ericsson engine, when working between 62° F. and 1000° F., the compression pressure being 60 lbs. per sq. in. absolute. (Work with 1 lb. weight of the gas.)

The calculation is left as an exercise for the reader; but his results may be checked from Figs. 228 and 229.

In Fig. 228 AB and CD are isothermals, the equations to which are $pv = 193$ and $pv = 540$ respectively

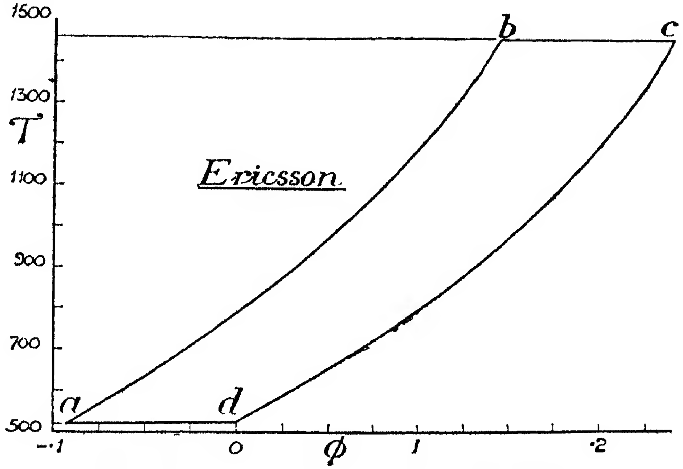


Fig. 229.—$\tau\phi$ Diagram for Ericsson Engine.

Exercises 41.—On the Construction and Use of the PV and $\tau\phi$ Diagrams.

1. Construct a $\tau\phi$ chart, the temperature range being 120° F. to 380° F.; and by the use of this chart solve the problems in Exercises **2** to **6**.

2. Steam ·42 dry at 350° F. expands adiabatically to 140° F. What is now its dryness fraction?

3. Three hundred heat units are added to a sample of steam dry at 310° F. Find the dryness after the addition of the heat.

The steam is now allowed to expand by throttling to 185° F; find the number of heat units that must be added so that the steam becomes dry saturated at this lower temperature.

4. Draw the Carnot cycle, the upper pressure being 150 lbs. per sq. in. absolute, and the lower being 14·7 lbs. per sq. in. absolute.

5. Show on the chart constant volume lines for volumes 5, 10, 15 and 20 cu. ft. respectively.

6. Draw constant heat lines in the superheat area for steam dry saturated at 250° F. and 65° F. respectively.

7. Draw on a PV diagram the adiabatic line mentioned in Exercise **2**, working with 1 lb. of steam. The equation of this expansion line being $pv^n = C$, find the value of n

(*a*) Directly from the diagram.

(*b*) Using Zeuner's rule, viz. $n = 1{\cdot}035 + {\cdot}1q$, q being the initial dryness.

8. Draw constant-dryness lines for dryness fractions of ·2 and ·3 respectively.

9. Calculate the dryness fraction for which the constant-dryness line is straight; assuming that $L = 1437 - {\cdot}7\tau$ and $\phi_\tau = \log_e \frac{\tau}{461}$

CHAPTER X

THE DETERMINATION OF LAWS

It is often necessary to embody the results of experiments or observation in concise forms, with the object of simplifying the future use of these results. Thus the draughtsman concerned with the design of steam engines might collect the results of research concerning the connection between the weight of an engine and its horse-power, and then express the relation between these variable quantities in the form of a law. He might, however, prefer to plot a chart, from which values other than those already known might be read off. The object of this chapter is to show how to fit the best law to correlate sets of quantities: and before proceeding with this chapter the reader should refer back to Chapter IV, where a method of finding a law connecting two quantities was demonstrated. The results of the experiments there considered gave straight lines as the result of directly plotting the one quantity against the other, and from the straight line the law was readily determined.

The values of the quantities obtained in experiments, except in special cases, do not give straight lines when plotted directly the one against the other, but, by slight changes in the form of one or both, straight lines may be obtained as the result of plotting. **The general scheme then is to first reduce the results to a "linear" or "straight-line" equation, to plot the straight line and then to calculate the values of the constants.**

The general equation of the straight line may be stated as—

$$Y = aX + b$$

$$\text{or (Vertical)} = a\text{ (Horizontal)} + b$$

where a is the slope of the line. It is the only "curve" for which the slope is constant; hence the reason for our method of procedure.

e. g., suppose we know that two quantities P and Q are connected by an equation of the form—

$$P^3 = aQ^2 + b.$$

We can rewrite this as—

$$\overline{\underline{P}} = a\overline{\underline{Q}} + b$$

where $\overline{\underline{P}} = P^3$ and $\overline{\underline{Q}} = Q^2$

and this equation is then of the straight-line form. Therefore by plotting $\overline{\underline{P}}$ against $\overline{\underline{Q}}$ a straight line must result.

Conversely, if the plotting of P^3 against Q^2 gives a straight line the equation must be of the form—

$$P^3 = aQ^2 + b.$$

In dealing with the results of any original work there will probably be no guide as to the form of equation, and much time will therefore be spent in experimenting with the different methods of plotting until a straight-line form is found. Sometimes the shape of the curve plotted from the actual values themselves will give some idea of the form of the equation, but a great deal of experience is needed before the various curves can be distinguished with certainty.

It will be found of great value to work according to the scheme of substitutions here suggested, for by the judicious use of the method much of the difficulty will be removed. Thus small or large letters stand for the original quantities, and large or "bar" letters respectively stand for the corresponding "plotting" quantities.

e. g., we are told that given values of x and y are connected by an equation of the type—

$$y = bx^2 + c.$$

If we write Y for y and X for x^2 the equation becomes—

$$Y = bX + c$$

which is of the straight-line form required. The change here made is extremely simple but very effective.

Again, suppose the equation $H = aD^n$ is given as the type.

Seeing that a power occurs we must take logs: thus—

$$\log H = \log a + n \log D.$$

As this equation stands, it is not apparent that it is of the straight-line form; but by rewriting it as

$$\overline{\underline{H}} = A + n\overline{\underline{D}}$$

where $\overline{\underline{H}}$ (H bar) $= \log H$, $A = \log a$ and $\overline{\underline{D}} = \log D$,

it is seen to be of the standard linear form.

We shall deal in turn with the various types of equation that occur most frequently.

Laws of the Type $y = a + \frac{b}{x}$; $y = a + bx^2$, etc.

Example 1.—The following quantities are connected by a law of the form—

$$y = ax^3 + b$$

x	0	2	5	9	10
y	-8	-5	31	212	291

Test the truth of this statement and find the values of a and b.

If we write the equation $y = ax^3 + b$ as $Y = aX + b$, which is permissible provided that $Y = y$ and $X = x^3$; then if the law is true, a straight line should result when Y is plotted against X.

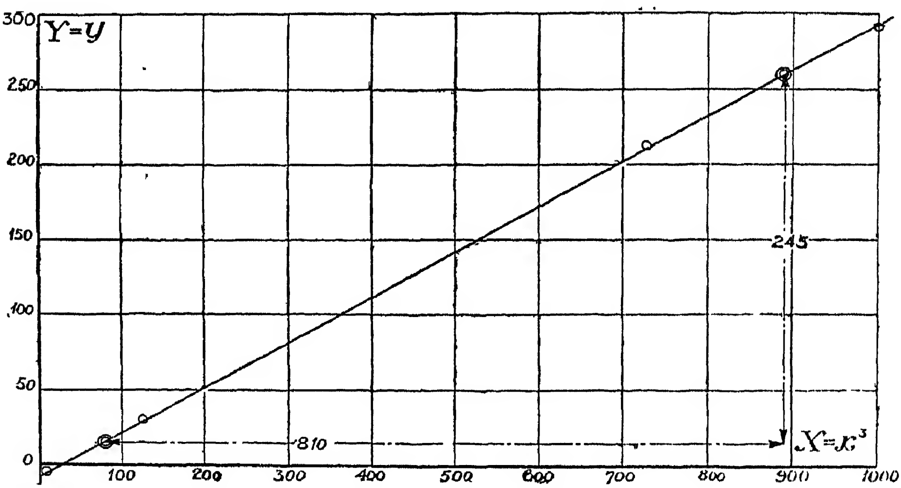


Fig. 230.—Determination of Law for Equation of $y = ax^3 + b$ type.

Hence the table for the plotting reads :—

$X = x^3$	0	8	125	729	1000
$Y = y$	-8	-5	31	212	291

Plotting these values, as shown in Fig. 230, we find that a straight line passes well through the points; and therefore the statement as to the form of equation is correct.

Selecting two convenient points on the curve—

$$\left.\begin{array}{l} X = 80 \text{ when } Y = 15 \\ X = 890 \text{ when } Y = 260 \end{array}\right\}$$

and

Inserting values—

$$260 = 890a + b \quad \ldots \ldots \quad (1)$$

$$15 = 80a + b \quad \ldots \ldots \quad (2)$$

Subtracting— $245 = 810a$

$\therefore \quad a = \cdot 302$

Substituting in (2)— $15 = 24\cdot 2 + b$

$b = -9\cdot 2$

$\therefore \quad Y = \cdot 302X + (-9\cdot 2)$

i. e., $y = \cdot 302x^3 - 9\cdot 2.$

Alternatively, a and b might be found from the graph; since $a = \text{slope} = \frac{245}{810} = \cdot 302$; and $b =$ intercept on vertical axis through 0 of $X = -9\cdot 2$. $\therefore \quad Y = \cdot 302X + (-9\cdot 2)$

and $\underline{y = \cdot 302x^3 - 9\cdot 2.}$

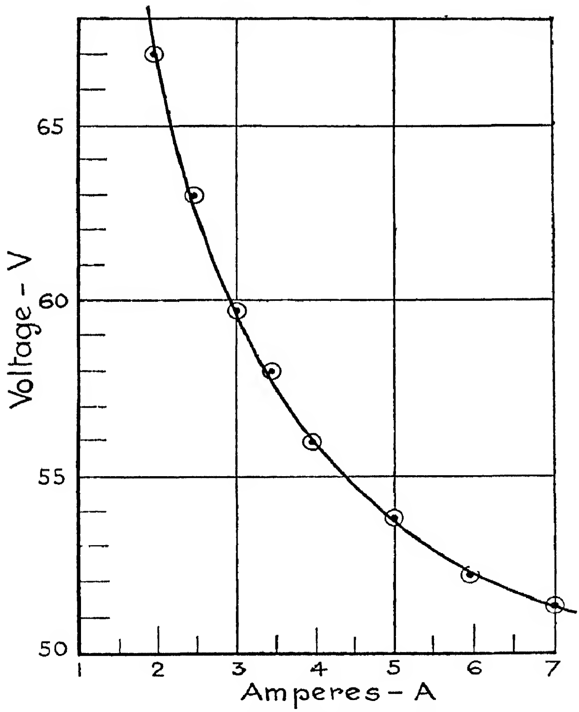


Fig 231.—Law connecting Volts and Amperes of Electric Arc.

Example 2.—An electric arc was connected up in series with an adjustable resistance. The following readings of the volts V and the amperes A were taken, the length of arc being kept constant and the resistance in the circuit being varied :—

V	67	63	59·7	58	56	53·8	52·2	51·4
A	1·95	2·46	3	3·44	3·96	4·99	5·95	7

Find the law connecting V and A.

By plotting V against A, as in Fig. 231, a curve is obtained which shows clearly that the connection between V and A must be of an inverse rather than a direct character, since A increases as V decreases.

Hence a good suggestion is to plot $\frac{1}{A}$ against V, or, in other words, to assume an equation of the form—

$$V = b + \frac{c}{A}$$

Rewriting this equation as $\overline{\underline{V}} = b + c\overline{\underline{A}}$, we see that this is the equation for a straight line.

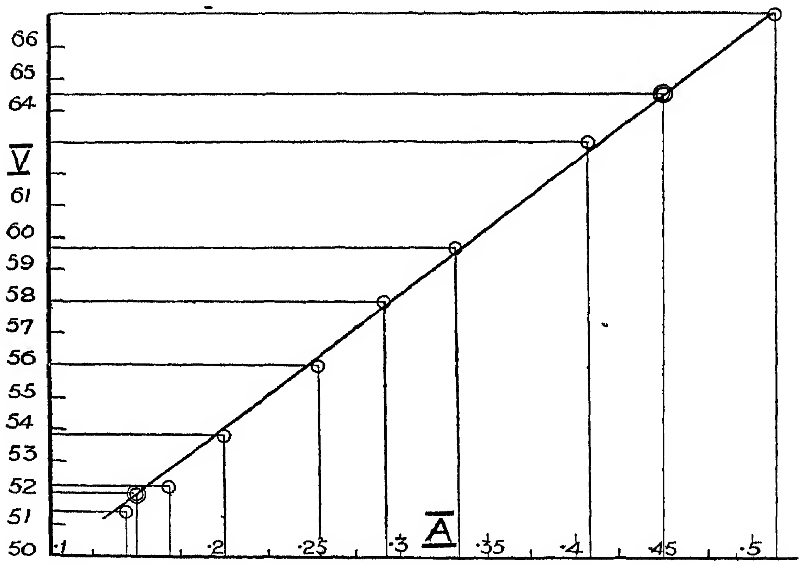


Fig. 232.—Law connecting Volts and Amperes of Electric Arc.

The plotting table will then be as follows :—

$\overline{\underline{V}}$	67	63	59·7	58	56	53·8	52·2	51·4
$\overline{\underline{A}}$	·513	·407	·333	·291	·253	·2	·168	·143

The values of $\overline{\underline{A}}$, *i. e.*, reciprocals of values of A, are obtained from the slide rule. To do this, invert the slide so that the B scale is now adjacent to the D scale. Then the product of any number on the B scale with the number level with it on the A scale equals unity, *i. e.*, if the numbers are read on the B scale, their reciprocals are read on the A scale.

The plotting of $\overline{\underline{V}}$ against $\overline{\underline{A}}$ gives a straight line (see Fig. 232).

Selecting two sets of values—

$$\left.\begin{aligned} \underline{\mathrm{V}} &= 52 \quad \text{when } \overline{\mathrm{A}} = \cdot 15 \\ \text{and} \qquad \underline{\mathrm{V}} &= 64\cdot 5 \quad \text{when } \overline{\mathrm{A}} = \cdot 45 \end{aligned}\right\}$$

Inserting values—

$$64\cdot 5 = b + \cdot 45c \quad \ldots \ldots \quad (1)$$
$$52 = b + \cdot 15c \quad \ldots \ldots \quad (2)$$

Subtracting—

$$12\cdot 5 = \cdot 3c$$
$$\therefore \quad c = 41\cdot 7$$

and by substitution in (1)—

$$64\cdot 5 = b + 18\cdot 75$$

whence

$$b = 45\cdot 75$$
$$\therefore \quad \underline{\mathrm{V}} = 45\cdot 75 + 41\cdot 7\overline{\mathrm{A}}$$
$$\text{or} \quad \mathrm{V} = 45\cdot 75 + \frac{41\cdot 7}{\mathrm{A}}$$

Notice that this problem could have been attacked in a slightly different way.

$$\mathrm{V} = b + \frac{c}{\mathrm{A}}$$

Multiplying through by A—

$$\mathrm{AV} = b\mathrm{A} + c$$

but the product of amps and volts gives watts (W).

$$\therefore \quad \mathrm{W} = b\mathrm{A} + c.$$

Therefore a straight line results if the power (watts) is plotted against the current (amperes).

The table for the plotting would then read :—

A	1·95	2·46	3	3·44	3·96	4·99	5·95	7
W = AV	130·5	155	179·1	199·4	221·5	269	311	359·8

and thence the procedure is as before.

Laws of the Type $y = ax^n$.—If there is no guide to the form of equation, it is most usual to assume it to be $y = ax^n$, or, in more special cases, $y = ax^n + b$; this latter form embracing those already discussed. To avoid the quite unnecessary expenditure of time in searching for the form, this will be indicated before each example or set of like examples.

If $y = ax^n$, then, by taking logs—

$$\log y = \log a + n \log x$$
$$\text{or} \qquad \mathrm{Y} = \mathrm{A} + n\mathrm{X}$$

the large letters being written for the logs of the corresponding small ones.

This last form is the equation of a straight line, the co-ordinates of the points thereon being X and Y, *i. e.*, log x and log y. Accordingly, if corresponding values of two quantities are given, and it is

thought that they are connected by an equation of the type with which we are now dealing, a new or "plotting" table must be made, in which the given values are replaced by their logarithms. These must next be plotted, and if a straight line passes through or near the points, the form of equation is the correct one.

The values of the constants n and a may be found, as before, by either of two methods: (a) by simultaneous equations, or (b) by working directly from the graph.

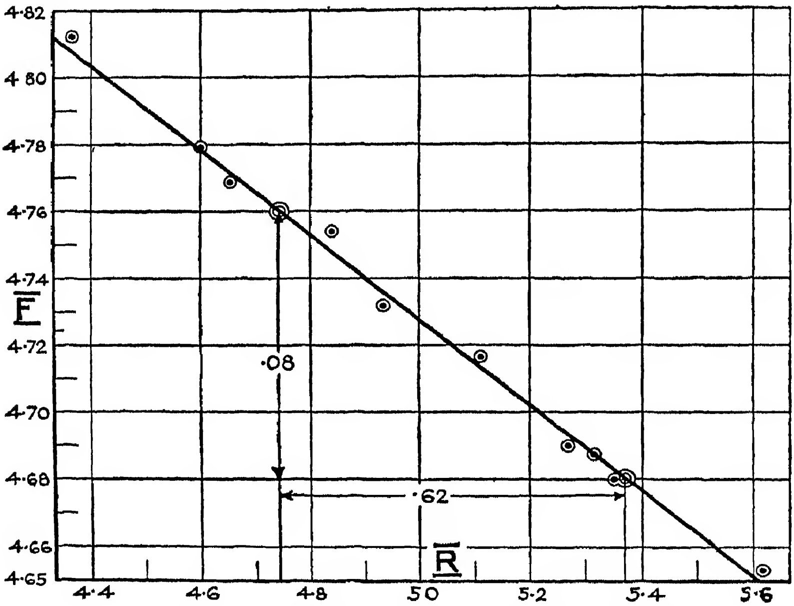


Fig. 233.—Endurance Tests on Mild Steel Rods.

To illustrate by an example:—

Example 3.—In some endurance tests on mild steel rod the following results were obtained:—

Maximum skin stress F in lbs per sq. in.	45200	47500	48700	49000	52100	54000	56750	58700	60150	64800
Revolutions to fracture R	420000	223300	207300	186200	128600	85400	69000	45000	40000	23200

Find the connection between F and R in the form $F = aR^n$.

$$F = aR^n$$

In the log form— $\log F = \log a + n \log R$

or $\overline{F} = A + n\overline{R}$

where $\overline{F} = \log F, \quad A = \log a \quad \text{and} \quad \overline{R} = \log R$

The table of values reads :—

$\overline{\underline{F}} = \log F$ . .	4 6551	4·6767	4 6875	4·6902	4·7168	4·7324	4·7540	4·7686	4·7793	4·8116
$\overline{\underline{R}} = \log R$. .	5 6232	5·3489	5·3166	5 2700	5·1093	4 9315	4 8388	4 6532	4·6021	4·3655

Plotting from this table, we see from Fig. 233 that a straight line passes well through the points.

To find the values of n and a :—

By method (*a*).—Select two convenient points on the line, giving the values—

$$\overline{\underline{F}} = 4{\cdot}68 \text{ when } \overline{\underline{R}} = 5{\cdot}36$$

and $$\overline{\underline{F}} = 4{\cdot}76 \text{ when } \overline{\underline{R}} = 4{\cdot}74$$

Inserting values— $$4{\cdot}76 = A + 4{\cdot}74n \quad \text{. . . . . (1)}$$

$$4{\cdot}68 = A + 5{\cdot}36n \quad \text{. . . . . (2)}$$

Subtracting— $$\cdot 08 = -\cdot 62n$$

whence $$n = -\frac{\cdot 08}{\cdot 62} = -\cdot 129$$

Substituting $-\cdot 129$ in place of n in equation (1)—

$$4{\cdot}76 = A + (-\cdot 129 \times 4{\cdot}74)$$

$$\therefore \quad A = 5{\cdot}37$$

but $A = \log a$ and therefore a = antilog of $5{\cdot}37 = 234400$

$$\therefore \quad \underline{F = 234400R^{-\cdot 129}}$$

By method (*b*).— $$\overline{\underline{F}} = A + n\overline{\underline{R}}$$

Hence if $\overline{\underline{R}}$ be plotted horizontally n is the slope of the resulting line. In measuring the slope, ordinary scales must be used, since the question of logs does not arise at all; and from the equation it is observed that n is a small letter, and therefore represents a number and not a log.

A is the intercept on the vertical axis through the zero of the $\overline{\underline{R}}$ scale, and since the zero of any log scale is the reading corresponding to 1, A is the intercept on the vertical axis through 1 on the scale of R. Obviously in the example under notice, it would be impossible to show this axis on the diagram, at the same time choosing a reasonable scale for $\overline{\underline{R}}$; and consequently method (*a*) is the better.

The slope of the line $= -\dfrac{\cdot 08}{\cdot 62} = -\cdot 129$.

$$\therefore \quad n = -\cdot 129$$

In many practical examples it is only the value of the exponent that is of importance, so that only the slope of the line is required. The slide rule may be used to great advantage in this connection, since its scales are scales of logarithms : and therefore there is no need to consult the log tables, for the logs of the given quantities

are plotted directly from the rule. After plotting, the slope is calculated, both horizontal and vertical distances being measured in centimetres or in inches, the scales on the rule being used: this slope is the value n.

Note.—If the B scale of the rule is used for both horizontal and vertical measurements, then the slope $= \dfrac{\text{difference of vertical}}{\text{difference of horizontal}}$, the same units being employed for both lengths.

If, however, a more open scale is required, say, for the vertical, *i. e.*, the B scale is used for the horizontal and the C scale for the vertical, then the vertical difference must be divided by 2 before comparing with the horizontal difference.

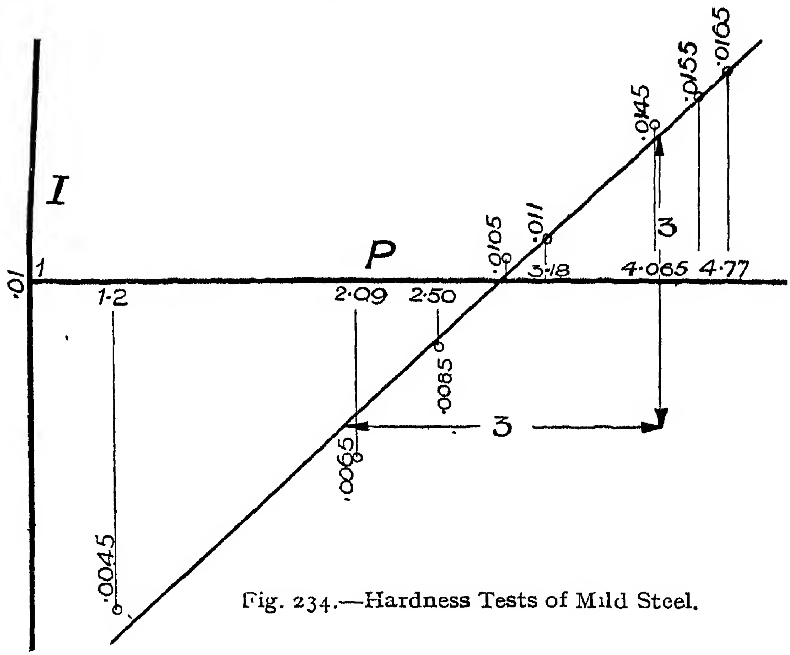


Fig. 234.—Hardness Tests of Mild Steel.

Example 4.—As a result of some tests for hardness, on mild steel, the following figures were obtained :—

Pressure (tons per inch width) . .	1·2	2·09	2·50	2·925	3·18	4·065	4·46	4·77
Indentation (ins.) .	·0045	·0065	·0085	·0105	·011	·0145	·0155	·0165

If i = indentation in inches, p = tons per inch width, and c is a constant for the material, $ci = p^n$.

Find the value of n.

For the actual plotting, shown in Fig. 234, the C scale of the slide rule was used along both axes, and therefore $n = \text{slope} = \frac{3}{3} = 1$.

For— $ci = p^n$

In the log form— $\log c + \log i = n \log p$

or $\log i = n \log p - \log c$

$$I = nP - C$$

and $n = \dfrac{\text{vertical difference}}{\text{horizontal difference}}$, if I is plotted vertically and P horizontally.

Laws of the Type $y = ae^{bx}$, where $e = 2{\cdot}718$, the base of natural logs. We have already seen that many natural phenomena may be expressed mathematically by an equation of the type $y = ae^{bx}$; so also is it possible that an equation of this type may best fit a series of observations so as to correlate them.

If— $y = ae^{bx}$

then $\log y = \log a + bx \log e$

and, since $\log e$ is a constant and equal to $\cdot 4343$,

$$\log y = \log a + \cdot 4343bx$$

or $Y = A + Cx$

where $Y = \log y$, $A = \log a$, and $C = \cdot 4343b$.

$Y = A + Cx$ is the equation of a straight line of slope C, and whose intercept on the vertical axis through the zero of the horizontal scale is A; provided that Y, *i. e.*, $\log y$, is plotted against x.

In the cases in which this law applies we have to employ both direct and log values in the same plotting, and hence there is little advantage in using the slide rule; in fact, it seems better to take the logs required from the tables only. Also, in finding the constants, simultaneous equations must be formed and solved.

Example 5.—The following are the results of Beauchamp Tower's experiments on friction of bearings. The speed was kept constant, corresponding values of the coefficient of friction and the temperature being shown in the table :—

t	120	110	100	90	80	70	60
μ	·0051	·0059	·0071	·0085	·0102	·0124	·0148

Find values of a and b in the equation $\mu = ae^{bt}$ for the set of results given.

$$\mu = ae^{bt}$$

In the log form $\log \mu = \log a + bt \log e = \log a + \cdot 4343bt$

or $M = A + Ct$

where $M = \log \mu$, $A = \log a$, and $C = \cdot 4343b$.

Hence the plotting table reads :—

t	120	110	100	90	80	70	60
$M = \log \mu$	$\bar{3}\cdot7076$	$\bar{3}\cdot7709$	$\bar{3}\cdot8513$	$\bar{3}\cdot9294$	$\bar{2}\cdot0086$	$\bar{2}\cdot0934$	$\bar{2}\cdot1703$

In plotting the values of M it should be remembered that $\bar{3}\cdot7076$ is $-3 + \cdot7076$, and that therefore the marking for $\bar{3}\cdot7076$ on the vertical scale is *above* that for $\bar{3}$, to the extent of $\cdot7076$ unit.

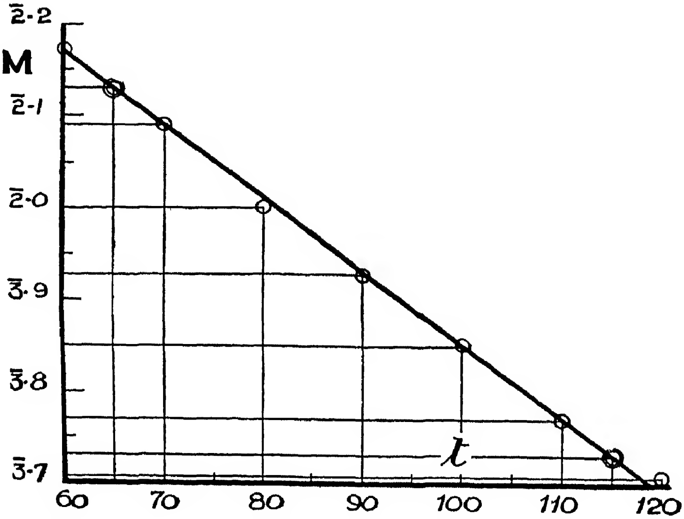


Fig. 235.—Experiments on Friction of Bearings

Plotting these values, as in Fig. 235, we find the straight line that best fits the points. Selecting two sets of values of M and t—

$$\left.\begin{array}{ll} \text{viz.,} & M = \bar{2}\cdot13 \text{ when } t = 65 \\ \text{and} & M = \bar{3}\cdot73 \text{ when } t = 115 \end{array}\right\}$$

we substitute these values in the equation $M = A + Ct$.

Thus—

$$\bar{2}\cdot13 = A + 65C \quad \ldots \quad (1)$$

$$\bar{3}\cdot73 = A + 115C \quad \ldots \quad (2)$$

Subtracting—

$$\cdot40 = -50C$$

$$\therefore \quad C = -\frac{\cdot40}{50} = -\cdot008$$

but

$$C = \cdot4343b$$

$$\therefore \quad b = \frac{C}{\cdot4343} = -\frac{\cdot008}{\cdot4343} = -\cdot0184.$$

Substituting for C in (1)

$$\bar{2}\cdot13 = A + (-1\cdot195)$$

whence $A = \bar{1}\cdot325$

and $a = \text{antilog of } A = \cdot2113$

$$\therefore \quad \underline{\mu = \cdot2113e^{-\cdot0184t}}$$

Laws of the Type $y = a + bx + cx^2$.—Suppose that given values of x are plotted against those of y and instead of the straight line a fairly well-defined curve suits them best. The curve is most likely to be a portion of some parabola, if not of the types of the two previous paragraphs. Its equation may then be of the form $y = a + bx + cx^2 + dx^3 \ldots \ldots$, any terms of which may be absent. This case thus includes types already discussed (e.g., $y = a + bx^2$, and $y = a + dx^3$). If nothing is stated to the contrary, and it is thought that the curve is some form of parabola, it is usually sufficiently accurate to assume as its equation—

$$y = a + bx + cx^2.$$

In this equation there are three constants a, b and c; and to determine them in any case three equations must be stated.

If, then, the equation is to be of this type, plot the given values, sketch in the best smooth curve to pass well amongst the points, and select three convenient points on this curve: the three equations can now be formed and solved in the manner indicated in Chapter II. If possible, one point should be on the y axis, for then $x = 0$ and $y = a + 0 + 0$; or the value of y is such that the value of the unknown a is found directly.

Example 6.—Readings were taken as follows in a calibration of a thermo-electric couple :—

Temperature C.° (T) . .	0	490	840	1003
E.M.F. (microvolts) (E) .	0	3152	5036	5773

Find (a) a formula connecting E and T in the form—

$$E = a + bT + cT^2$$

and hence (b) an expression, enabling values of T to be calculated from any value of E.

The plotting of the values from the table is shown in Fig. 236. Selecting three sets of values—

$$\left.\begin{array}{ll} E = -150 & \text{when } T = 0 \\ E = 2600 & \text{when } T = 400 \\ \text{and} \quad E = 5800 & \text{when } T = 1000 \end{array}\right\}$$

$$\therefore \quad a = -150 \{\text{for } -150 = a + 0 + 0\}$$

$$5800 = -150 + 1000b + 10^6c \quad \ldots \ldots \ldots \ldots \quad (1)$$

$$2600 = -150 + 400b + 16 \times 10^4c \quad \ldots \ldots \ldots \quad (2)$$

Multiplying (1) by 4 and (2) by 10 and subtracting—

$$23200 = -600 + 4000b + 4{,}000{,}000c.$$

$$26000 = -1500 + 4000b + 1{,}600{,}000c$$

Subtracting—
$$-2800 = 900 + 2{,}400{,}000c$$
$$-\frac{3700}{2{,}400{,}000} = c$$
$$c = -\cdot 00154$$
Substituting in (1)—
$$5800 + 150 = 1000b - 1540$$
whence
$$b = 7{\cdot}49.$$
$$\therefore \quad \underline{E = -150 + 7{\cdot}49T - \cdot 00154T^2.}$$

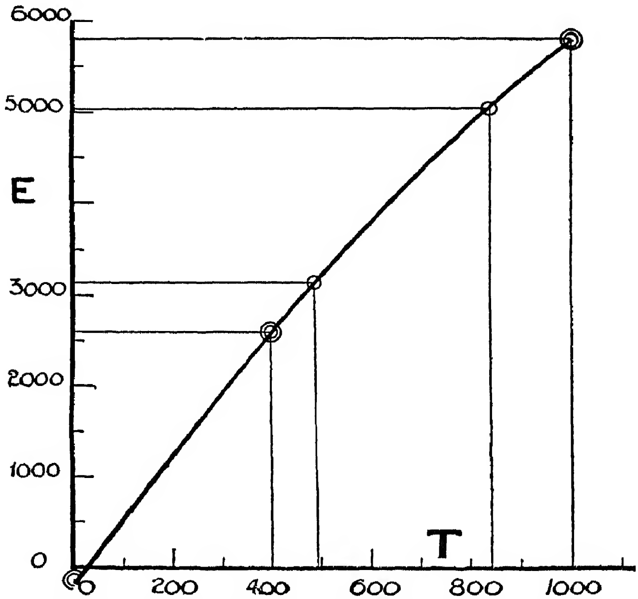


Fig. 236.—Calibration of a Thermo-Electric Couple.

To find an expression for T, solve the quadratic—
$$\cdot 00154T^2 - 7{\cdot}49T + (150 - E) = 0.$$
Thus—
$$T = \frac{7{\cdot}49 \pm \sqrt{56 - \cdot 00616(150 - E)}}{\cdot 00308}$$
$$= \underline{2430 \pm 325\sqrt{55{\cdot}08 + \cdot 00616E}}.$$

Equations of Types other than the Foregoing.—Very occasionally one meets with laws in the form $y = a + bx^n$, $y = b(x + a)^n$, $y = a + be^{nx}$, or $y = ax^nz^m$. These may be dealt with in the following manner :—

(*a*) Type $y = a + bx^n$.

This may be written : $y - a = bx^n$ or $Y = bx^n$ and is of the type already discussed ; but for the change from the one form to the other to be effective, the value of a must be known.

a is the value of y when $x = 0$, so that if possible the curve with y plotted against x should be prolonged to give this value; and it is worth while to sacrifice the scale to a certain extent to allow of this being done.

Otherwise select two points on the curve, draw the tangents there, and measure their slopes. Let the slopes be s_1 and s_2 when x has the values x_1 and x_2 respectively.

Then n, b and a can be calculated from—

$$\left.\begin{aligned} n &= \frac{\log s_1 - \log s_2}{\log x_1 - \log x_2} + 1 \\ b &= \frac{s_1}{n x_1^{n-1}} \\ a &= y_1 - b x_1^n \end{aligned}\right\}$$

(*b*) Type $y = b(x + a)^n$.

If $X = x + a$, then $y = bX^n$, a standard type already discussed.

When $y = 0$, $x + a = 0$ or $x = -a$, so that the value of x where the curve crosses the x axis is $-a$. Values of b and n can then be found in the ordinary way.

An alternative, but rather tedious, method is as follows :—

Select three sets of values of x and y, viz. x_1, x_2, x_3, and y_1, y_2, and y_3.

Then let— $$Y = \frac{\log y_1 - \log y_2}{\log y_1 - \log y_3}$$

and $$A = \frac{\log (x_1 + a) - \log (x_2 + a)}{\log (x_1 + a) - \log (x_3 + a)}$$

Then $Y = A$, because
$$\begin{aligned} \log y_1 &= \log b + n \log (x_1 + a) \\ \log y_2 &= \log b + n \log (x_2 + a) \\ \log y_3 &= \log b + n \log (x_3 + a) \end{aligned}$$

Whence by subtraction—

$$\log y_1 - \log y_2 = n\{\log (x_1 + a) - \log (x_2 + a)\}$$

and $$\log y_1 - \log y_3 = n\{\log (x_1 + a) - \log (x_3 + a)\}$$

{By division n is eliminated.}

For various values of a plot values of $(Y - A)$ until this equals 0; thus the required value of a is found: and values of n and b can now be obtained by logarithmic plotting.

(*c*) Type $y = a + be^{nx}$.

Plot y against x; select two points on the curve and draw the tangents there: call the slopes of these s_1 and s_2.

Then—
$$n = \frac{\log s_1 - \log s_2}{\cdot 4343(x_1 - x_2)}$$
$$b = \frac{s_1}{ne^{nx_1}}$$
$$a = y_1 - be^{nx_1}$$

(*d*) Type $y = ax^n z^m$.

The method of dealing with this form of equation will be demonstrated in the following examples:—

Example 7.—Assuming that the loss of head h in a unit length of pipe in which water is flowing with a mean velocity v can be expressed in the form—
$$h = cv^{3-n}d^{-n}$$
find the numerical values of c and n expressed in feet and second units for a pipe of 4″ diameter and 28 ft. long, using the experimental data of the annexed table:—

Loss of head in feet	·58	1·064	1·635
Discharge in lbs./min.	1550	2138	2690

The corresponding values of h, i. e., loss per foot, will be found by dividing the first line in the table by 28, and are ·0207; ·0381; ·0584 respectively.

To find the velocity—
$$1550 \text{ lbs. per min.} = \frac{1550}{60 \times 62\cdot 4} \text{ cu. ft./sec.}$$
$$= \cdot 415 \text{ cu. ft./sec.}$$
$$\text{Area of } 4'' \text{ diam. pipe} = \cdot 0873 \text{ sq. ft.}$$
$$\therefore \quad \text{Velocity} = \frac{\cdot 415}{\cdot 0873} = 4\cdot 75 \text{ ft./sec.}$$

Similarly, when $Q = 2138$, $v = 6\cdot 54$, and when $Q = 2690$, $v = 8\cdot 22$.

Now— $h = cv^{3-n}d^{-n}$
$$\therefore \quad \log h = \log c + (3 - n)\log v - n \log d \qquad \left\{\begin{array}{l} d = \cdot 3333 \\ \log d = \bar{1}\cdot 5228 \end{array}\right\}$$
$$= \log c + (3 - n)\log v - n \times \bar{1}\cdot 5228$$
$$= \log c + (3 - n)\log v + \cdot 477n \quad \ldots \ldots \ldots \quad (1)$$

Selecting two convenient points on the curve shown in Fig. 237, which is obtained by plotting h against v—
$$h = \cdot 03 \text{ when } v = 5\cdot 8$$
$$h = \cdot 047 \text{ when } v = 7\cdot 3$$
and substituting in (1) we have the equations—
$$\bar{2}\cdot 6721 = \log c + (3 - n) \times \cdot 8633 + \cdot 477n \quad \ldots \ldots \quad (2)$$
$$\bar{2}\cdot 4771 = \log c + (3 - n) \times \cdot 7634 + \cdot 477n \quad \ldots \ldots \quad (3)$$

Subtracting $\cdot 195 = (3 - n) \cdot 0999$

$$= \cdot 3 - \cdot 1n$$

$$\cdot 1n = \cdot 3 - \cdot 195 = \cdot 105$$

$$n = 1 \cdot 05.$$

Substituting in (2)—

$$\bar{2}\cdot 6721 = \log c + (1\cdot 95 \times \cdot 8633) + (\cdot 477 \times 1\cdot 05)$$

$$= \log c + 1\cdot 682 + \cdot 501$$

$$\log c = \bar{4}\cdot 489$$

$$\therefore \quad c = \cdot 0003083$$

Hence— $$h = \cdot 0003083 \frac{v^{1\cdot 95}}{d^{1\cdot 05}}$$

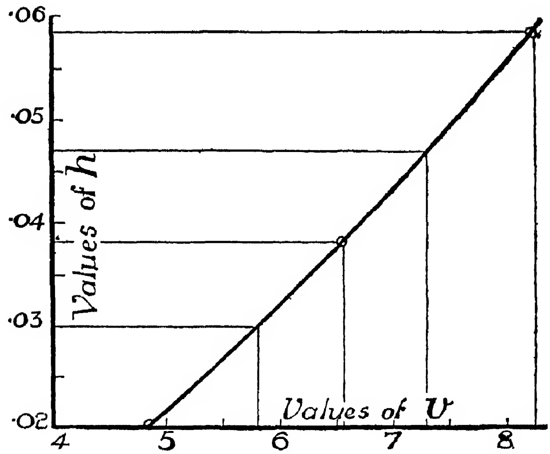


Fig. 237.—Experiment on Loss of Head in Pipe.

Alternatively, we might have proceeded from (1) in the following manner: Plot $\log h$ against $\log v$; find the slope of the resulting straight line, this being the value of $3 - n$; find also the intercept on the vertical axis through 0 of the horizontal scale which gives the value of $\log c - n \log d$, in which everything is known except c, and then calculate the value of c.

Example 8.—During experiments on the loss of head in a 6″ diam. pipe on a measured length of 10 ft. the following observations were made :—

Experiment	Quantity (Gals per min)	Loss of head (ins)
1	294	1·72
2	441	3·66
3	588	6·14
4	735	9·18

Assuming that the loss of head in feet per foot run $= \frac{\mu v^n}{d^m}$ and that $m + n = 3$, find values of n, μ and m.

$$m + n = 3$$

$$\therefore \qquad m = 3 - n$$

$$\therefore \qquad h = \frac{\mu v^n}{d^{3-n}}$$

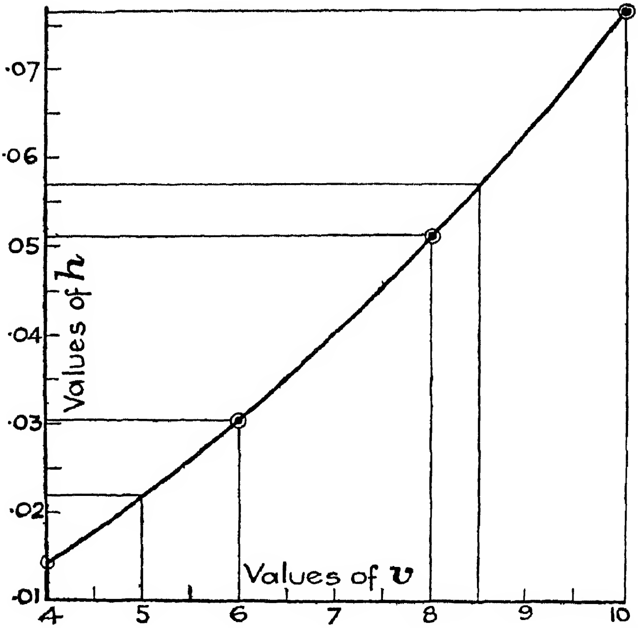


Fig. 238.—Experiments on Loss of Head in 6″-diameter Pipe.

$$d = 6'', \text{ area} = \cdot 196 \text{ sq. ft.}$$

$$294 \text{ gals. per min.} = \frac{294}{6\cdot 24 \times 60} \text{ cu. ft. per sec.}$$

$$= \cdot 785 \text{ cu. ft. per sec.}$$

Hence—

$$\text{velocity} = \frac{\cdot 785}{\cdot 196} = 4 \text{ ft. per sec.}$$

Similarly—

Q	294	441	588	735
v	4	6	8	10

Each value of loss of head (in ins.) must be divided by 10×12 to bring it to feet per foot, so that our final table reads :—

v (ft. per sec.)	4	6	8	10
h (ft. per foot)	·0143	·0305	·0512	·0765

Plot h against v (Fig. 238) and select two convenient points on the graph, viz.—

$$\left.\begin{array}{l} v = 5, \quad h = \cdot 022 \\ v = 8\cdot 5, \ h = \cdot 057 \end{array}\right\}$$

Now— $$h = \mu \frac{v^n}{d^{3-n}} \qquad \left\{\begin{array}{r} d = \cdot 5 \\ \log d = \bar{1}\cdot 699 \\ = -\cdot 301 \end{array}\right\}$$

$$\therefore \quad \log h = \log \mu + n \log v + (n - 3) \log d.$$

Substituting the above values—

$$\bar{2}\cdot 7559 = \log \mu + \cdot 9294n + (3 - n) \times \cdot 301 \quad \ldots \ldots \quad (1)$$
$$\bar{2}\cdot 3424 = \log \mu + \cdot 699n \ + (3 - n) \times \cdot 301 \quad \ldots \ldots \quad (2)$$

Subtracting—

$$\cdot 4135 = \cdot 2304n$$
$$n = \frac{\cdot 4135}{\cdot 2304} = 1\cdot 8.$$

Substituting in (1)—

$$\bar{2}\cdot 7559 = \log \mu + 1\cdot 672 + \cdot 361$$
$$\log \mu = \bar{4}\cdot 723$$
$$\therefore \quad \mu = \cdot 0005284$$

Hence— $$\underline{h = \cdot 000528 \frac{v^{1\cdot 8}}{d^{1\cdot 2}}}$$

Exercises 42.—On the Determination of Laws.

[In the following exercises it should be understood that "finding the law" means finding the constants in the equation.]

1. Find the law to express the following results of a test on an arc lamp, in the form—

$$W = m + nA$$

where W = watts = volts × amps.

V (volts) .	65	72	62	68	64	66	68·4
A (amps)	8·5	5	9·2	8	9·0	10·5	6·5

2. The law connecting μ and v, for the following figures, has the form—

$$\mu = a + b\sqrt{v}.$$

Find this law, which connects μ the coefficient of friction between belt and pulley, with v the velocity of the belt in feet per minute.

v	500	1000	2000	4000	6000
μ	·29	·33	·38	·45	·51

3. The working loads for crane chains of various diameters are given in the table. Find a law connecting W and d of the form—

$$W = a + bd^2.$$

Diam. d	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
Load on chain W (tons)	·20	·45	·81	1·27	1·83	2·49	3·25

4. Bazin gives the following results on the discharge over a weir; H being the head and m being a coefficient :—

H	·164	·328	·656	·984	1·312	1·64
m	·448	·432	·421	·417	·414	·412

If $m = a + \frac{b}{H}$, find the law connecting m and H.

5. The table of allowance for the difference l between the hypotenusal and horizontal measurements per 66 ft. chain in land surveying is given for various angles of slope a:—

$a°$ . .	5	6	7	8	9	10	15	20	25	30	35	40
l (links) .	·4	·6	·7	1	1·2	1·5	3·4	6·0	9·4	13·4	18·1	23·4

The connection between l and a can be expressed by a law of the form $l = ba^2$. Find this law.

6. The following table gives the weight W of cast-iron pedestals for various diameter of shaft d ;—

d (ft.) .	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	2
W (lbs.)	18·005	18·017	18·138	18·464	19·1	26

Find a law of the form $W = ad^3 + b$ to connect W and d.

7. The results of experiments at Northampton Institute with model aeroplanes were as follows :—

Space (ft.) .	1	2·4	4·4	6	7·6	11·2	15·6	20·4
Time (secs.)	·2	·4	·6	·7	·8	1·0	1·2	1·4

Find the law connecting S and t in the form $S = Kt^n$.

8. Find a law connecting horse power H with speed v in the form $H = av^n$, the following values being given :—

v	20·1	24·9	30·2
H	1054	2135	3850

9. Given the following values of torque T, and angle of twist θ, find a law connecting these quantities in the form $T = a\theta^n$.

T (lbs. in.)	800	850	900	950	1000	1050	1100	1150	1200
θ (degrees)	10·4	12·53	15·41	19·2	23·67	29·28	35·58	42·49	51·2

10. If d = diam. of rivet, t = thickness of plate, and $d = at^n$, find values of a and n to agree with the figures :—

t	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	1
d	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{7}{8}$	$\frac{15}{16}$	$1\frac{1}{16}$	$1\frac{1}{8}$	$1\frac{3}{16}$	$1\frac{1}{4}$

11. The following are results of a test on a Marcet Boiler :—

t° F.	320	315	311	307·5	303	300	297	293	287	281	277	271	265	258	251	244	240
Gauge pressure	88	80	75	70	64	60	55	50	45	40	35	30	25	20	15	12·5	10

Find a law connecting the absolute temperature τ $(t + 460)$ and the absolute pressure p (gauge + 15) in the form $\tau = ap^n$.

12. h and v are connected by a law of the form $h = av^n$. Find this law if corresponding values of h and v are as in the table :—

v	8·04	11·67	14·43	17·41	19·90
h	3·03	6·11	9·07	12·21	15·62

13. As a result of Odell's experiments on the torque required to keep a paper disc of diam. 22″ rotating at various speeds we have the following :—

Torque T (lbs. ins.)	·33	·56	·875	1·29	1·76	2·4
R.P.M. (n) . . .	400	500	600	700	800	900

Assuming that $T = an^m$, find the values of a and m.

14. The following figures were obtained in a calibration test of the discharge of water through an orifice :—

Head H .	2·2	1·8	1·4	1·1	·8	·6
Quantity Q	8·9	8·03	7·23	6·4	5·5	4·85

The law connecting H and Q has the form $Q = aH^n$. Find this law.

15. Find a law, of the form $v = a\mathrm{H}^n$, connecting the values :—

H	25	40	60	100	150	250	350
v	1119	1414	1732	2238	2740	3535	4180

16. l and t are connected by a law of the form $l = at^2 + b$. Find this law when corresponding values of l and t are :—

t	1·87	1·76	1·67	1·61	1·49	1·27	1·11	·79
l	34·5	30	28	25	21	16	12	6

17. The resistance R of a carbon filament lamp was measured at various voltages V, with the following results :—

V (volts)	62	64	66	68	70	72	74	76	78	80	82	84	86	88	90	92	94
R (ohms)	73	72·7	72·1	71·7	70·7	70·4	70·1	69·7	69·2	68·5	68·4	67·7	67·2	67·2	66·6	66·3	66·2

Find the values of a and b in the equation $\mathrm{R} = \frac{a}{\mathrm{V}} + b$.

18. The following are results of a test on a 100-volt carbon filament lamp. Find values for a and b as for Ex. 17 above.

V (volts) .	54	60	65	70	75	80	85	90	95	100
A (amps)	·67	·77	·86	·94	1·04	1·11	1·21	1·3	1·4	1·5

(Values of R must first be calculated from $\mathrm{R} = \frac{\mathrm{V}}{\mathrm{A}}$)

19. The difference between the apparent and the true levels owing to the curvature of the earth are given by—

Distance in feet d	300	600	900	1200	1500	1800	2400	3000	3900
Difference of level h (ins.)	·026	·103	·231	·411	·643	·925	1·645	2·57	4·344

Find a law for this having the form $h = \mathrm{K}d^n$.

20. If $pv^n = \mathrm{C}$, find n and C from the given values :—

v	1	2	3	4	5
p	205	114	80	63	52

21. y and x are connected by a law of the form $y = ax^2 + bx + c$. Given that—

x	0	4	10
y	15	16·8	18·75

find values of a, b and c.

22. Coker and Scoble give the following results of a test on a thermo-electric couple :—

Hot junction temp. T (C.°)	0	327	419	657
E.M.F. E (millivolts) .	·015	3·84	4·5	6·32

Find the coefficients in the equation $E = a+bT+cT^2$.

23. Find a law connecting E and T, in the form $E = a+bT+cT^2$, for the case in which—

T	0	490	840	1003	1283
E	0	3·152	5·036	5·773	6·382

24. The results of some experiments by Edge with a Napier car were—

Area of wind-resisting surface A (sq. ft.)	42	38	34	32	28	24	22	18	16	12
Speed V in m.p.h.	47·9	52·9	54	55 5	57·6	62·5	64·2	70·3	75	79

The law fitting these results has the form $A = a+bV+cV^2$; find this law.

25. Given the equation $R = a+bV+cV^2$, and a table of the corresponding values of R and V, find the values of a, b and c.

R	0	9·3	21	35
V	16	14	12	10

26. The velocity of the Mississippi river was measured at various depths with the results :—

Proportional depth D below surface	0	·1	·2	·3	·4	·5	·6	·7	·8	·9
Velocity (ft. per sec.)	3·195	3·23	3·253	3·261	3·252	3·228	3·181	3·127	3·059	2·976

If v and D are connected by a law of the form $v = a+bD+cD^2$, find this law.

27. Find values of a and b in the equation $y = ae^{bx}$ for the following case :—

x	1	1·5	2	2·5	3	3·5	4	4·5
y	13·28	15·04	17·53	19·80	23·11	26	30·5	34·4

In Exercises 28 to 30 the law is $T = 20e^{\mu\theta}$.

28.

T	22·2	24·66	28·86	35·56
θ	·524	1·047	1·833	2·880

Find μ.

29.

T	23·4	27·38	34·66	47·44
θ	·524	1·047	1·833	2·880

Find μ.

30.

T	24·66	30·42	41·64	63·26
θ	·524	1·047	1·833	2·880

Find μ.

31. Find values of a and b in the equation $y = ae^{bx}$ when values of y and x are as in the table :—

x	2·30	3·10	4	4·92	5·91	7·2
y	33	39·1	50·3	67·2	85·6	125

32. The following particulars were obtained from an experiment on the flow through a V notch. Determine a formula connecting the quantity Q with the head H for the notch ($Q = aH^n$)—

Quantity (cu. ft. per sec.)	1·12	·88	·72	·17
Head (ft.)	·900	·815	·757	·422

33. The given values of x and y are connected by a law of the form

$$y = \frac{x}{a+bx}$$

x	6·3	7·5	8	9·7	10	12
y	8·9	21·42	40	—31·2	—25·1	—12

Determine this law.

CHAPTER XI

THE CONSTRUCTION OF PRACTICAL CHARTS

It has been seen that the correlation of two variables constitutes a *graph*. If two or more interdependent variables are plotted on the same axes so as to solve by intercepts problems of all conditions of related variability, the result is a *chart*. Charts may be classified as (*a*) correlation charts or graphs, (*b*) ordinary intercept charts, or (*c*) alignment charts.

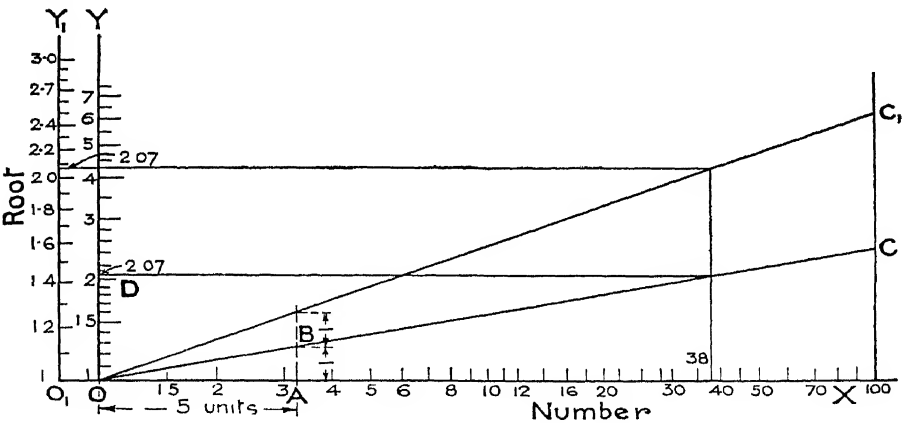


Fig 239.—Chart giving Fifth Roots

Correlation Charts may be regarded as forms of the graphs already treated, but specially adapted for particular circumstances. The modification in the construction of the graph frequently consists of the substitution of a straight line in place of a curve, the former being far the easier to draw, and when powers occur, this necessitates logarithmic plotting.

Example 1.—Construct a chart to read the fifth roots of all numbers up to 100.

Along OX and OY in Fig. 239 mark out log scales, using the B scale of the slide rule for both directions. The scale of numbers being along

OX, extend this axis to show 100 at its highest reading. Set off OA = 5 units, say $2\frac{1}{2}''$, and set off AB = 1 unit, *i.e.*, $\frac{1}{2}''$. Join OB and produce to C.

Then to find the fifth root of 38, erect a perpendicular through 38 on the horizontal scale to meet OC and project horizontally to meet OY in D, *i.e.*, at the reading 2·07 : then $\sqrt[5]{38} = 2{\cdot}07$.

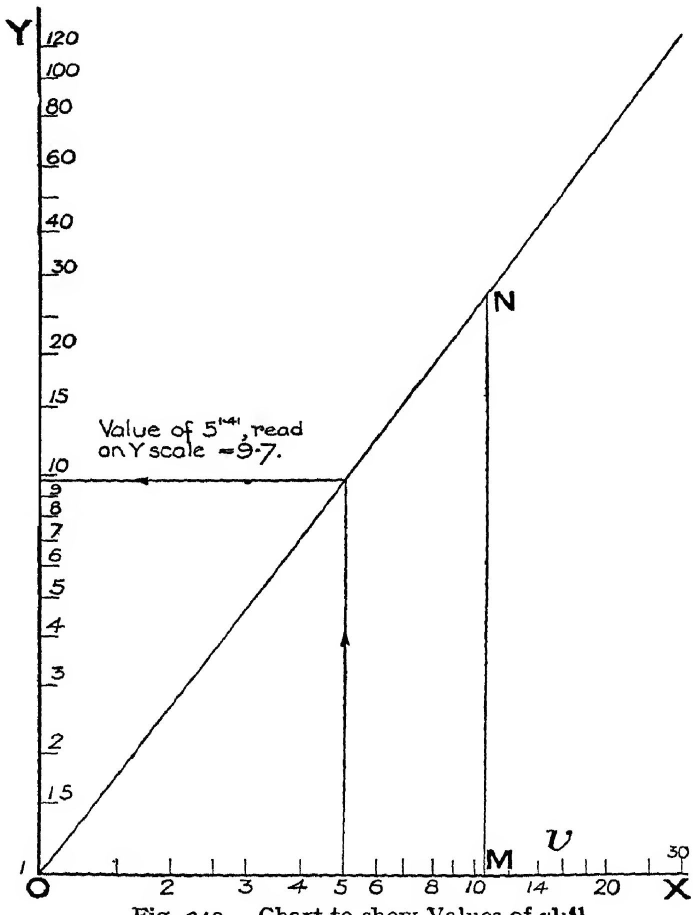


Fig. 240.—Chart to show Values of $v^{1{\cdot}41}$.

The value of the exponent is thus the slope of the line, and hence this method can be used to great advantage when the power is somewhat awkward to handle otherwise.

Example 2.—In calculating points on an expansion curve, it was required to find values of $v^{1{\cdot}41}$, v ranging from 1 to 30. Construct a chart by means of which the value of $v^{1{\cdot}41}$ for any value of v within the given range can be determined.

In Fig. 240 draw the axes OX and OY at right angles, and starting from 1 at the point O set out log scales along both axes; the same scale of the slide rule being used throughout.

Make OM = 1 unit of length and MN = 1·41 units of length (*i. e.*, actual distances): join ON and produce to cover the given range. Then for $v = 5$, $v^{1\cdot 41} = 9\cdot 7$, the method of obtaining this value being indicated on the diagram.

If it be desired to have a more open scale along one axis, allowance must be made in the following way:—

Referring to *Example* 1, suppose that the B scale of the slide rule is used for the scale of numbers and the C scale of the rule for the scale of roots. Then the slope of the line OC_1 (Fig. 239) must be made $= \frac{2}{5}$ and not $\frac{1}{5}$. The scale for roots for this case is shown to the left of the diagram, viz., along O_1Y_1.

Ordinary Intercept Charts.—A combination of two or more graphs is often of far greater usefulness than the separate graphs, since intercepts can then be read directly and from the one chart.

Intercept charts may take various forms, and the following examples illustrate some of the types:—

Example 3.—Construct a chart to give the horse-power transmitted by cast-iron wheels for various pitches and at various speeds. The speeds vary from 100 to 1500 ft. per min. and the pitch from $\frac{1}{2}$ in. to 4 ins.

Working with the units as stated, and allowing for the whole pressure to be carried by any one tooth at a time, the formula reduces to

$$H = \frac{p^2 V}{110}$$

This formula might be written as $H = p^2 \times \frac{V}{110}$ or $H = \frac{p^2}{110} \times V$, so

that if p is constant— $H \propto V$

or if V is constant— $H \propto p^2$

We may thus draw on one diagram (see Fig 241) a number of graphs: for on the assumption that $p = 2$, say, $H = \frac{4V}{110} = \cdot 0364V$, and this relation may be represented by a straight line. By varying p other lines may be obtained, and as they are all straight lines passing through the origin (for $H = 0$ when $V = 0$) only one point on each need be calculated, though as a check it is safer to make the calculation for a second point.

E. g., when $p = \frac{1}{2}''$ and $V = 440$, $H = 1$, giving a point on the line. Plot values of V vertically and H horizontally. Join the origin to the

point for which H = 1 and V = 440 and produce this line to cover the given range. Indicate that this is the line for pitch = $\frac{1}{2}''$.

For $p = 2''$ and V = 440, H = 16.

Hence join the origin to the point (16,440); produce this line and mark it for $p = 2''$. By two simple calculations in each case a number of such lines may be drawn, say for each $\frac{1}{4}''$ difference of pitch.

To use the chart.—To find the H.P. transmitted when the pitch is $3\frac{1}{4}''$ and the velocity is 560 ft. per min.: Draw a horizontal through 560 on the V scale to meet the sloping line marked $p = 3\frac{1}{4}''$, and project from the point so obtained to the scale of H, where the required value, viz., 54, is read off.

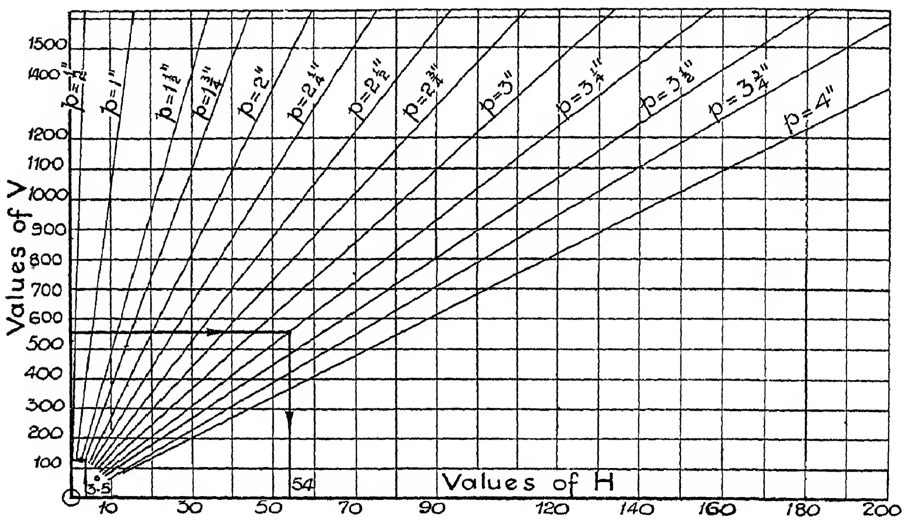


Fig. 241.—Chart giving H.P. transmitted by Cast-iron Wheels.

Again, if the pitch is $1\frac{3}{4}''$, what speed is necessary if $3\frac{1}{2}$ H.P. is to be transmitted? Draw a vertical through 3·5 on the H scale to meet the line marked $p = 1\frac{3}{4}''$ and project to the vertical scale, meeting it in V = 125.

In an exactly similar fashion a most useful chart might be constructed to give values of the rectangular moments of inertia for rectangular sections of various sizes. Since I (moment of inertia of a rectangular section) $= \frac{1}{12}bh^3$, then I $\propto$ b if h is constant. Then for each value of h a straight line can be drawn, and the chart can be used in the same way as before.

Example 4.—Construct a chart to give the diameters of crank shaft necessary, when subjected to both bending and twisting actions, the

greatest stress allowable in the material being 6000 lbs. per sq. in. Given that—

$$\text{Equivalent twisting moment} = T_e = M + \sqrt{M^2+T^2}$$

$$\text{and also} \qquad T_e = \frac{\pi}{16} f D^3$$

where $f = 6000$ and D = diam. of shaft in inches.

Although there are three variables, viz., M, T and D, one simple chart suffices; it being constructed in the following manner:—

Referring to Fig. 242, select an axis OY near the centre of the page, and along this axis set out the scale of torque in lbs. ins. Along the

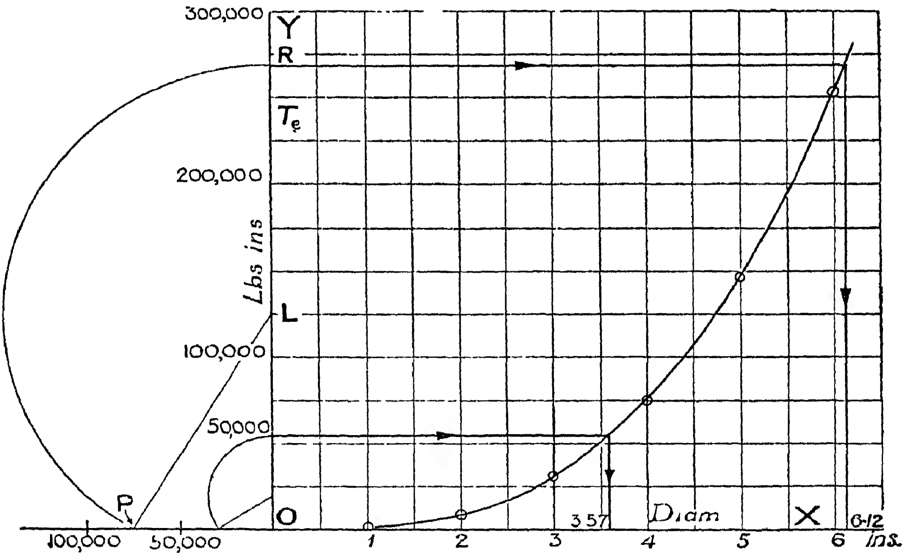


Fig. 242.—Chart to give Diameters of Crank-shaft subjected to Stresses.

horizontal axis OX indicate a scale for diameters, taking the maximum value as 6·5″. Two of the three variables may be combined by the following device: Suppose T = 75000 lbs. ins. and M = 125000 lbs. ins.; then set off along OP a distance to represent T, using the same scale as along OY; make OL to represent M. With centre L and radius LP strike an arc to cut OY in R. Then OR = T_e, since—

$$\begin{aligned} OR = OL+LR &= OL+LP \\ &= OL+\sqrt{(LO)^2+(OP)^2} \\ &= M+\sqrt{M^2+T^2} = T_e. \end{aligned}$$

Now T_e and D are connected by an equation which can be represented by a curve, and—

$$T_e = \frac{\pi \times 6000 D^3}{16} = 1176 D^3.$$

For this curve the plotting table is—

D	1	2	3	4	5	6
D^3	1	8	27	64	125	216
$T_e = 1176D^3$	1176	9420	31800	75300	147000	254000

By plotting T_e against D complete the chart.

For use: let it be asked what diameter of shaft is required which is to be subjected to a bending moment of 125000 lbs. ins. and a twisting moment of 75000 lbs. ins.

Set off OP = 75000 and OL = 125000: with centre L and radius LP strike the arc PR. Draw the horizontal through R to meet the curve and thence project vertically to the scale of D, where the diameter is read as 6·12 ins. Again, if M = 20000, and T = 30000, then D = 3·57, the method of obtaining this value being as before.

A chart representing an equation similar to that in *Example* 1 might be constructed in a slightly different and better manner; thus:—

Example 5.—Construct a chart to show the quantity of water flowing through pipes of various diameters, the velocity of flow also varying.

Let Q_1 = quantity in cu. ft. per sec. = area in sq. ft. × velocity of flow in ft. per sec.

then Q_2 = quantity in cu. ft. per min. = $60Q_1$

and Q = quantity in lbs. per min. = 60 × 62·4 × area in sq. ft. × velocity in. ft. per sec.

so if the diameter is given in inches and the rate of flow in ft. per sec.—

$$Q = \frac{60 \times 62{\cdot}4 \times \text{area} \times \text{velocity}}{144} = \frac{60 \times 62{\cdot}4 \times \pi d^2 v}{4 \times 144} = 20{\cdot}4d^2v$$

where d = diam. of pipe in inches, and v = velocity of flow in ft. per sec.

We will assume a maximum diameter of 6 ins., and a maximum velocity of 10 ft. per sec.

Draw two axes at right angles in Fig. 243, the vertical axis being in the middle of the horizontal. Along OX_1 indicate a scale of diameters, the range being 0 to 6, and along OX indicate a scale of quantities, the range being 0 to 7500, to include the maximum value of Q, viz. 7350, the value of the product $20{\cdot}4 \times 6^2 \times 10$. Along OY set out values of $20{\cdot}4d^2$, the maximum value being $20{\cdot}4 \times 36 = 735$; and draw the curve having the equation $y = 20{\cdot}4d^2$, a table for which is:—

d	0	1	2	3	4	5	6
$20{\cdot}4d^2$	0	20·4	81·6	183·6	326	510	735

thus obtaining the curve OA.

In the right-hand division of the diagram lines must be drawn of various inclinations, the slopes depending on the values given to v.

E. g., if $v = 2$, when the value of y (*i. e.*, $20{\cdot}4d^2$) is 500, the value of Q is 1000, therefore join the origin to the point for which $Q = 1000$, $y = 500$, and mark this as the line for $v = 2$. The diagram is completed by the lines for $v = 1, 3, 4 \ldots\ldots 10$.

Use of the chart.—To find the discharge when the pipe is $2\frac{1}{2}''$ diam. and the velocity of flow is 5 ft. per sec.: Erect a perpendicular from $2\frac{1}{2}$ on the d scale to meet the curve OA; then move across on the horizontal till the line for $v = 5$ is met; and a vertical from this point on to the scale of Q gives the required value, viz. 637 lbs. per min.

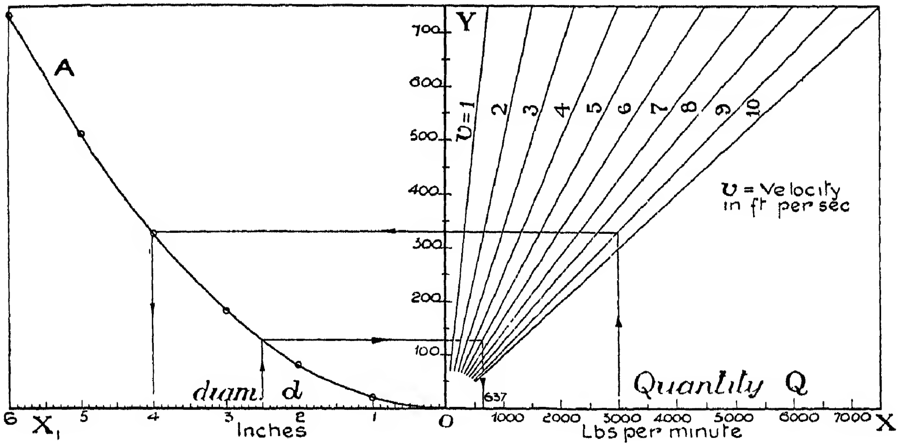


Fig. 243.—Chart to give Quantity of Water flowing through Pipes.

Again, if the quantity is 3000 lbs. per min. and the velocity is 9 ft. per sec, to find the diameter: Erect a perpendicular through 3000 on the Q scale to meet the line marked $v = 9$: draw a horizontal through this point to cut the curve, and finally drop a perpendicular on to the scale of diameters. The diameter required is seen to be $4''$.

If desired, the scale of Q may be modified to show values of Q_1 (cu. ft. per sec.) or Q_2 (cu. ft per min.).

Example 6.—The weight in lbs. of a cylindrical pressure tank with flat heads (allowing for manhole, nozzles, and rivet-heads) may be expressed, approximately, by $W = 10DT(L+D)$, where L = length in feet, D = diam., in feet, and T = thickness of shell in sixteenths of an inch. Construct a chart to show weights for tanks of any diameter up to 5 ft. and lengths up to 30 ft.; the maximum thickness of metal to be $\frac{1}{2}''$.

Let $W = 10DT(L+D) = W_1T$, *i. e.*, $W_1 = 10D(L+D)$.

On the left of the diagram (see Fig. 244) no notice is taken of the thickness, *i. e.*, W_1 is plotted against $(L+D)$ for various values of D. A number of straight lines result, since $W_1 \propto (L+D)$.

Along OX_1 indicate the scale from 0 to 35 for $(L+D)$, and along OY the scale for W_1 from 0 to 1750. The scale along OX will be that for W, the maximum value required being 8×1750, *i. e.*, 14000 lbs.

For the left-hand portion.—Suppose $D = 2$, then for $L = 30$

$$W_1 = 10 \times 2 \times (30+2) = 640.$$

Join the origin to the point for which $(L+D) = 32$ and $W_1 = 640$, and mark this as the line for $D = 2$. Proceed similarly for other values of D.

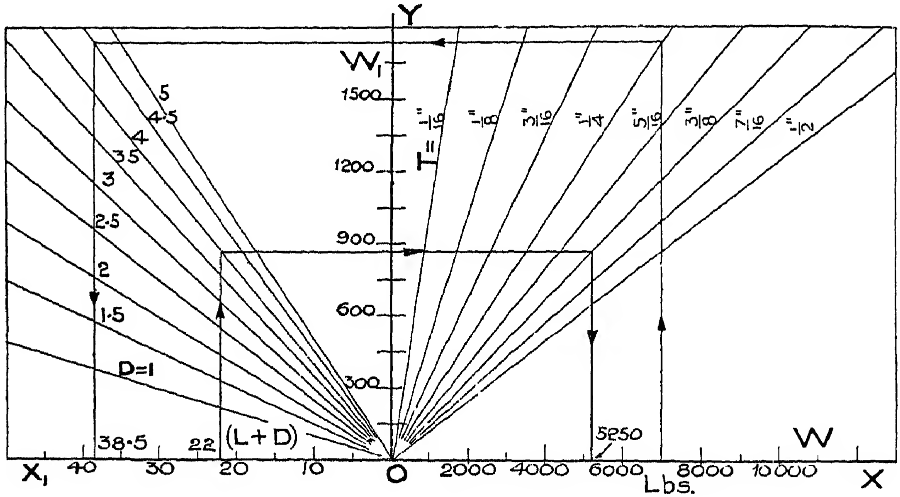


Fig. 244.—Chart to give Weights of Pressure Tanks.

For the right-hand portion.—Suppose $T = \frac{1}{2}''$, *i. e.*, $\frac{8}{16}''$.

When— $W_1 = 1000$, $W = W_1T = 1000 \times 8 = 8000$.

Join the point for which $W = 8000$, $W_1 = 1000$ to the origin, and mark this as the line for $T = \frac{1}{2}''$. Draw lines for $T = \frac{1}{8}''$, $\frac{1}{4}''$, etc., in a similar manner.

To use the chart.—Let it be required to find the weight of a tank of length 18 ft. and of diameter 4 ft., with thickness of shell $\frac{3}{8}''$.

Here $(L+D) = 18+4 = 22$. Hence erect an ordinate through 22 on the scale of $(L+D)$ to meet the line for $D = 4$; draw a horizontal to meet the line for which $T = \frac{3}{8}''$; then project to OX, and the value of W is read off as 5250 lbs.

Again, what will be the length of the tank, of diameter $4\frac{1}{2}$ ft., the thickness of shell being $\frac{1}{4}''$, and the weight 7000 lbs.?

Erect a perpendicular through 7000 on the scale of W to meet the sloping line for which $T = \frac{1}{4}''$, and draw a horizontal to meet the line for which $D = 4{\cdot}5$. A perpendicular through this point cuts OX_1 in the point for which $L+D = 38{\cdot}5$, but as $D = 4{\cdot}5$, then L must $= 34$ ft.

Example 7.—The next chart involves a considerable amount of calculation, which, however, once done serves for all cases. We wish to find the volume of water in a cylindrical tank for various depths and various lengths.

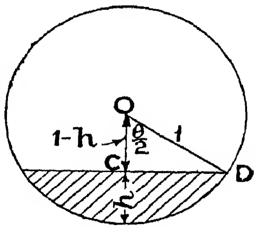


Fig. 245.

Preliminary calculation.—Let the depth of the water be h (Fig. 245).

Then $OC = r - h$, or, taking the radius as 1 ft., $1 - h$.

$$\text{Let—} \qquad \angle DOC = \frac{\theta}{2}, \text{ then } \cos\frac{\theta}{2} = \frac{1-h}{1} = 1-h$$

$$E.\ g., \text{ for} \qquad h = {\cdot}1, \quad \cos\frac{\theta}{2} = 1-{\cdot}1 = {\cdot}9 = \cos 25^\circ 50'$$

$$\therefore \qquad \frac{\theta}{2} = 25^\circ 50', \quad i.\ e.,\ \theta = 51^\circ 40'$$

Now, the area of the cross-section of the water = area of segment

$$= \frac{1^2}{2}(\theta - \sin\theta)$$

$$= \frac{\theta - \sin\theta}{2}$$

where θ is expressed in radians—

$$i.\ e., \qquad \theta \text{ (radians)} = \frac{\theta}{57{\cdot}3} \text{ (degrees)}$$

Hence our table, giving areas of cross-section for different heights, may be arranged as follows; h being expressed as a fraction of the radius—

h	$\cos\frac{\theta}{2}$	$\frac{\theta}{2}$	θ°	θ (radians)	$\sin\theta$	$\theta - \sin\theta$	Area
0	1	0	0	0	0	0	0
·2	·8	36°52′	73°44′	1·287	·960	·327	·164
·4	·6	53°8′	106°16′	1·855	·960	·895	·448
·6	·4	66°25′	132°50′	2·310	·733	1·583	·792
·9	·1	84°16′	168°32′	2·94	·199	2·741	1·371
1·2	—·2	101°32′	203°4′	3·545	—·392	3·837	1·919
1·5	—·5	120°	240°	4·186	—·866	5·052	2·526
1·7	—·7	134°26′	268°52′	4·70	—·999	5·699	2·85
2·0	—1	180°	360°	6·284	0	6·284	3·142

Plot a curve with h horizontally and areas vertically, as in Fig. 246.

$$\text{Now volume} = \text{area} \times \text{length}$$

and for a length of 10 ft. and area 3 sq. ft. the volume = 30 cu. ft. Hence join the origin to the point for which $V = 30$, $A = 3$, and mark this as the line for $l = 10$. Add other lines for different values of l as before.

If the chart is to be made perfectly complete, a number of curves must be drawn in the left-hand portion, one for each separate value of the diameter. For diam. = 4 ft., ordinates of the curve would be $\left(\frac{4}{2}\right)^2$

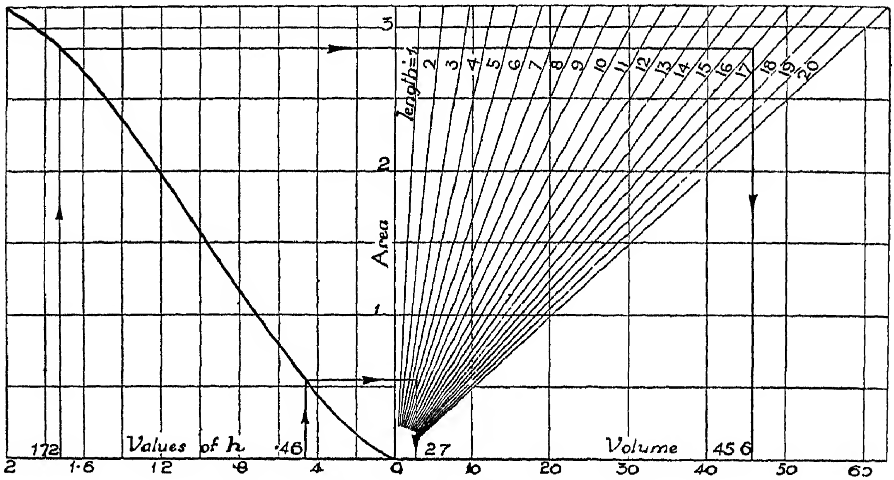


Fig. 246.—Chart giving Volume of Water in Cylindrical Tanks

i. e., four times those of the curve for $d = 2$ as already drawn. This tends to cramp the scale, so that it is preferable to work from the one curve and to multiply afterwards, remembering that the variation will be as the squares of the diameters.

E. g., if diam. = 2 ft., $h = \cdot 46$ ft., and $l = 5$ ft., then vol. = 2·7 cu. ft., the lines for this being shown on the diagram.

But if the diam. = 6 ins., $h = \cdot 46 \times$ radius, and $l = 5$ ft., then—

$$\text{vol.} = 2\cdot 7 \times \left(\frac{\frac{1}{2}}{2}\right)^2$$

$$= \cdot 169 \text{ cu. ft.}$$

Again, if $h = 1\cdot 72 \times$ radius, diam. = 5 ft., and length = 16 ft., to find the volume proceed as indicated on the diagram. The volume for 2 ft. diam. is 45·6, so that the volume for 5 ft. diam.—

$$= 45\cdot 6 \times \left(\frac{5}{2}\right)^2 = 285 \text{ cu. ft.}$$

The following construction may reasonably be introduced as a chart :—

Example 8.—Resistances of 54 and 87 ohms respectively are joined in parallel; what is the combined resistance of these ?

This question may be worked graphically in the following manner—

Draw OA and OB, Fig. 247, lines making 120° with one another. Along OA set off a distance to represent 54 ohms, thus obtaining the point E, and along OB set off OF to represent 87 ohms to the same scale. Bisect the angle AOB by the line OC.

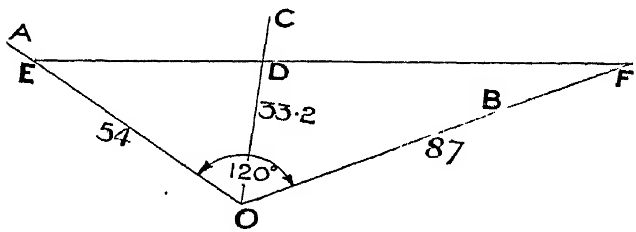


Fig. 247.

Join EF to intersect OC at D. Then OD measures, to the same scale as that used along OA and OB, the combined resistance, and it is found to be 33·2 ohms.

Alignment Charts.*—In these charts two or more variables are set out along vertical axes, which are so spaced, and for which the scales are so chosen, that complicated formulæ may be evaluated by the simple expedient of drawing certain crossing lines. Then for the same connection between the variables, one chart will give all possible values of all of them within the range for which the chart is designed. Thus transposition and evaluation of formulæ become unnecessary; and, in fact, the charts can be used in a perfectly mechanical manner by men whose knowledge of the rules of transposition is a minimum.

Referring to our work on straight line graphs, we see that the general equation of a straight line is $Y = aX+b$. By suitably choosing the values of a and b we may write this equation in the form $AX+BY = C$; and it is with the equation in this form we wish to deal.

Plotting generally is to most minds connected inseparably with two axes at right angles : that is certainly the easiest arrangement of the axes when two variables only are concerned. Suppose,

* For fuller treatment of these charts see *Line Charts for Engineers*.

now, that three, four, or even eight or nine variables occur; then our method fails us, and in such a case it is found that vertical axes only can be used with advantage.

It is not our intention to fill the book with alignment charts, for examples of these intensely practical aids may be found in the technical periodicals; what is intended is that the theory of the

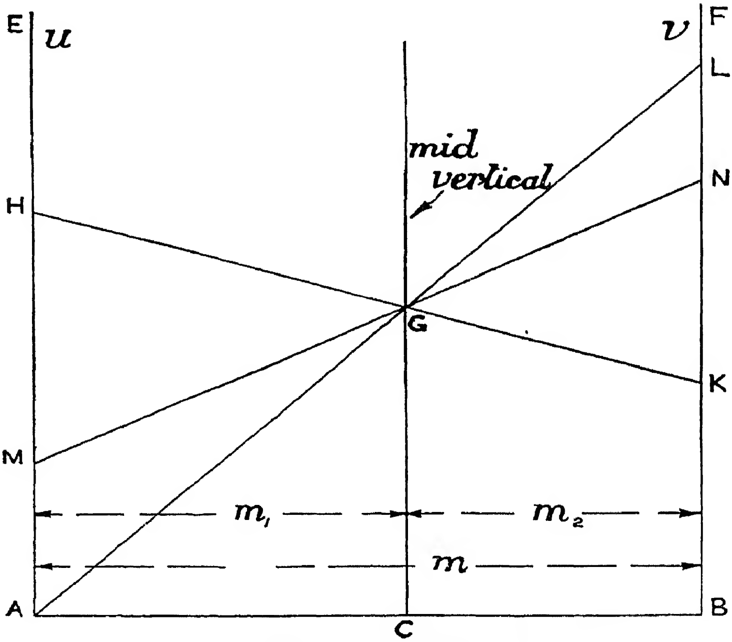


Fig. 248.—Principle of Alignment Charts.

charts should be grasped, so that any one can construct a chart to suit his own particular needs and conditions.

Let us consider firstly the simplest case, viz. $x+y=c$, or, as we shall write it, $u+v=c$ (u and v being adopted for the sake of clearness, since both the u and the v axes are to be vertical, whereas axes for x and y are horizontal and vertical respectively).

Draw two verticals AE and BF (Fig. 248) any convenient distance apart, and let AE be the axis of u and BF be the axis of v. Draw also the horizontal AB, which is to be the line on which the zeros of the scales along the u and v axes lie.

Assume some value for c and calculate values of u and v for two cases; set off along AE these values of u to a scale of l_1 units per inch, and along BF these values of v to a scale, say, of l_2 units per inch. Let AH represent the value of u when v has the value

represented by BK, and AM the value of u corresponding to the value of v represented by BN. Join HK and MN to intersect at G, and through G draw a vertical GC, which will be referred to throughout as the *mid-vertical.*

Then— AH represents the first value of u; call it u_1;
and BK represents the first value of v_1; call it v_1.

Similarly AM and BN represent u_2 and v_2 respectively, and since $u+v = c$ for all values of u and v, $u_1+v_1 = c$ and $u_2+v_2 = c$.

AH, AM, BK, and BN are actual distances on the paper, hence $l_1 \times \text{AH} = u_1$, $l_1 \times \text{AM} = u_2$, $l_2 \times \text{BK} = v_1$, and $l_2 \times \text{BN} = v_2$.

Substituting in the equations $u_1+v_1 = c$ and $u_2+v_2 = c$,

$$(l_1 \times \text{AH}) + (l_2 \times \text{BK}) = c \quad \ldots \ldots \quad (1)$$

and

$$(l_1 \times \text{AM}) + (l_2 \times \text{BN}) = c \quad \ldots \ldots \quad (2)$$

From the figure—

$$\text{AH} = \text{AM}+\text{MH} \quad \ldots \ldots \quad (3)$$

$$\text{BK} = \text{BN}-\text{NK} \quad \ldots \ldots \quad (4)$$

By multiplication of (3) by l_1 and (4) by l_2, we obtain the equations—

$$\text{AH} \times l_1 = (\text{AM} \times l_1) + (\text{MH} \times l_1) \quad \ldots \ldots \quad (5)$$

$$\text{BK} \times l_2 = (\text{BN} \times l_2) - (\text{NK} \times l_2) \quad \ldots \ldots \quad (6)$$

By similar figures—

$$\frac{\text{MH}}{\text{NK}} = \frac{\text{AC}}{\text{CB}}, \quad \text{whence NK} = \frac{\text{MH} \times \text{CB}}{\text{AC}} \quad \ldots \ldots \quad (7)$$

Add equations (5) and (6), then—

$$(\text{AH} \times l_1)+(\text{BK} \times l_2) = (\text{AM} \times l_1)+(\text{MH} \times l_1)+(\text{BN} \times l_2)-(\text{NK} \times l_2)$$

and by substitution for NK its value found in equation (7)—

$$(\text{AH} \times l_1)+(\text{BK} \times l_2) = (\text{AM} \times l_1)+(\text{MH} \times l_1)+(\text{BN} \times l_2) - \left(\frac{\text{MH} \times \text{CB}}{\text{AC}} \times l_2\right)$$

$$= (\text{AM} \times l_1)+(\text{BN} \times l_2)+\text{MH}\left(l_1 - \frac{\text{CB}}{\text{AC}} \times l_2\right)$$

i e., by substitution from (1) and (2)—

$$c = c+\text{MH}\left(l_1-\frac{\text{CB}}{\text{AC}} \times l_2\right)$$

Hence $\text{MH}\left(l_1-\frac{\text{CB}}{\text{AC}} \times l_2\right)$ must equal zero, so that either—

$$\text{MH} = 0 \quad \text{or} \quad l_1-\frac{\text{CB}}{\text{AC}} \times l_2 = 0.$$

Accordingly, since MH is not zero—

$$l_1 = \frac{CB}{AC} \times l_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Let the lengths AB, AC and CB be represented by m, m_1 and m_2 respectively, then equation (8) may be written $l_1 = \frac{m_2 l_2}{m_1}$

Also— $$l_1 + l_2 = \frac{m_2 l_2}{m_1} + l_2 = \frac{(m_2 + m_1)}{m_1} l_2 = \frac{m l_2}{m_1}$$

whence $\frac{m_1}{m} = \frac{l_2}{l_1 + l_2}$ and by similar reasoning $\frac{m_2}{m} = \frac{l_1}{l_1 + l_2}$

Any pairs of values of u and v to suit the equation $u + v = c$ might have been chosen, and the same argument might have been applied, so that as long as the scales for the u and v axes and the constant c remain the same, the ratio $\frac{m_1}{m_2}$ will hold, *i. e.*, there can only be the one mid-vertical. Also G will be a fixed spot, since it is vertically over C, and any one crossline satisfying the equation $u + v = c$ will give the position of G. The length of GC is thus fixed. Let it represent the constant c to some scale, say the scale of l_3 units per inch. A relation between l_3, l_1 and l_2 can now be found.

GC is an actual length, representing c to the scale of l_3 units per inch—

$$\therefore \qquad GC \times l_3 = c.$$

Substituting in (1) and (2)—

$$(l_1 \times AH) + (l_2 \times BK) = l_3 \times GC$$
and
$$(l_1 \times AM) + (l_2 \times BN) = l_3 \times GC.$$

Calculate the value of v when $u = 0$, and plot BL to represent this value; join AL, then this line passes through G, by the argument already given.

When $u = 0$, $v = c$, so that BL actually represents c,

$$\text{or} \qquad BL \times l_2 = c.$$
$$\text{But—} \qquad GC \times l_3 \text{ also} = c$$
$$\therefore \qquad BL \times l_2 = GC \times l_3.$$

By similar triangles—
$$= \frac{AC}{AB} \times BL \times l_3$$
$$= \frac{m_1}{m} \times BL \times l_3$$

$$\text{or} \qquad l_2 = \frac{m_1}{m} \times l_3.$$

Now—
$$\frac{m_1}{m} = \frac{l_2}{l_1+l_2}$$
$$\therefore \qquad l_2 = \frac{l_2}{l_1+l_2} \times l_3$$
or
$$l_3 = l_1+l_2$$

i. e., the scale along the mid-vertical is the sum of the scales along the outside axes.

The student of mechanics may be helped by the analogy of the case of parallel forces. If weights of W_1 and W_2 are hung at the ends of a bar of length l, their resultant W_3 is the sum of the separate weights, and acts at a point which divides the length into two parts in the inverse proportions of the weights. Thus, in Fig. 249, if C is the point of action of the resultant W_3—

$$\frac{AC}{CB} = \frac{W_2}{W_1}$$

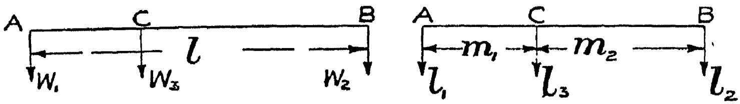


Fig. 249. Fig 249*a*.

This is exactly the same kind of thing as we have in connection with the scales along the three axes, for we may replace W_1, W_2 and W_3 by l_1, l_2 and l_3 respectively, and we get the bar loaded as in Fig. 249*a*.

We can now proceed to the more general case, viz., that in which the equation is $au+bv = c$.

Use may be made of the same diagram (Fig. 248) as that used for the simpler equation, viz., $u+v = c$. To do this, however, the scale of u must be opened out "a" times, and that of v opened "b" times; the distance BL, which formerly represented c, now representing $\frac{c}{b}$, since it shows the value of v when u is zero.

Accordingly, if l'_1 and l'_2 are the new scales along AE and BF—

$$l'_1 = \frac{l_1}{a} \quad \text{and} \quad l'_2 = \frac{l_2}{b}$$

Hence,

$$\text{the scale along GC} = l_1+l_2$$
$$= al'_1+bl'_2$$

and
$$\frac{m_1}{m_2} = \frac{l_2}{l_1} = \frac{bl'_2}{al'_1}$$

l'_1 and l'_2 would be the actual scales used. For a general statement, therefore, we can regard these as l_1 and l_2 and the scale along GC as l_3; so we sum up our results in the forms—

$$l_3 = al_1 + bl_2$$
$$\frac{m_1}{m_2} = \frac{bl_2}{al_1}$$

where l_1, l_2 and l_3 are the *actual* scales used.

These results might also be summarised in the following way :—

If the general equation is $au + bv = c$, then the scale of c (along

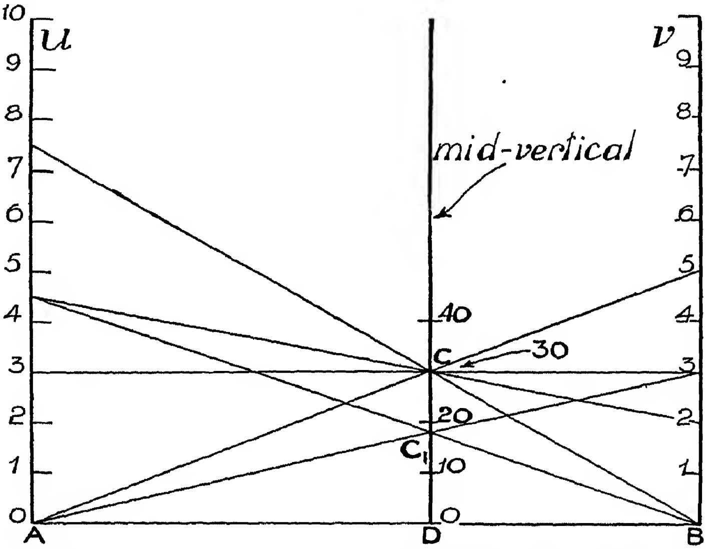


Fig. 250.—Alignment Chart for Equation $4u + 6v = 30$.

the mid-vertical) = "a" times the scale of u + "b" times the scale of v, and the division of AB at C is such that—

$$\frac{\text{CB}}{\text{AC}} = \frac{a \text{ times the } u \text{ scale}}{b \text{ times the } v \text{ scale}}$$

To illustrate by some numerical examples :—

Let us first deal with the equation $4u+6v = 30$.

To construct a chart for this equation, draw two vertical lines, as in Fig. 250, fairly well apart, say 6″ (this distance being simply a matter decided by the size of the paper and the degree of accuracy desired). Number from the same horizontal line scales for u and v, and let the two vertical scales be equal in value, viz.—

$$l_1 = l_2 = 2 \text{ (units per inch)}.$$

Then the mid-vertical must be so placed that $\frac{m_1}{m_2} = \frac{bl_2}{al_1}$

or $$\frac{m_1}{m_2} = \frac{6\times 2}{4\times 2} = \frac{3}{2}$$

i. e., $$m_1 = \frac{3}{5}\times m = \frac{3}{5}\times 6'' = 3{\cdot}6''$$

or the mid-vertical is $3{\cdot}6''$ distant from the axis of u.

Also the scale along the mid-vertical is fixed, since l_3 is given by al_1+bl_2, i. e., $l_3 = (4\times 2)+(6\times 2) = 20$, or $1''$ represents 20 units. If AB is the horizontal on which the zero of the scale of u and also that of the scale of v lies, number from the point D the scale along the mid-vertical, and indicate the marking for the constant term in the equation, viz., 30.

If $u = 7{\cdot}5$, $v = 0$, and it will be noticed that if a line is drawn from $7{\cdot}5$ on the scale of u through the point C (30 on the mid-vertical), it intersects the axis of v at the point B, *i. e.*, at the point for which $v = 0$. Similarly, if $v = 5$, then $u = 0$, and the line joining 5 on the axis of v to 0 on the axis of u passes through the point C.

Hence if a value of u, say, is given, the value of v to satisfy the equation $4u+6v = 30$ can be readily obtained by drawing a straight line through that given value of u and the point C, and noting its intersection with the axis of v:—*e. g.*, to find the value of v when $u = 3$: join 3 on the axis of u to C and produce to cut the axis of v; read off this value of v, viz., 3, and this is the solution required.

As an illustration of the fact that the alteration in the value of c alone alters the position of the point C on the mid-vertical and not the position of the mid-vertical, let us deal with the equation $4u+6v = 18$. Working with the same scales, join $4{\cdot}5$ on the axis of u to 0 on the axis of v, since if $u = 4{\cdot}5$, $v = 0$. This line passes through the point C_1 numbered 18 on the mid-vertical. To find the value of v when $u = 0$, join 0 on the axis of u to 18 on the mid-vertical and produce to cut the axis of v in the point 3; then the required value of v is 3.

Example 9.—Construct a chart to read values of t in the formula $t = {\cdot}7d + {\cdot}005D$, where t = thickness at edge of a pulley rim, d = thickness of belt, and D = diameter of pulley, all in inches. d is to range from $\cdot1''$ to $\cdot5''$ and D from $3''$ to $10''$.

Construction of the chart (see Fig. 251).—Draw two verticals, say $5''$ apart (as in the original drawing for Fig. 251). Let values of d be

set out on the left-hand vertical. The range of d being $\cdot 4''$, let $4''$ represent this value, so that $1'' = \cdot 1$ unit or $l_1 = \cdot 1$. The range of D is $7''$, so let $3\frac{1}{2}''$ represent this, so that $1'' = 2$ units or $l_2 = 2$.

Also— $a = \cdot 7$ and $b = \cdot 005$, so that

$$\begin{aligned} l_3 &= al_1 + bl_2 \\ &= (\cdot 7 \times \cdot 1) + (\cdot 005 \times 2) \\ &= \cdot 07 + \cdot 01 = \cdot 08 \end{aligned}$$

i. e., $1'' = \cdot 08$ unit of t, along the mid-vertical.

To fix the position of the mid-vertical—

$$\frac{m_2}{m_1} = \frac{al_1}{bl_2} = \frac{\cdot 7 \times \cdot 1}{\cdot 005 \times 2} = \frac{7}{1}$$

so that the mid-vertical is $\frac{1}{8} \times 5''$, *i. e.*, $\cdot 625$ from the axis of d.

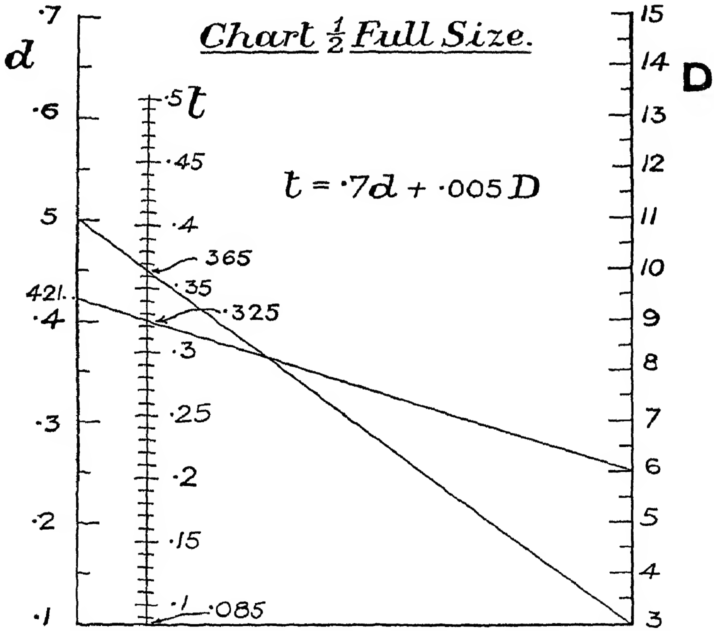


Fig. 251.—Alignment Chart giving Thickness at Edge of Pulley Rim.

The zero on the t scale will lie on the line joining the zero on the axis of d to that on the axis of D. We are not, however, bound to enlarge the diagram to allow this line to be shown; in fact, in a great number of cases the line of zeros or virtual zeros is quite outside the range of the diagram. As a matter of convenience let $\cdot 1$ on the d scale and 3 on the D scale be on the same horizontal; then, since $t = \cdot 085$ when $d = \cdot 1$ and $D = 3$, this horizontal will cut the mid-vertical at the point to be numbered $\cdot 085$. The scales along the three axes can now be set out, and the chart is complete.

Use of the chart.—To find the value of t when $d = \cdot 5$ and $D = 3$, join $\cdot 5$ on the d scale to 3 on the D scale to intersect the mid-vertical in the point $\cdot 365$; then the required value of t is $\cdot 365$. Again, if

$D = 6$ and $t = \cdot 325$, the value of d is found by joining 6 on the axis of D to ·325 on the axis of t and producing the line to cut the axis of d in ·421; the required value of d thus being ·421.

To carry this work a step further:—Most of the formulæ encountered in practice contain products, many in addition containing powers and roots. By taking logs, the multiplications are converted to additions, and the methods of chart construction already detailed can be applied with slight modifications.

To deal with a simple case, by way of introduction:—

Chart giving Horse-power supplied to Electric Motor.

Example 10.—Construct a chart to give the horse-power supplied to an electric motor, the amperage ranging from 2 to 12 and the voltage from 110 to 240. $\left(\text{Watts} = \text{Amps} \times \text{Volts and H P.} = \frac{\text{Watts}}{746}\right)$

Taking initials to represent the quantities—

$$W = AV \text{ and } H = \frac{AV}{746}$$

$$\text{or} \qquad 746H = AV.$$

Taking logs throughout—

$$\log 746 + \log H = \log A + \log V.$$

Let $\log 746 + \log H = C$, then if for $\log A$ we write $\overline{\underline{A}}$ and for $\log V$ we write $\overline{\underline{V}}$, the equation becomes $\overline{\underline{A}} + \overline{\underline{V}} = C$, which is of exactly the same form as $au + bv = c$, where $a = b = 1$.

Hence— $$l_3 = al_1 + bl_2 = l_1 + l_2$$

and $$\frac{m_2}{m_1} = \frac{al_1}{bl_2} = \frac{l_1}{l_2}$$

In order that the scale along the mid-vertical may be the sum of the scales along the outside axes, the mid-vertical must be so placed that it divides the distance between the outside axes in the inverse proportion of the scales thereon. By the scales, it must now be clearly understood that 1″ represents so many units of logarithms and not units of the actual quantities.

Slide rule scales will often be found convenient for small diagrams. If the B scale is used, 9·86″ (the length from index to index) would represent 2 units (*i. e.*, log 100 − log 1), whilst if the C scale is used, 9·86″ would represent 1 unit.

If a log scale is not used, the best plan is to tabulate the numbers, their logarithms, and corresponding lengths, before indicating the scales on the diagram. One setting of the slide rule will then serve for the conversion of the logs to distances, according to the scales chosen.

In this case A varies from 2 to 12, *i. e.*, log A varies from ·301 to 1·0792, a range of about ·8 units; and a fairly open scale will result if $1'' = \frac{1}{5}$ unit is chosen, *i. e.*, $l_1 = \cdot 2$.

For V the range is 110 to 240, so that the range in the logs is 2·0414 to 2·3802, or about ·35 unit; and accordingly let $l_2 = \cdot 1$.

Then— $$l_3 = l_1 + l_2 = \cdot 2 + \cdot 1 = \cdot 3$$

and $$\frac{m_2}{m_1} = \frac{l_1}{l_2} = \frac{\cdot 2}{\cdot 1} = \frac{2}{1}$$

In the original drawing (Fig. 252) m, the distance between the outside axes, was taken as 6″; hence $m_1 = \frac{1}{2+1}$ of 6″, *i. e.*, 2″, or the mid-vertical must be placed 2″ from the axis of A.

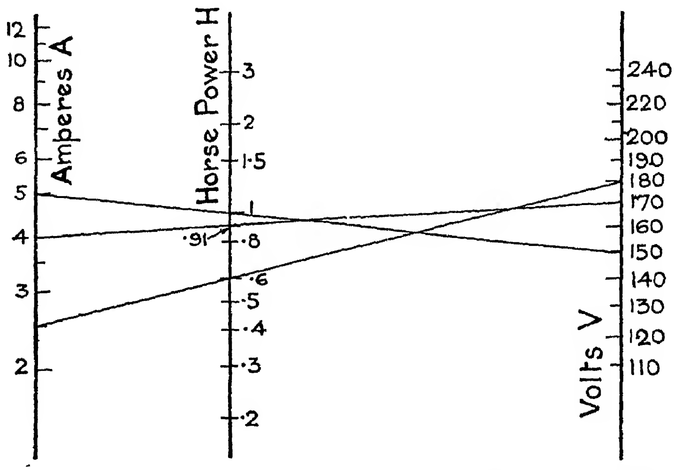


Fig. 252.—Chart giving H.P. supplied to Electric Motors.

Preliminary tabulation for the graduation of the outside axes reads :—

For the A *axis.*

A	2	2·5	3	3·5	4	5
log A	·301	·3979	·4771	·5441	·6021	·6990
Diff. of logs	0	·097	·176	·243	·301	·398
Actual distance from base line (ins.) . .	0	·485	·88	1·22	1·51	1·99

6	7	8	9	10	11	12
·7782	·8451	·9031	·9542	1·0	1·0414	1·0792
·477	·544	·602	·653	·699	·740	·778
2·39	2·72	3·01	3·27	3·5	3·7	3·89

The marking for 2 is first fixed, and then all distances are measured from that: thus to find the position of the point to be marked 4, log 4—log 2 = ·301, and since 1″ = ·2 units the actual distance from 2 to 4 must be $\frac{\cdot 301}{\cdot 2}$, viz., 1·51″.

For the V *axis*

V	110	120	130	140	150	160
log V	2·0414	2·0792	2·1139	2·1461	2·1761	2·2041
Diff. of logs	0	·0378	·0725	·1047	·1347	·1627
Actual distance from base line (ins.) . .	0	·378	·725	1·047	1·347	1·627

170	180	190	200	210	220	230	240
2·2304	2·2553	2·2788	2·3010	2·3222	2·3424	2·3617	2·3802
·1890	·2139	·2374	·2596	·2808	·3010	·3203	·3388
1·890	2·139	2·374	2·596	2·808	3·01	3·203	3·388

The fourth line in the latter table is obtained by division of the third line by ·1, since $l_2 = \cdot 1$.

The scales can now be indicated along their respective axes, and the mid-vertical may be drawn. It is not convenient in this particular example to join the zero of each of the outside scales, which would necessitate the axes being extended to show 1 on the A scale and 1 on the V scale, since log 1 = 0. If such a line were drawn, however, it would be the line on which the *virtual* zero of the scale of H would lie. Then the virtual zero would be $\frac{1}{746}$ since when $A = V = 1$, $H = \frac{1 \times 1}{746}$. It is, therefore, the best plan to locate some convenient point on the mid-vertical to serve as a zero. Thus, join 5 on the A scale to 149·2 on the V scale; and mark the point of intersection of this line with the mid-vertical as 1, since

$$H = \frac{5 \times 149\cdot 2}{746} = 1.$$

For other graduations, tabulate thus :—

H	1	1·5	2	3	·8	·6	·5	·4	·3	·2
log H	0	·1761	·301	·4771	$\bar{1}$·9031	$\bar{1}$·7782	$\bar{1}$·699	$\bar{1}$·602	$\bar{1}$·477	$\bar{1}$·301
Diff. of logs	0	·1761	·301	·4771	−·0969	−·2218	−·301	−·398	−·523	−·699
Actual distance from 1·0 (ins.)	0	·586	1·0	1·59	−·32	−·738	−1	−1·32	−1·74	−2·33

The fourth line is obtained from the third by division by ·3, since $l_s = \cdot 3$. Marking in these numbers along the H axis, the chart is complete.

Use of the chart.—To find the H.P. supplied if the current is 4 amps and the pressure is 170 volts. Join 4 on the A scale to 170 on the axis of V: this line passes through ·91 on the H axis, and therefore the required value of H is ·91.

Again, if H = ·6 and current = 2·5, what is the voltage? Join 2·5 on axis of A to ·6 on the H axis, and produce the line to cut the V axis in 179; therefore V = 179.

{It should be noted that the chart is not crowded with figures, because clearness is desired. Charts to be used frequently, and from which great accuracy is desired, should be drawn to a much larger scale.}

At a first reading one may be tempted to comment on the length of calculation necessary to perform what is, after all, a very simple operation: it must be borne in mind, however, that (*a*) a most simple example has been chosen as an illustration, and (*b*) a chart once constructed by this method may be used very many times in a perfectly mechanical way.

So many formulæ contain powers, that we must now investigate the effect of the exponents on the scales, etc., of these charts, and the modification in the construction due to them.

Flow of Water through Circular Pipes.

Example 11.—If water is flowing through a pipe of diameter d inches, at the rate of v ft. per sec., then the quantity Q in lbs. per sec. is obtained from—

$$Q = \frac{62\cdot 4}{144} \times \frac{\pi}{4} d^2 v = \cdot 34 d^2 v$$

Transposing— $\frac{Q}{\cdot 34} = d^2v.$

In the log form— $\log Q - \log \cdot 34 = \log d^2 + \log v$

i. e., $\log Q - \log \cdot 34 = 2 \log d + \log v$

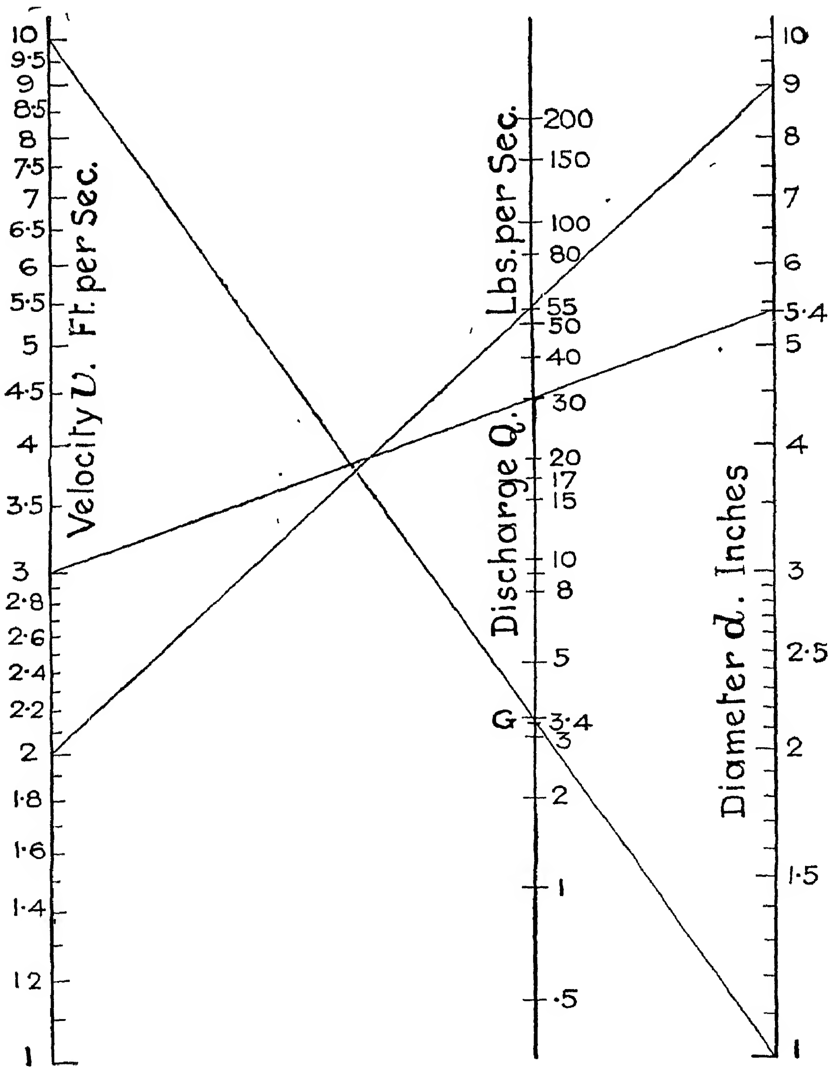


Fig. 253.—Chart giving the Flow of Water through Circular Pipes.

Let— $C = \log Q - \log \cdot 34$, then $\log v + 2 \log d = C$

or $V + 2D = C$

i. e., in comparison with the standard form, $a = 1$ and $b = 2$.

Assume the range of pipe diameters to be 1" to 9", and the range of the velocity of flow to be 1 to 10 ft. per sec. Then the same scale will be convenient for both axes. Let $l_1 = l_2 = \frac{1}{4 \cdot 93}$, *i. e.*, use the B scale of the slide rule.

Now—
$$\frac{m_1}{m_2} = \frac{bl_2}{al_1} = \frac{2 \times \frac{1}{4 \cdot 93}}{1 \times \frac{1}{4 \cdot 93}} = \frac{2}{1}$$

so that if m is taken as 6" (as in the original drawing for Fig. 253), $m_1 = \frac{2}{3} \times 6''$, *i. e.*, 4", or the mid-vertical is 2" removed from the axis of v.

Also—
$$l_3 = al_1 + bl_2 = \left(1 \times \frac{1}{4 \cdot 93}\right) + \left(2 \times \frac{1}{4 \cdot 93}\right) = \frac{3}{4 \cdot 93} = \cdot 607,$$

or 1" = ·607 unit along the axis of Q.

Draw the axes of v, Q and d and graduate the outside ones, using the B scale of the slide rule. In Fig. 253 the 1 of each scale is on a horizontal, but it is quite immaterial where the graduations begin.

To select a starting-point on the mid-vertical, join 10 on the axis of v to 1 on the axis of d, and call the point of intersection with the mid-vertical G.

Now, $Q = \cdot 34 d^2 v$, and therefore for the particular values of d and v chosen, $Q = \cdot 34 \times 1^2 \times 10 = 3 \cdot 4$.

G is therefore at the position to represent 3·4 lbs. per sec.

The table for the graduation of the mid-vertical will then be :—

Q	3·4	3	2	5	8	10	15	20	30	40	50	80	100
log Q	·532	·477	·301	·699	·903	1	1·176	1·301	1·477	1·602	1·699	1·903	2
Diff. of logs	0	−·055	−·231	·167	·371	·468	·644	·769	·945	1·07	1·167	1·371	1·468
Distance above or below G	0	−·09	−·38	·27	·61	·77	1·06	1·26	1·55	1·75	1·92	2·25	2·41

The fourth line is obtained by multiplying the third by 1·64 or by dividing it by ·607, since $l_3 = \cdot 607$.

Use of the chart.—Find the discharge through a pipe of 9" diam. when the flow is at the rate of 2 ft. per sec. Join 2 on the axis of v to 9 on the axis of d, to intersect the axis of Q at 55; then the required quantity is 55 lbs. per sec.

Again, what diameter of pipe is required if the discharge is 30 lbs./sec.

and the rate of flow is 3 ft./sec.? Join 3 on the v scale to 30 on the Q scale and produce the line to cut the axis of d in 5·4; then the required diameter is 5·4″.

To illustrate the question of scales further, consider the following cases :—

Example 12.—Show how to decide upon the scales for the chart giving the values of T, f and d in the equation $T = \frac{\pi}{16} f d^3$, referring to the torsion of shafts.

The equation may be written $\frac{16T}{\pi} = fd^3$, *i. e.*, $5{\cdot}1T = fd^3$, and by taking logs throughout—

$$\log 5{\cdot}1 + \log T = \log f + 3 \log d.$$

Write this $\log f + 3 \log d = C$, then $F + 3D = C$ (the large letters being written to represent logs). Thus $a = 1$ and $b = 3$.

Hence—

$$\text{if } l_1 = 5, \text{ say, and } l_2 = 2,\ l_3 = al_1 + bl_2 = (1 \times 5) + (3 \times 2) = 11$$

$$\text{and} \qquad \frac{m_1}{m_2} = \frac{bl_2}{al_1} = \frac{3 \times 2}{1 \times 5} = \frac{6}{5} \quad \text{or} \quad m_1 = \frac{6}{11} \text{ of } m.$$

Similarly for $pv^n = C$, where n may have values such as ·9, 1·37, 1·41, etc.

Here— $\log p + n \log v = \log C$

i. e., $P + nV \quad = \overline{C}$

so that $a = 1,\ b = n$.

Hence— $l_3 = (1 \times l_1) + (n \times l_2) = l_1 + nl_2$

and $\frac{m_1}{m_2} = \frac{bl_2}{al_1} = \frac{nl_2}{l_1}$

Questions involving more complicated formulæ can be dealt with by a combination of charts. From the above work it will be seen that when three axes are employed, three variables may be correlated, or one axis is required for each variable. However many variables occur, they may be connected together in threes, so that the graph work is merely an extension of that with the three axes.

Chart giving Number of Teeth in Cast-iron Gearing.

Example 13.—To construct a chart giving the number of teeth necessary for strength in ordinary cast-iron gearing.

Given that— $$T = \frac{791H}{Np^3}$$

where T = No. of teeth in wheel, N = revs. per min.
H = H.P. transmitted, p = pitch.

$$Tp^3 = 791\frac{H}{N}$$

so that $\qquad Tp^3 = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

and also $\qquad \dfrac{791H}{N} = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$

i.e., two charts can be constructed, and by suitably choosing the scales and the positions of the axes the charts may be made interdependent.

For chart (1), let $l_1 = \dfrac{1}{4\cdot93}$ unit of T and let $l_2 = \dfrac{1}{4\cdot93}$ unit of p, *i. e.*, use the B scale of the slide rule for both the T and the p axes.

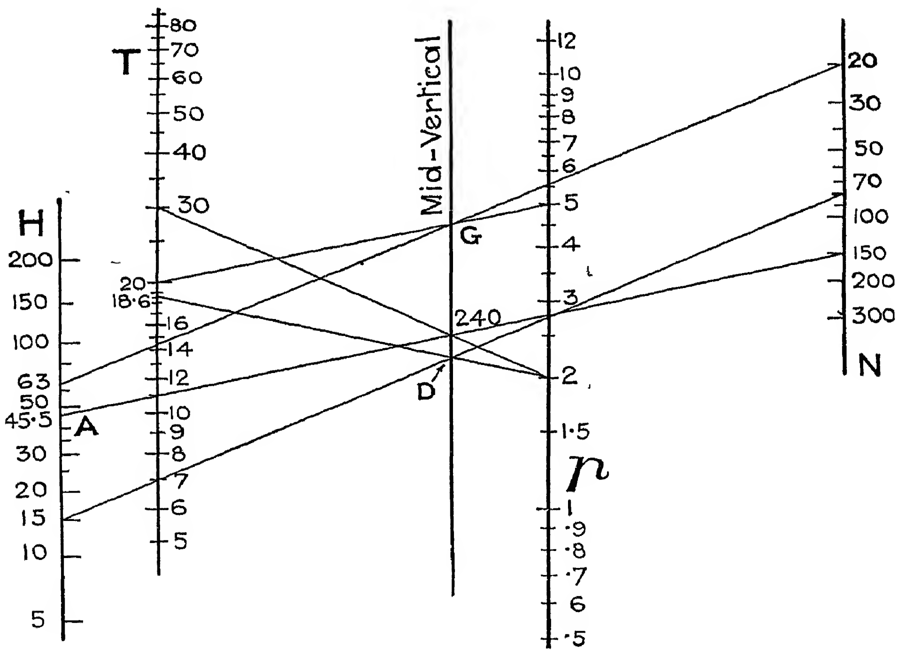


Fig. 254.—Chart giving Number of Teeth necessary in Cast-iron Gearing.

Then, since $\quad \log T + 3 \log p = \log C, \quad l_3 = \left(1 \times \dfrac{1}{4\cdot93}\right) + \left(3 \times \dfrac{1}{4\cdot93}\right)$

$$= \frac{4}{4\cdot93} = \cdot811$$

Also— $\qquad \dfrac{m_1}{m_2} = \dfrac{3 \times \dfrac{1}{4\cdot93}}{1 \times \dfrac{1}{4\cdot93}} = \dfrac{3}{1}, \quad$ so that $m_1 = \dfrac{3}{4}$ of m.

Draw two axes for T and p respectively 4″ apart, as drawn in the original drawing for Fig. 254, and also put the mid-vertical 3″ from the axis of T. The last is simply a connecting-link between charts (1) and (2), and therefore no graduations need be shown upon it.

Along the axis of T mark off readings (using the B scale of the slide rule) for, say, 6 up to 80 and along the axis of p, readings from 1 to 8. Join 2 on the p axis to 30 on the axis of T, and note the point of intersection with the mid-vertical; this must be marked 240, since $30 \times 2^3 = 240$.

For chart (2), we already have the mid-vertical and its scale. We must now proceed to find scales for the axes of H and N, *i. e.*, the usual process is reversed.

Suppose the range of H is 5 to 100 and that of N is 20 to 150, and we decide to use the same scale for both, say l_4.

Then— $\frac{H}{N} = \frac{C}{791}$ or $\log H - \log N = \log C - \log 791$

i. e., $a = 1$, whilst $b = -1$. If, however, the numbering for N is placed in the opposite direction to that for H, we may say that $b = 1$.

Hence— $l_3 = l_4 + l_4 = 2l_4$ or $l_4 = \frac{l_3}{2} = \frac{\cdot 811}{2} = \cdot 406$

also— $\frac{m_1}{m_2} = \frac{l_2}{l_1} = \frac{l_4}{l_4} = 1$

i. e., the mid-vertical, which has already been drawn, must be midway between the axes of H and N. For convenience let $m_1 = m_2 = 4''$.

Then for N the tabulation is as follows :—

N	20	30	40	50	60	70	80	90	100	150	200
log N	1·301	1·477	1·602	1·699	1·778	1·845	1 903	1·954	2	2·176	2·301
Diff. of logs	0	·176	·301	·398	·477	·544	·602	·653	·699	·875	1
Distance from mark for 20	0	·432	·74	·98	1·17	1·34	1·48	1·61	1·72	2·16	2 46

To obtain the fourth line from the third divide by ·406, for $l_4 = \cdot 406$.

Some little trouble may arise in the placing of the marking for 20 conveniently: thus in our case we have marked 20 fairly high up on the paper.

Join any point on the N axis, say 150, to the 240 on the mid-vertical, and produce this line to cut the axis of H in the point A. We must now find the reading for A.

$$C = 240 \quad \text{and} \quad C = \frac{791H}{N}, \quad \text{but } N = 150$$

$$\text{hence} \qquad H = \frac{150 \times 240}{791} = 45 \cdot 5.$$

Thus we can graduate the axis of H from 45·5 as zero.

The table for the graduation of the H scale is :—

H	45·5	5	10	15	20	25
log H	1·658	·699	1	1·176	1·301	1·398
Diff. of logs. . .	0	—·959	—·658	—·482	—·357	—·260
Distance from A	0	—2·36	—1·62	—1·19	—·88	—·64

30	35	40	50	60	80	100
1·477	1·544	1·602	1·699	1·778	1·903	2
—·181	—·114	—·056	·041	·120	·245	·342
—·445	—·28	—·14	·1	·295	·602	·84

l_4 = ·406, so that the fourth line is obtained by dividing the figures in the third line by ·406.

Use of the chart.—Suppose N = 20, T = 20, p = 5, and the value of H is to be found. Join 20 on the T axis to 5 on the axis of p; and let this line intersect the mid-vertical at G. Join 20 on the N axis to G, and produce the line to cut the axis of H in 63. Then the required value of H is 63.

Again, if H = 15, N = 80, and p = 2, we are to find the value of T. Join H = 15 to N = 80 to cut the mid-vertical at D. Join p = 2 to D, and produce to cut the axis of T in 18·6 : then the required value of T is 18·6.

The lines must join values of either T and p or H and N, because the chart was so constructed.

Exercises 43.—On Alignment Charts.

1. Construct a chart giving values of u and v to satisfy the equation $2u+7v = 52$, the range of v being 2 to 12.

2. Construct a chart to give values of u and v to satisfy the equation $1{\cdot}2v-{\cdot}64u = {\cdot}85$, u ranging from 5 to 20.

(To allow for the minus sign, either the mid-vertical may be placed *outside* the axes of u and v, as for *unlike* parallel forces, or the numbering on the u scale may be downward, whilst that on the v scale is upward.)

3. The thickness of boiler shell necessary if the working pressure is p lbs. per sq. in., the diameter of the boiler is d inches, and the allowable stress is f lbs. per sq. in., is found from $t = \frac{pd}{2f}$. Taking the value of f as 10000, construct a chart to give values of t, the range of diameter being 1′-6″ to 6 ft., and the pressure varying from 40 to 150 lbs. per sq. in. What is the thickness when the diameter is 2′-3″ and the working pressure is 85 lbs. per sq. in.? If the thickness is $\frac{1}{4}$″ and the diameter is 4′-6″, what is the working pressure?

4. According to the B.O.T. rule the permissible working pressure in a boiler having a Fox's corrugated steel furnace is $P = \frac{875t}{D}$, where t = thickness of plate in sixteenths of an inch and D is the internal diameter in inches. Construct a chart to give values of P for boilers of diameters ranging from 2 ft. to 5 ft., the thickness of the shell varying between $\frac{1}{32}''$ and $\frac{3}{4}''$.

5. The diameter in inches for a round shaft to transmit horse-power H at N revs. per min. (for a steel shaft) is given by $d = 2{\cdot}9\sqrt[3]{\frac{H}{N}}$. If N varies from 15 to 170 and H from $\frac{1}{4}$ to 10, construct a chart to show all the diameters necessary within this range.

6. For tinned copper wire the fusing current C is found from $C = 6537d^{1{\cdot}403}$, where d is the diameter in inches. Construct a chart to read the diameter of wire necessary if the fusing current is between 22 and 87 amperes.

7. Hodgkinson's rule for the breaking load for struts is—

$$P = \frac{Ad^{3{\cdot}6}}{L^{1{\cdot}7}}$$

where d = diameter in inches and L = length in feet, A being a constant. Construct a chart to give the breaking load for cast-iron struts with rounded ends, the diameters ranging from 2″ to 15″ and the lengths from 6 ft. to 20 ft. The value of A for solid cast-iron pillars with rounded ends is 14·9.

8. Construct a chart to give the points on an adiabatic expansion line of which the equation is $pv^{1{\cdot}37} = 560$, the range of pressure being 14·7 lbs. per sq. in. to 160 lbs. per sq. in.

9. The coefficient of friction between a certain belt and pulley was ·32. If the angle of lap varies from 30° to 180°, construct a chart to give the tensions at the ends of the belt, the smaller tension varying from 50 lbs. wt. to 100 lbs. wt. Given that $T = te^{\mu\theta}$, μ being the coefficient of friction, and θ being the angle of lap in radians. [Note that the scales for T and t will be log scales, but that for θ will be one of numbers only.]

10. If P = safe load in tons carried by a chain, d = diameter of stock, and f = safe tensile stress, then for a chain with open links

$$P = {\cdot}4d^2f.$$

If f varies between 4 and 10 tons per sq. in., and the diameter of the stock ranges from $\frac{1}{4}''$ to 2″, construct a chart to give the safe load for any combination of f and d.

CHAPTER XII

VARIOUS ALGEBRAIC PROCESSES, MOSTLY INTRODUCTORY TO PART II

Continued Fractions.—Consider the fraction—

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{2}{5+\frac{4}{7}}}}$$

or, as it would usually be written as a continued fraction—

$$\frac{1}{2+}\;\frac{1}{3+}\;\frac{2}{5+}\;\frac{4}{7}$$

Its true value would be found by simplification, working from the bottom upwards; thus, $5+\frac{4}{7}=\frac{39}{7}$

$$\frac{2}{\frac{39}{7}}=\frac{14}{39}; \qquad 3+\frac{14}{39}=\frac{131}{39}$$

$$\frac{1}{\frac{131}{39}}=\frac{39}{131}; \qquad 2+\frac{39}{131}=\frac{301}{131}$$

and

$$\frac{1}{\frac{301}{131}}=\frac{131}{301}$$

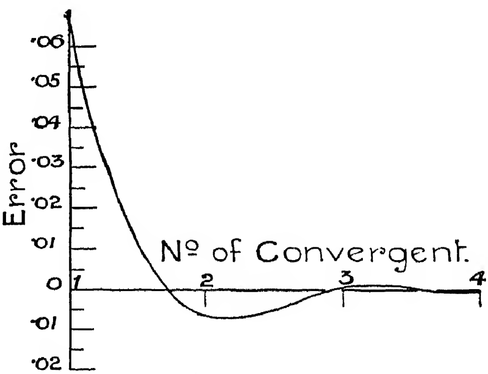


Fig. 255.

i. e., the true value of the fraction, known as a *continued fraction*, is $\frac{131}{301}$

Conversely, if the resulting fraction $\frac{131}{301}$ is given, various approximations can be made for it by taking any portion of the continued fraction, the correct order being maintained.

Thus, for example, a first approximation would be $\frac{1}{2}$, which is too large; the second approximation is $\frac{1}{2+\frac{1}{3}}=\frac{3}{7}$, which is too small, but is nearer the correct value.

The approximations, or *convergents*, are alternately too large and too small, but the error becomes less as more terms of the fraction are taken into account. To illustrate this fact a curve is plotted dealing with the above fraction, in which the ordinates are errors and the abscissæ the numbers of the convergents. (Fig. 255.)

Occasionally in engineering practice a fraction such as $\frac{131}{301}$ occurs which is not convenient to deal with practically, so that one seeks for some more convenient fraction which is a fair approximation to that given. The following example introduces such a case :—

Example 1.—A dividing head on a milling machine is required to be set for the angle $19°25'1''$ with great accuracy.

For the Brown & Sharpe dividing heads, 40 turns of the crank make one revolution of the spindle, and there are three index plates with number of holes as follows—

$$\left.\begin{matrix}15, 16, 17, 18, 19, 20\\ 21, 23, 27, 29, 31, 33\\ 37, 39, 41, 47, 49\end{matrix}\right\}$$

Thus one turn of the index crank would give an angle of $\frac{360}{40}$, *i.e.*, 9°. Evidently two turns will be required for 18°, and $1°25'1''$ has then to be dealt with. Expressing this as a fraction of 9°, the proportion of one turn is found.

Now— $$1°25'1'' = \frac{5101}{60} \text{ mins.}$$

hence the fraction of one turn required $= \frac{5101}{60 \times 9 \times 60} = \frac{5101}{32400}$

We wish to reduce this fraction to its best approximation having a denominator between 15 and 49, according to the numbers of holes as above.

Proceed as though finding the G.C.M. of 5101 and 32400.

Thus—

```
5101)32400(6
      1794)5101(2
           1513)1794(1
                 281)1513(5
                      108)281(2
                           65)108(1
                               43)65(1
                                  22, etc.
```

Then the continued fraction is--

$$\frac{1}{6+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{5+}\ \frac{1}{2+}\ \frac{1}{1+}\cdots$$

1st convergent $= \frac{1}{6}$, 2nd $= \frac{2}{13}$, 3rd $= \frac{3}{19}$, 4th $= \frac{17}{108}$, 5th $= \frac{37}{235}$; these being found by simplification of the fraction, a method which is a trifle laborious. The 3rd convergent might have been found from the 2nd in the following way—

Numerator of 3rd convergent = {numerator of 2nd convergent × denominator of last fraction added} + {numerator of 1st convergent × numerator of last fraction added}.

Denominator of 3rd convergent = {denominator of 2nd convergent × denominator of last fraction added} + {denominator of 1st convergent × numerator of last fraction added}.

In this case—

1st convergent $= \frac{1}{6}$, 2nd $= \frac{2}{13}$ and the next fraction $= \frac{1}{1}$

$$\therefore\quad \text{3rd convergent} = \frac{(2 \times 1) + (1 \times 1)}{(13 \times 1) + (6 \times 1)} = \frac{3}{19}$$

Now from the 2nd and 3rd convergents the 4th convergent may be found; for the 2nd convergent $= \frac{2}{13}$, the 3rd $= \frac{3}{19}$, and the next fraction $= \frac{1}{5}$.

$$\therefore\quad \text{the 4th convergent} = \frac{(3 \times 5) + (2 \times 1)}{(19 \times 5) + (13 \times 1)} = \frac{17}{108}$$

To obtain the 5th convergent: the 3rd convergent $= \frac{3}{19}$, the 4th $= \frac{17}{108}$, and the next fraction $= \frac{1}{2}$

$$\therefore\quad \text{the 5th convergent} = \frac{(17 \times 2) + (3 \times 1)}{(108 \times 2) + (19 \times 1)} = \frac{37}{235}$$

For the purpose of the question we require the convergent with denominator between 15 and 49: the only one is $\frac{3}{19}$. Therefore, it would be best to take two complete turns together with 3 holes on the 19-hole circle. The error in so doing is very small. Thus—

$$\frac{5101}{32400} = \cdot 15744, \quad \text{whilst } \frac{3}{19} = \cdot 15790$$

i.e., the error is 46 in 15744 or $\frac{46}{15744} \times 100\,\%$

$= \cdot 292\,\%$ **too large.**

Example 2.—Find a suitable setting of the dividing head to give $88°21'45''$.

$$\text{No. of turns of index crank} = \frac{88°21'45''}{9°} = \frac{21207}{2160} = 9\frac{1767}{2160}$$

$$= 9\frac{589}{720}$$

Hence 9 complete turns are necessary together with $\frac{589}{720}$ of a turn.

To find a convenient convergent for $\frac{589}{720}$:—

```
589)720(1
    131)589(4
         65)131(2
              1)65(65
```

$\therefore$ The fraction $= \frac{1}{1+}\ \frac{1}{4+}\ \frac{1}{2+}\ \frac{1}{65}$

The 1st convergent $= \frac{1}{1}$, the 2nd convergent $= \frac{4}{5}$

so that the 3rd convergent $= \dfrac{(4 \times 2) + (1 \times 1)}{(5 \times 2) + (1 \times 1)} = \dfrac{9}{11}$

also the 4th convergent $= \dfrac{(9 \times 65) + (4 \times 1)}{(11 \times 65) + (5 \times 1)} = \dfrac{589}{720}$

Thus the best convergent for our purpose $= \frac{9}{11}$, and 27 holes on the 33-hole circle would give this ratio.

Therefore, 9 complete turns together with 27 holes on the 33-hole circle are required.

An interesting example concerns the convergents of π.

Example 3.—To 5 places of decimals the value of π is $3{\cdot}14159$; what fractions may be taken to represent this?

$$3{\cdot}14159 = 3\frac{14159}{100000}$$

```
14159)100000(7
         887)14159(15
              5289
               854)887(1
                    33)854(25
                        194
                         29)33(1
                             4
```

$$i.e., \pi = 3 + \frac{1}{7+}\ \frac{1}{15+}\ \frac{1}{1+}\ \frac{1}{25+}\ \frac{1}{1+} \dots$$

The 1st convergent $= 3$, the 2nd convergent $= \frac{22}{7}$

$$\text{and hence the 3rd convergent} = \frac{(22 \times 15) + (3 \times 1)}{(7 \times 15) + (1 \times 1)} = \frac{333}{106}$$

$$\text{the 4th convergent} = \frac{(333 \times 1) + (22 \times 1)}{(106 \times 1) + (7 \times 1)} = \frac{355}{113}$$

$$\text{the 5th convergent} = \frac{(355 \times 25) + (333 \times 1)}{(113 \times 25) + (106 \times 1)} = \frac{9208}{2931}$$

The values of these convergents in decimals are—

3, 3·14286, 3·14151, 3·14159 +, and 3·14159 −, respectively.

A rule often given for a good setting of the slide rule for multiplication or division by π is :—Set 355 on the one scale level with 113 on the other, etc. The reason for this is seen from the above investigation; $\frac{355}{113}$ as a value for π being far more accurate than, say, $\frac{22}{7}$

Partial Fractions.—Consider the fractions—

$$\frac{2}{x-4}, \quad \frac{4}{2x-7}, \text{ and their sum.}$$

To find their sum, *i. e.*, to combine them to form one fraction, the L.C.D. is found, viz., $(x - 4)(2x - 7)$ or $2x^2 - 15x + 28$; the numerators are multiplied by the quotients of the respective denominators into the L.C.D., and the results are added to form the final numerator.

Thus—
$$\frac{2}{x-4} + \frac{4}{2x-7} = \frac{4x-14+4x-16}{2x^2-15x+28}$$
$$= \frac{8x-30}{2x^2-15x+28}$$

The fraction last written may be spoken of as the complete fraction, for which $\frac{2}{x-4}$ and $\frac{4}{2x-7}$ are the *partial* fractions.

It is often necessary to break up a fraction into its partial fractions: they are easier to handle, and operations may be performed on them that could not be performed on the complete fraction.

To resolve into partial fractions, proceed in the manner outlined in the following examples :—

Example 4.—Resolve $\frac{8x - 30}{2x^2 - 15x + 28}$ into partial fractions.

$$\frac{8x - 30}{2x^2 - 15x + 28} = \frac{8x - 30}{(2x - 7)(x - 4)} = \frac{A}{(x - 4)} + \frac{B}{(2x - 7)}$$

where A and B have values to be found.

Reduce to a common denominator, $(2x-7)(x-4)$, and calling this D—

$$\frac{8x-30}{D}=\frac{A(2x-7)+B(x-4)}{D}$$

Equating the numerators—

$$8x-30=A(2x-7)+B(x-4).$$

This relation must be true for all values of x: accordingly let $x=4$, this particular value being chosen so that the term containing B vanishes, and one unknown only remains.

Then— $$32-30=A(8-7)+B(4-4)$$
or $$2=A.$$

Now let the term containing A be made to vanish by writing $3\frac{1}{2}$ in place of x—

Then— $$28-30=A(7-7)+B(3\tfrac{1}{2}-4)$$
$$-2=-\tfrac{1}{2}B$$
$$B=4$$

$$\therefore \text{ the fraction }=\frac{2}{x-4}+\frac{4}{2x-7}$$

Example 5.—Express $\frac{5+4x}{3x-8}$ as a sum of two or more fractions.

The numerator and denominator are here both of the same degree; in such cases divide out until the numerator is of one degree lower than the denominator.

Now suppose— $\frac{A}{B}=C$ with D remainder

then the fraction $\frac{A}{B}=C+\frac{D}{B}$

Applying to our example, by actual division the quotient $=\frac{4}{3}$ and the remainder $=\frac{47}{3}$: hence the fraction $=\frac{4}{3}+\frac{47}{3(3x-8)}$

Example 6.—(*a*) Find the sum of—

$$\frac{4}{(2x+1)}-\frac{7x}{5(x+1)^2}-\frac{3}{(x+1)}$$

and (*b*) resolve $\frac{-24x^2-12x+5}{5(2x+1)(x+1)^2}$ into partial fractions.

(*a*) $$\frac{4}{2x+1}-\frac{7x}{5(x+1)^2}-\frac{3}{x+1}$$
$$=\frac{4\times 5(x+1)^2-7x(2x+1)-3\times 5(x+1)(2x+1)}{5(2x+1)(x+1)^2}$$
$$=\frac{20x^2+40x+20-14x^2-7x-30x^2-45x-15}{D}$$
$$=\frac{-24x^2-12x+5}{5(2x+1)(x+1)^2}$$

(*b*) To resolve $\frac{-24x^2-12x+5}{5(2x+1)(x+1)^2}$ into partial fractions, therefore, it is necessary to consider the possibility of the existence of $(x+1)$ as a denominator, in addition to $(x+1)^2$, for $(x+1)$ is included in $(x+1)^2$.

Let the fraction $= \frac{A}{5(2x+1)} + \frac{Bx}{(x+1)^2} + \frac{C}{(x+1)}$

[Bx is written in place of B, so that the numerator shall be of degree one less than the denominator, *i. e.*, all terms of the numerator, when over the same denominator, will then be of the same degree.]

Thus the fraction $= \frac{A(x+1)^2 + 5Bx(2x+1) + 5C(2x+1)(x+1)}{D}$

Equating numerators—

$$-24x^2 - 12x + 5 = A(x+1)^2 + 5Bx(2x+1) + 5C(2x+1)(x+1)$$

Let $x = -1$ {*i.e.*, terms containing $(x+1)$ are thus made to vanish}

$$\therefore \quad -24+12+5 = 0+5B(-1)(-1)+0$$
$$-7 = +5B$$
$$B = -\frac{7}{5}$$

Let— $\quad x = -\frac{1}{2} \quad \{i.\,e.,\ 2x+1 = 0\}$

$$\therefore \quad -6+6+5 = A\left(\frac{1}{2}\right)^2+0+0$$
$$\therefore \quad 5 = \frac{1}{4}A$$
$$A = 20.$$

The numerators must be *identically* equal, *i. e.*, term for term; therefore the coefficients of x^2 must be equated.

Thus—

$$-24 = A + 10B + 10C = 20 - 14 + 10C \quad \left\{\text{for } A = 20 \text{ and } B = -\frac{7}{5}\right\}$$
$$\therefore \quad 10C = -30$$
$$C = -3$$
$$\therefore \quad \text{the fraction} = \frac{20}{5(2x+1)} - \frac{7x}{5(x+1)^2} - \frac{3}{x+1}$$
$$= \underline{\frac{4}{(2x+1)} - \frac{7x}{5(x+1)^2} - \frac{3}{(x+1)}}$$

Example 7.—Resolve $\frac{9x-17}{(2x-3)(x^2+5x+9)}$ into partial fractions.

Let the fraction $= \frac{A}{(2x-3)} + \frac{Bx+C}{(x^2+5x+9)}$

$$= \frac{A(x^2+5x+9) + (Bx+C)(2x-3)}{(2x-3)(x^2+5x+9)}$$

Equating the numerators—

$$9x-17 = A(x^2+5x+9) + (Bx+C)(2x-3)$$

Let— $x = \frac{3}{2}$ *i. e.*, let $2x - 3 = 0$

Then— $$13\tfrac{1}{2} - 17 = A\left(\frac{9}{4} + \frac{15}{2} + 9\right) + 0$$

$$-3\tfrac{1}{2} = \frac{75}{4}A$$

$$\therefore \quad A = -\frac{14}{75}$$

Equating the coefficients of x^2, and as no terms on the L.H.S. contain x^2, its coefficient $= 0$,

$$0 = A + 2B = -\frac{14}{75} + 2B$$

$$\therefore \quad 2B = \frac{14}{75}$$

$$B = \frac{7}{75}$$

Equating the coefficients of x on the two sides of the equation—

$$9 = 5A - 3B + 2C = -\frac{14}{15} - \frac{7}{25} + 2C$$

$$\therefore \quad 2C = 9 + \frac{14}{15} + \frac{7}{25}$$

$$= \frac{766}{75}$$

$$\therefore \quad C = \frac{383}{75}$$

$$\therefore \quad \text{the fraction} = \frac{7x + 383}{75(x^2 + 5x + 9)} - \frac{14}{75(2x - 3)}$$

Limiting Values, or Limits.—Let it be required to find the value of the fraction $\dfrac{2x-2}{4x^2+x-5}$ when $x = 1$.

When $x = 1$, $\dfrac{2x-2}{4x^2+x-5} = \dfrac{0}{0}$ if x be replaced by 1.

We can give no definite value at all to $\dfrac{0}{0}$; it might indicate anything, and therefore we must find some other method for dealing with cases such as this.

Let us calculate the value of the fraction F when x is slightly less than 1, say when x has the value ·9 :—

Then— $$F = \frac{1{\cdot}8-2}{3{\cdot}24+{\cdot}9-5} = \frac{-{\cdot}2}{-{\cdot}86} = {\cdot}2326.$$

When x has a value nearer to 1, say ·95

$$F = \frac{1{\cdot}9-2}{3{\cdot}61+{\cdot}95-5} = \frac{-{\cdot}1}{-{\cdot}44} = {\cdot}2273.$$

Now let us take values of x slightly in excess of 1.

$$\text{When } x = 1\cdot05, \quad F = \frac{2\cdot1-2}{4\cdot41+1\cdot05-5} = \frac{\cdot1}{\cdot46} = \cdot2174.$$

$$\text{When } x = 1\cdot1, \quad F = \frac{2\cdot2-2}{4\cdot84+1\cdot1-5} = \frac{\cdot2}{\cdot94} = \cdot2127.$$

Therefore for values of x in the neighbourhood of 1 the fraction has perfectly definite values, and consequently it is unreasonable to suppose that there is no value of F for $x = 1$. If we plot a curve, as in Fig. 256, of F against x, we see from it, assuming that it is continuous (and there is nothing to negative this supposition) that the value of F when $x = 1$ is $\cdot2222$.

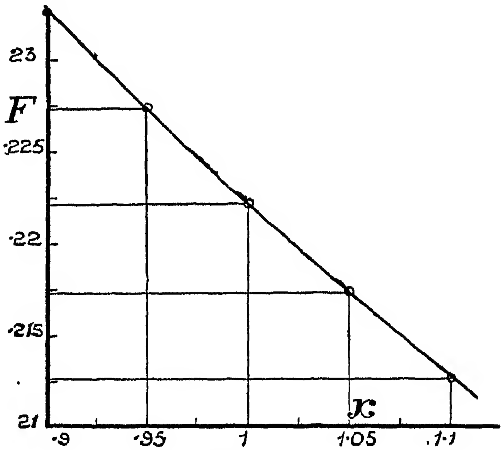


Fig. 256.

We say, then, that the limiting value of F when x approaches 1 is $\cdot2222$, or—

$$\underset{x\to1}{\mathrm{L}}\ \frac{2x-2}{4x^2+x-5} = \cdot2222.$$

To obtain this value without the aid of a graph we might take values of x closer and closer to 1 and see to what figure the value of F was tending—

e. g., when $x = \cdot99$, $F = \cdot2232$
when $x = \cdot995$, $F = \cdot2227$.

This method, besides being somewhat laborious, is not definite enough.

As an alternative method :—if x does not actually equal 1 but differs ever so slightly from it, $(x - 1)$ does not equal 0, and therefore we may divide numerator and denominator by it.

$$\text{Thus—} \qquad F = \frac{2(x-1)}{(x-1)(4x+5)} = \frac{2}{(4x+5)}$$

As x approaches more and more nearly to 1, this last fraction becomes more nearly $= \frac{2}{9}$ *and in the limit* when $x = 1$, $F = \frac{2}{9}$.

Later on we shall see that this method of obtaining a value or limit by "approaching" it is of great utility and importance.

Example 8.—Corresponding values of y and x are given in the table :—

x	3·9	3·94	3·97	4·02	4·05	4·1
y	30·42	31·04	31·52	32·32	32·80	33·62

Required the probable value of y when $x = 4$.

When x has values slightly under 4, those of y are increasing fairly uniformly; thus for an increase of x from 3·9 to 3·94 (*i. e.*, ·04) the increase of y is ·62, or the rate of increase is $\frac{\cdot 62}{\cdot 04}$, *i. e.*, 15·5, and whilst x increases a further ·03 unit, y increases ·48 unit, or the rate of change of y compared with x is $\frac{\cdot 48}{\cdot 03}$, *i. e.*, 16. Thus y is increasing at a rather greater rate as the value of x increases. This is confirmed by dealing with values of x greater than 3·97: we might tabulate the differences of x and of y thus :—

Change in x.	Change in y.	Rate of change of y.
3·97 to 4·02, *i. e.*, ·05	·80	$\frac{\cdot 80}{\cdot 05} = 16$
4·02 to 4·05, *i. e.*, ·03	·48	$\frac{\cdot 48}{\cdot 03} = 16$
4·05 to 4·1, *i. e.*, ·05	·82	$\frac{\cdot 82}{\cdot 05} = 16{\cdot}4$

Therefore, as nearly as we can estimate, when x has the value 4, y has a value very slightly over $\frac{\cdot 03}{\cdot 05}$ of ·80, *i. e.*, slightly more than ·48 above its value when $x = 3{\cdot}97$. Hence the value of y when $x = 4$ is most probably $31{\cdot}52 + {\cdot}48$, *i. e.*, 32.

This result is further illustrated by the graph (see Fig. 257).

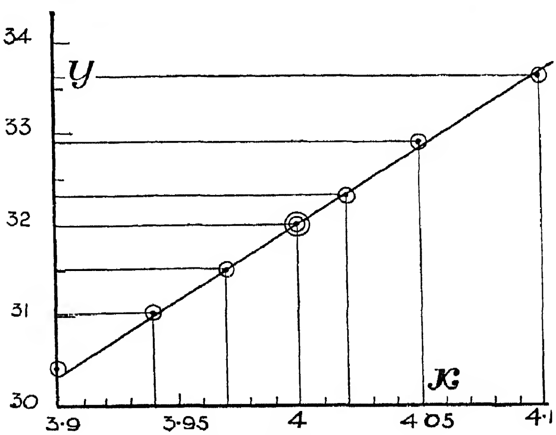


Fig. 257.

Example 9.—Find the value of—

$$\frac{2x^2 + 18x + 28}{12x^3 + 26x^2 - 76x - 160} \quad \text{when } x = -2.$$

The fraction $F = \dfrac{2(x^2 + 9x + 14)}{2(6x^3 + 13x^2 - 38x - 80)} = \dfrac{(x+2)(x+7)}{(x+2)(6x^2 + x - 40)}$

[$(x + 2)$ is tried as a factor, use being made of the Remainder Theorem, to which reference is made on p. 55.]

$$\therefore \quad F = \frac{(x+7)}{6x^2+x-40} = \frac{-2+7}{24-2-40} = \frac{5}{-18}$$

$$= \underline{-\frac{5}{18}} \text{ when } x = -2.$$

Example 10.—Find the limiting value of $\dfrac{(x + a)^4 - x^4}{a}$ when $a = 0$.

By direct substitution of 0 for a we again arrive at the indeterminate form $\dfrac{0}{0}$.

Proceeding along other lines—

$$F = \frac{(x + a)^4 - x^4}{a} = \frac{x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4 - x^4}{a}$$

$$= \frac{4x^3a + 6x^2a^2 + 4xa^3 + a^4}{a}$$

If a is to equal 0, and the value of F is then required, this value must differ extremely slightly from the value if calculated on the assumption that a is infinitely near to 0 but not exactly so.

If a is not zero, we may divide by it—

$$\text{then} \qquad F = 4x^3 + 6x^2a + 4xa^2 + a^3.$$

Hence, the limiting value to which F approaches as a is made nearer and nearer to zero is $4x^3$, for all the terms containing a may be made as small as we please by sufficiently decreasing a.

$$\therefore \quad \underset{a \to 0}{\mathrm{L}} \frac{(x + a)^4 - x^4}{a} = \underline{4x^3.}$$

$\underset{a \to 0}{\mathrm{L}} \dfrac{(x + a)^4 - x^4}{a} = 4x^3$ is the abbreviation recognised for the statement: "The limiting value to which the fraction $\dfrac{(x + a)^4 - x^4}{a}$ approaches as a approaches more and more nearly to o, is $4x^3$."

Example 11.—Find the limiting value of $\dfrac{\sin \theta}{\theta}$ when $\theta = 0$, it being given that—

$$\sin \theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \quad \ldots\ldots\ldots \quad \left\{\begin{matrix}\theta \text{ being measured} \\ \text{in radians}\end{matrix}\right\}$$

Adopting this expansion—

$$F = \frac{\sin \theta}{\theta} = \frac{\theta - \dfrac{\theta^3}{6} + \dfrac{\theta^5}{120}}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}$$

and $\underset{\theta \to 0}{\mathrm{L}} \dfrac{\sin \theta}{\theta} = 1$, as terms containing θ^2 and higher powers of θ must be very small compared with 1.

This result is of great importance : for small angles we may replace the sine of an angle by the angle itself (in radians). This rule is made use of in numerous instances. Thus when determining the period of the oscillation of a compound pendulum swinging through small arcs, an equation occurs in which $\sin\theta$ is replaced by θ; the change being legitimate since θ, the angular displacement, is small.

Exercises 44.—On Continued Fractions, etc.

1. Find the first 4 convergents of 8·09163. By how much does the 3rd convergent differ from the true value?

2. Find the 5th convergent of $\frac{1}{2+}\ \frac{2}{5+}\ \frac{7}{10+}\ \frac{1}{6+}\ \frac{3}{8}$.

3. Convert $\frac{481}{5043}$ into a continued fraction. What is the 3rd convergent?

4. Express as a continued fraction the decimal fraction ·08172.

5. Using the dividing head as in *Example* 1, p. 449, an angle of $59°14'5''$ is required to be marked off accurately. How many turns and partial turns would be required for this?

6. Similarly for an angle of $73°2'19''$.

7. Similarly for an angle of $5°19'3\frac{1}{2}''$.

8. It is desired to cut a metric screw thread on a lathe on which the pitch of the leading screw is measured in inches. To do this two change wheels have to be introduced in the train of wheels to give the correct ratio. If 1 cm. = ·3937″, find the number of teeth in each of the additional wheels, *i. e.*, find a suitable convergent for the decimal fraction ·3937.

On Partial Fractions.

9. Express $\frac{3x+8}{x^2+7x+6}$ as a sum or difference of simpler fractions.

10–16. Resolve the following into partial fractions—

10. $\frac{2}{6x^2+19x+15}$ **11.** $\frac{3x+5}{x^2-3x-88}$ **12.** $\frac{x(x+1)}{x^2-3x+2}$

13. $\frac{6x^2-9x+30}{(x-5)(x^2+2x-8)}$ **14.** $\frac{-22x^2-179x-240}{6x^3+15x^2-57x-126}$

15. $\frac{3x+2}{x^3+2x^2-x-2}$ **16.** $\frac{2x-3}{(x-3)(x^2+3x+3)}$

On Limiting Values.

17. Find the limiting value of $\frac{x^2-4x-5}{x^2+9x+8}$ when $x = -1$.

18. Determine $\underset{x\to 2}{\mathrm{L}}\ \frac{x^3+3x^2-17x+14}{x^2+2x-8}$

19. Show exactly what is meant by the statement—

$$\underset{x\to a}{\mathrm{L}}\ \frac{x^2-6ax+5a^2}{x^2+9ax-10a^2} = -\frac{4}{11}$$

20. Determine the limiting value of the sum of the series 16, 8, 4, 2, etc.

21. A body is moving according to the law, space $= 4 \times (\text{time})^3$. By taking small intervals of time in the neighbourhood of 2 secs., and thus calculating average velocities, deduce the actual velocity at the end of 2 secs.

22. If $e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots$; find the limiting value of the fraction $\frac{e^x - 1}{x}$ when $x = 0$.

23. If $\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4}$; and $\sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5}$. Find $\underset{\theta \to 0}{\mathrm{L}} \sin\theta$, $\underset{\theta \to 0}{\mathrm{L}} \cos\theta$, and by combination of these results $\underset{\theta \to 0}{\mathrm{L}} \frac{\tan\theta}{\theta}$. Hence show that no serious error is made when calling the taper of a cotter the angle of the cotter.

24. Find— $\underset{a \to b}{\mathrm{L}} \frac{4a^4 - 4b^4}{5a^2 - 15b^2 + 10ab}$

Permutations and Combinations.—Without going deeply into this branch of algebra, we can summarise the principal or most useful rules.

By the permutations of a number of things is understood the different arrangements of the things taken so many at a time, regard being paid to the order in these different arrangements.

By the combinations of a number of things is understood the different selections of them taken so many at a time.

e. g., a firm retains 12 men for their motor-van service. There are 6 vans and 2 men are required for each, 1 to be the driver. By simply arranging the men in pairs, a number of groups or *combinations* is obtained. But if the first pair might be sent to any one of the 6 vans, *i. e.*, if regard is paid to the arrangement of the pairs, and if also either of any pair might drive, we get further arrangements. We are then dealing with *permutations*.

To make this example a trifle clearer: let the men be represented by A, B, C, D, etc. Then the different selections of the 12, taken 2 at a time, would be A and B, A and C, A and D . . ., B and C, B and D . . ., C and D . . ., and so on. But A and B might be in the 1st van or in any of the others, so that a number of different arrangements of pairs amongst the vans would result.

Also A might drive or B might, so that the arrangements in the vans themselves would be increased. As we might write it for one van, the different arrangements would be A (driver) and B, or B (driver) and A.

To find a rule for the number of permutations of n things taken at a time.

If one operation can be performed in n ways and (when that has been performed in any one of these ways), a second operation

can then be performed in p ways, the number of ways of performing the two operations in conjunction will be $n \times p$: *e. g.*, suppose a cricket team possesses 5 bowlers; then the number of ways in which a bowler for one end can be chosen is 5. That end being settled, there are 4 ways of arranging the bowler for the other end. For each of the 5 arrangements at the one end there can be 4 at the other end, so that the total number of different arrangements will be 5×4, *i. e.*, 20.

Suppose a choice of r things is to be made out of a total of n to fill up r places.

Then the 1st place can be filled in n ways.

For the 2nd place (the 1st being already filled) choice can only be made from $(n-1)$ things; hence the number of different ways in which the 1st and 2nd can together be filled is $n(n-1)$.

The 1st, 2nd and 3rd together can be filled in $n(n-1)(n-2)$ ways, and so on, so that all the r places can be filled in $n(n-1)(n-2)$ . . . to r factors.

When there are 3 factors, the last $= (n-2) = (n-3+1)$

When there are 4 factors, the last $= (n-3) = (n-4+1)$

$\therefore$ When there are r factors, the last $= (n-r+1)$

$\therefore$ **The number of permutations of n things taken r at a time—**

$$= {}^n\mathbf{P}_r = n(n-1)(n-2) \;.\;.\;.\;.\; (n-r+1).$$

For shortness this product is often written n_r.

If n places are to be filled from the n things the number of possible ways—

$$= {}^nP_n = n_n = n(n-1)(n-2) \;.\;.\;.\;.\; (n-n+2)(n-n+1)$$
$$= n(n-1)(n-2) \;.\;.\;.\;.\; 2.1$$

i. e., n_n is the product of all the integers to n: this is spoken of as *factorial* n and is written $\underline{|n}$ or $n!$

Thus— "factorial 4" $= \underline{|4} = 1.2.3.4 = 24.$

To find the number of combinations of n things taken r at a time, written ${}^nC_r$:—Obviously ${}^nC_r$ must be less than ${}^nP_r$, because groups of things may be altered amongst themselves to give different permutations. For groups of r things, the number of different arrangements in each group must be $\underline{|r}$ (r things taken r at a time); hence the number of permutations must $= \underline{|r} \times$ the number of combinations—

or $${}^nP_r = \underline{|r} \times {}^nC_r$$

i. e., $${}^n\mathbf{C}_r = \frac{{}^n\mathbf{P}_r}{\underline{|r}} = \frac{n_r}{\underline{|r}}$$

$$= \frac{n(n-1)(n-2) \;.\;.\;.\;.\; (n-r+1)}{1.2.3 \;.\;.\;.\;.}$$

If both numerator and denominator are multiplied by $\underline{|n-r}$

$$i.e., \text{ by } \quad 1.2.3 \ldots\ldots (n-r)$$

then—

$${}^nC_r = \frac{n(n-1)\ldots\ldots(n-r+1)\times(n-r)\ldots\ldots 2.1}{\underline{|r}\ \underline{|n-r}}$$

$$= \frac{\underline{|n}}{\underline{|r}\ \underline{|n-r}}$$

from which we conclude that—

$${}^nC_r = {}^nC_{n-r}$$

a result often useful.

The number of permutations of n things taken n at a time when p of them are alike and all the rest are different $= \frac{\underline{|n}}{\underline{|p}}$

The number of permutations of n things taken r at a time when each thing may be repeated once, twice, ... r times in any arrangement $= n^r$.

The total number of ways in which it is possible to make a selection by taking some or all of n things $= 2^n - 1$.

Example 12.—Find the values of ${}^6P_2$, ${}^9C_3$ and ${}^{15}C_{11}$.

$${}^6P_2 = 6(6-1) = \underline{30}$$

$${}^9C_3 = \frac{9_3}{\underline{|3}} = \frac{9.8.7}{1.2.3} = \underline{84}$$

$${}^{15}C_{11} = \frac{15_{11}}{\underline{|11}} \quad \text{or} \quad \frac{\underline{|15}}{\underline{|11}\ \underline{|4}} \quad \text{or} \quad \frac{15_4}{\underline{|4}}$$

$$= \frac{15.14.13.12.}{1.2.3.4} = \underline{1365.}$$

When n and r are nearly alike (as in this last case) and ${}^nC_r$ is required, we use the form ${}^nC_r = {}^nC_{n-r}$; and the arithmetical work is thus reduced.

Example 13.—There are six electric lamps on a tramcar direction board; find the number of different signs that may be shown by these.

If the lamps all show the same coloured light, the question resolves itself into finding the total possible arrangements of 6 lamps when any number of them are lighted.

Thus if 6 lamps are on, there is only one arrangement possible. If 5 lamps are on, these can evidently be placed amongst the six places in six different ways; or, in other words, the number of arrangements in this case is ${}^6C_5$ or ${}^6C_1$ $[{}^nC_r = {}^nC_{n-r}]$. If 4 lamps only are to be switched on, the possible arrangements will be ${}^6C_4$, *i. e.*, ${}^6C_2$, *i. e.*, 15.

Similarly the numbers of arrangements for the cases of 3, 2 and 1

lamp on will be ${}^6C_3$, ${}^6C_2$ and ${}^6C_1$ respectively: hence the total number of different arrangements giving the different signs will be—

$$1 + {}^6C_5 + {}^6C_4 + {}^6C_3 + {}^6C_2 + {}^6C_1 = 1 + 6 + 15 + 20 + 15 + 6 = \underline{63}.$$

This result could also have been obtained by making use of the rule given on p. 462 for the total number of ways in which it is possible to make a selection by taking *some* or *all* of n things.

Thus total $= 2^n - 1 = 2^6 - 1 = 64 - 1 = 63.$

If the lamps had been of different colours the number of different signs would be greatly increased, since the different sets of the above could be changed amongst themselves.

Example 14.—Twelve change wheels are supplied with a certain screw-cutting lathe; find the number of different arrangements of these, 4 being taken at a time, viz. for the stud, pinion, lathe spindle, and spindle of leading screw.

In this case the order in which the wheels are placed is of consequence; hence we are dealing with Permutations.

As there are 12 to be taken, 4 at a time, the total number of arrangements $= {}^{12}P_4 = 12.11.10.9 = \underline{11880}.$

The Binomial Theorem.—By simple multiplication it can be verified that—

$$(x+a)^2 = x^2+2ax+a^2$$
$$(x+a)^3 = x^3+3x^2a+3xa^2+a^3$$
$$(x+a)^4 = x^4+4x^3a+6x^2a^2+4xa^3+a^4$$

It is necessary to find a general formula for such expansions; $(x+a)$ is a two-term or binomial expression, and the expansion of $(x+a)^n$ is performed by means of what is known as the *Binomial Theorem*. For simple cases, such as the above, there is no need for the theorem, but for generality it is desirable that some rule should be found. The expansion of $(x+a)^{-3}$ could certainly be found by writing it as $\frac{1}{(x+a)^3}$ and then performing the division, an endless series resulting, but it would be a painfully laborious process.

Suppose the continued product of $(x+a)(x+b)(x+c) \ldots$ to n factors is required, n being a positive integer.

The 1st term is obtained by taking x out of each factor, giving x^n.

The 2nd term is obtained by taking x out of all brackets but one, and then taking one of the letters $a, b, c \ldots$ out of the remaining bracket. The 2nd term thus $= x^{n-1}(a+b+c+d \ldots$ to n terms).

The 3rd term is obtained by taking x out of all brackets but two, and combining with the products of the letters $a, b, c \ldots$ taken two at a time.

The 3rd term thus $= x^{n-2}(ab + ac + ad + \ldots + bc + \ldots$ to, say, p terms).

p is then the number of combinations of n letters taken two at a time—

$$i.\,e., \qquad p = \frac{n_2}{\lfloor 2} = \frac{n(n-1)}{1.2}$$

so that the 3rd term is found.

In the same way any particular term may be found.

Example 15.—Write down the value of the product—

$$(x-2)(x+4)(x+6)(x-7).$$

1st term $= x^4$ (i. e., x is taken out of each bracket).

2nd term $= x^3\{-2+4+6-7\} = x^3$ (x being taken out of all brackets but one).

$$\text{3rd term} = x^2\{(-2)\times(+4)+(-2)\times(+6)+(-2)\times(-7) +(+4)\times(+6)+(+4)\times(-7)+(+6)\times(-7)\}.$$
$$= x^2\{-8-12+14+24-28-42\} = -52x^2.$$
$$\text{4th term} = x\{(-2)(+4)(+6)+(-2)(+6)(-7)+(+4)(+6)(-7) +(-2)(+4)(-7)\}.$$
$$= x\{-48+84-168+56\} = -76x.$$
$$\text{5th term} = (-2)(+4)(+6)(-7) = 336.$$
$$\therefore \quad (x-2)(x+4)(x+6)(x-7) = \underline{x^4+x^3-52x^2-76x+336.}$$

Now let—

$b = c = d = \ldots = a$, then $(x+a)(x+b)(x+c)\ldots$ to n factors, becomes $(x+a)^n$.

Then 1st term of the expansion—

$$= x^n$$

the 2nd term of the expansion $= x^{n-1}(a + a + a \ldots$ to n terms)

$$= nx^{n-1}a$$

the 3rd term of the expansion $= x^{n-2}(a^2+a^2+a^2 \ldots$ to ${}^nC_2$ terms)

$$= \frac{n(n-1)}{1.2}x^{n-2}a^2$$

Similarly, the 4th term of the expansion—

$$= \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3$$

$$\therefore \quad (x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{1.2}x^{n-2}a^2 + \ldots a^n$$

Thus the indices of x and a together always add up to n, that of x decreasing by one each term. The numerical coefficients can be remembered in a somewhat similar fashion; the numerator

having a factor introduced which is one less than the last factor in the preceding numerator, whilst the denominator has an additional factor one more than the last factor in the preceding denominator, *i. e.*, a kind of equality is preserved.

The proof here given is of an elementary character, and only applies when n is a positive integer, but it can be proved that the theorem is true for all values of n, integral or fractional, positive or negative.

To find an expression for any particular term in the expansion :—

The 3rd term $= \frac{n_2}{\lfloor 2} x^{n-2}a^2$, *i. e.*, is distinguished by the 2's throughout, and is on that account called term $(2+1)$ or $T_{(2+1)}$

The 14th term is thus written $T_{(13+1)}$

Putting the terms in this form we are enabled to write down at a glance, *i. e.*, without full expansion, any particular term desired.

e. g., the 6th term $= T_{(5+1)} = \frac{n_5}{\lfloor 5} x^{n-5}a^5$.

The $(r+1)^{\text{th}}$ term is usually taken as the general term, and it is given by—

$$\frac{n_r}{\lfloor r} x^{n-r}a^r \quad \text{or} \quad \frac{n(n-1)(n-2) \ldots\ldots (n-r+1)}{1.2.3 \ldots\ldots r} x^{n-r}a^r$$

Example 16.—Find the 8th term of the expansion of $(x-2y)^{10}$.

$$\left.\begin{array}{ll} \text{Here} & n = 10 \\ & x = x \\ \text{and} & a = -2y \end{array}\right\} \text{ in comparison with the standard form.}$$

$$\text{Hence} \quad T_8 = T_{(7+1)} = \frac{10_7}{\lfloor 7} x^{10-7}(-2y)^7$$

$$= \frac{10_3}{\lfloor 3} x^3(-2y)^7 \qquad [\text{for } {}^{10}C_7 = {}^{10}C_{10-7} = {}^{10}C_3]$$

$$= \frac{10\ 9\ 8}{1.2.3} x^3(-128y^7) = \underline{-15360x^3y^7}.$$

Example 17.—Expand $(a-3b)^{\frac{1}{2}}$ to 4 terms.

[Whenever n is fractional or negative the expansion gives an infinite series, and therefore it is necessary to state how many terms are required.]

Comparing with the standard form—

$$\begin{aligned} x &= a \\ a &= (-3b) \\ n &= \tfrac{1}{2} \end{aligned}$$

H H

Hence the expansion—

$$= a^{\frac{1}{4}} + \tfrac{1}{4}a^{\frac{1}{4}-1}(-3b) + \frac{\frac{1}{4}(\frac{1}{4}-1)}{1.2}a^{\frac{1}{4}-2}(-3b)$$
$$+ \frac{\frac{1}{4}(\frac{1}{4}-1)(\frac{1}{4}-2)}{1.2.3}a^{\frac{1}{4}-3}(-3b)^3 + \ldots.$$
$$= a^{\frac{1}{4}} - \tfrac{3}{4}a^{-\frac{3}{4}}b + \tfrac{1}{4}\times -\tfrac{3}{4}\times\tfrac{1}{2}\times 9a^{-\frac{7}{4}}b^2$$
$$+\tfrac{1}{4}\times -\tfrac{3}{4}\times -\tfrac{7}{4}\times -\tfrac{27}{6}a^{-\frac{11}{4}}b^3$$
$$= \underline{a^{\frac{1}{4}} - \tfrac{3}{4}a^{-\frac{3}{4}}b - \tfrac{27}{32}a^{-\frac{7}{4}}b^2 - \tfrac{189}{128}a^{-\frac{11}{4}}b^3 \ldots.}$$

Example 18.—Expand $\left(3m-\frac{n}{5}\right)^{-4}$ to 3 terms.

Here— $x = 3m,\ a = -\frac{n}{5},\ n = -4$

Hence the expression—

$$= (3m)^{-4}+(-4)(3m)^{-4-1}\left(-\frac{n}{5}\right)$$
$$+ \frac{(-4)(-4-1)}{1.2}(3m)^{-4-2}\left(-\frac{n}{5}\right)^2 + \ldots.$$
$$= 3^{-4}m^{-4} + \frac{4\cdot 3^{-5}m^{-5}n}{5} + \frac{4\times 5\times 3^{-6}m^{-6}n^2}{2\times 25} + \ldots.$$
$$= \underline{\frac{1}{81m^4} + \frac{4n}{1215m^5} + \frac{2n^2}{3645m^6} + \ldots.}$$

The method of setting out the work in these examples (Nos. 17 and 18) should be carefully noted; the brackets inserted helping to avoid mistakes with signs, etc. Thus in the evaluation of $n(n-1)$ when $n = -4$ one is very apt to write down the result straight away as -4×-3, whereas its true value is $(-4)(-4-1)$, *i. e.*, $+20$.

Example 19.—In the Anzani aero engine the cylinder is "offset," *i. e.*, the cylinder axis does not pass through the axis of the crank shaft, but is "offset" by a small amount c. The length of the stroke is given by the expression $\sqrt{(l+r)^2-c^2} - \sqrt{(l-r)^2-c^2}$, where l = length of connecting-rod and r = length of crank. Show that—

$$\text{stroke} = 2r\left\{1 + \frac{1}{2}\frac{c^2}{l^2-r^2}\right\}$$

Dealing with the expression $\sqrt{(l+r)^2-c^2}$, we may rewrite it as $\{(l+r)^2-c^2\}^{\frac{1}{2}}$ and then expand by the binomial theorem.

Thus— $$\{(l+r)^2-c^2\}^{\frac{1}{2}} = \{(l+r)^2\}^{\frac{1}{2}} + \tfrac{1}{2}\{(l+r)^2\}^{\frac{1}{2}-1}\times(-c^2)$$
$$+ \frac{\frac{1}{2}(\frac{1}{2}-1)}{1\times 2}\{(l+r)^2\}^{-\frac{3}{2}}\times(-c^2)^2 + \ldots.$$
$$= (l+r) - \frac{c^2}{2(l+r)} + \text{terms containing as factors}$$

the fourth and higher powers of c; these terms being negligible, since c^4, c^6, etc., are very small.

In like manner it can be shown that—

$$\sqrt{(l-r)^2 - c^2} = (l-r) - \frac{c^2}{2(l-r)}$$

Hence—

$$\text{stroke} = (l+r) - \frac{c^2}{2(l+r)} - (l-r) + \frac{c^2}{2(l-r)}$$

$$= 2r - \frac{c^2}{2}\left\{\frac{1}{(l+r)} - \frac{1}{(l-r)}\right\}$$

$$= 2r\left\{1 + \frac{1}{2}\frac{c^2}{l^2 - r^2}\right\}$$

Comparing this result with the length of the stroke of the engine if not offset, we see that there is small gain in the length of the stroke; the increase being the value of $rc^2 \div l^2 - r^2$.

Use of the Binomial Theorem for Approximations.—Let us apply the Binomial Theorem to obtain the expansion for $(1+x)^n$.

Writing 1 in place of x, and x in place of a, in the standard form—

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+ \dots$$

If x is very small compared with 1, then x^2, x^3, and higher powers of x will be negligible in comparison. Hence—

$$\mathbf{(1+x)^n = 1+nx} \quad \textbf{when } x \textbf{ is very small.}$$

Example 20.—In an experiment on the flow of water through a pipe the head lost due to pipe friction was required. The true velocity was 10 f.p.s., but there was an error of ·2 f.p.s. in its measurement. What was the consequent error in the calculated value of the head lost, given that loss of head $\propto$ (velocity)^2?

Let H_c = calculated loss of head.

$$H_c = Kv^2 = K(10 + \cdot 2)^2 \qquad \{v \text{ being the measured velocity}\}$$
$$= K \times 10^2(1 + \cdot 02)^2$$

Making use of the above approximation—

$$H_c = 100K(1 + \cdot 02 \times 2)$$
$$= 100K(1 + \cdot 04)$$
$$\text{But true head lost} = K \times 10^2 = 100K$$
$$\therefore \quad \text{error} = 100K \times \cdot 04 \text{ or } \underline{4\ \%.}$$

Example 21.—Find the cube root of 998.

$$998 = 1000 - 2 = 1000(1 - \cdot 002)$$
$$\therefore \text{cube root of } 998 = 998^{\frac{1}{3}} = 1000^{\frac{1}{3}}(1 - \cdot 002)^{\frac{1}{3}}$$
$$= 10(1 - \tfrac{1}{3} \times \cdot 002)$$
$$= 10(1 - \cdot 0007) = \underline{9 \cdot 993.}$$

Example 22.—Find the value of 1005^4.

$$1005 = 1000(1+\cdot005)$$
$$\therefore \quad 1005^4 = 1000^4(1+\cdot005)^4 = 1000^4[1+(4\times\cdot005)]$$
$$= \underline{10^{12}\times 1\cdot02.}$$

With a little practice one can mentally extract roots or find powers for cases for which these approximations apply—

$$e.\,g., \quad \sqrt{98} = 9\cdot9$$

For 98 differs from 100 by 2, hence its square root differs by $\frac{1}{2}$ of $\cdot2$, *i. e.*, $\cdot1$ from 10.

Similarly, $(1\cdot03)^3 = 1\cdot09$ very nearly.

Further instances of approximation are seen in the following :—

$$(1+x)(1+y) = 1+x+y+xy = 1+x+y$$
when x and y are small

$$(1+x)(1+y)(1+z) = 1+x+y+z \text{ when } x, y \text{ and } z \text{ are small}$$

$$\frac{(1+x)}{(1+y)} = 1+x-y$$

$$\frac{(1+x)^3}{(1+y)^2} = 1+3x-2y$$

$$\frac{(1+x)^m}{(1+y)^n} = 1+mx-ny.$$

Example 23.—Find the value of $\dfrac{985 \times 5\cdot08}{1004}$

$$F = \frac{1000(1 - \cdot015) \times 5(1 + \cdot016)}{1000(1 + \cdot004)}$$
$$= 5(1 - \cdot015 + \cdot016 - \cdot004) = \underline{4\cdot985.}$$

Example 24.—If l = measured length of a base line in a survey
L = correct or geodetic length, *i. e.*, length at mean sea-level
h = height above mean sea-level at which the base line is measured
and r = mean radius of the earth

Then— $$\frac{L}{l} = \frac{r}{r+h}$$

and it is required to find a more convenient expression for L.

$$L = \frac{lr}{r+h}, \text{ whence } L = \frac{l}{1+\frac{h}{r}} = l\left(1+\frac{h}{r}\right)^{-1}$$
$$= \underline{l\left(1-\frac{h}{r}\right)}$$

since h is very small compared with r.

Exponential and Logarithmic Series.

Applying the Binomial Theorem to $\left(1+\frac{1}{m}\right)^m$

$$\left(1+\frac{1}{m}\right)^m = 1+m\cdot\frac{1}{m}+\frac{m(m-1)}{1.2}\,\frac{1}{m^2}+\frac{m(m-1)(m-2)}{1.2.3}\times\frac{1}{m^3}+\ldots\ldots$$

$$= 1+1+\frac{\frac{m}{m}\left(1-\frac{1}{m}\right)}{1.2}+\frac{\frac{m}{m}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{1.2.3}+\ldots\ldots$$

$$= 1+1+\frac{\left(1-\frac{1}{m}\right)}{1.2}+\frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{1.2.3}+\ldots\ldots$$

Suppose now that m is increased indefinitely, then $\frac{1}{m}$, $\frac{2}{m}$, etc., become exceedingly small, and may be neglected.

Hence when m is infinitely large—

$$\left(1+\frac{1}{m}\right)^m = 1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\ldots$$

This is the case of compound interest with the interest very small but added to the principal at extremely short intervals of time. The letter e is written for this series—

$$i.\,e.,\quad \mathbf{e} = \mathbf{1}+\mathbf{1}+\frac{\mathbf{1}}{\underline{|2}}+\frac{\mathbf{1}}{\underline{|3}}+\frac{\mathbf{1}}{\underline{|4}}$$

[If it is any aid to the memory this statement may be written—

$$e = \frac{1}{\underline{|0}}+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\ldots]$$

In like manner, $\left(1+\frac{1}{m}\right)^{mx}$ would be e^x if m were infinitely large.

But—

$$\left(1+\frac{1}{m}\right)^{mx} = 1+mx\cdot\frac{1}{m}+\frac{mx(mx-1)}{1.2}\,\frac{1}{m^2}+\frac{mx(mx-1)(mx-2)}{\underline{|3}\cdot m^3.}+\ldots\ldots$$

$$= 1+x+\frac{x\left(x-\frac{1}{m}\right)}{\underline{|2}}+\frac{x\left(x-\frac{1}{m}\right)\left(x-\frac{2}{m}\right)}{\underline{|3}}+\ldots\ldots$$

$$= 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots \text{ (when } m \text{ is very large)}$$

$$\therefore\quad e^x = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$$

To obtain a more general series, *i.e.*, one for a^x, where a has any value whatever, let $a = e^k$, so that $\log_e a = k$.

Then— $$a^x = e^{kx}$$

The series for e^{kx} can be obtained from that for e^x by writing kx in place of x.

Then— $$a^x = 1 + kx + \frac{(kx)^2}{\underline{|2}} + \frac{(kx)^3}{\underline{|3}} + \dots .$$

and substituting for k its value we arrive at the important result—

$$\mathbf{a^x = 1 + x \log a + \frac{(x \log a)^2}{\underline{|2}} + \frac{(x \log a)^3}{\underline{|3}} + \dots .}$$

This is known as the **Exponential Series.**

A further series may be deduced from this, by the use of which natural logarithms can be calculated directly; common logarithms being in turn obtained from the natural logs by multiplying by the constant ·4343.

For let— $$a = 1 + y$$

Then by employing the exponential series—

$$(1+y)^x = 1 + x \log_e(1+y) + \frac{\{x \log_e (1+y)\}^2}{\underline{|2}} + \dots .$$

It is now required to obtain a series for $\log_e(1 + y)$, which can be done by equating coefficients on the two sides.

The left-hand side may be expanded by the Binomial Theorem, giving—

$$(1+y)^x = 1 + xy + \frac{x(x-1)y^2}{1.2} + \frac{x(x-1)(x-2)y^3}{1.2.3} + \dots .$$

Now x occurs in every term except the first, and the coefficient of x in the second term $= y$.

The third term is $\frac{1}{2}(x^2y^2 - xy^2)$; and the coefficient of x is $-\frac{y^2}{2}$

The fourth term is $\frac{1}{6}(x^3y^3 - 3x^2y^3 + 2xy^3)$; and the coefficient of x is $\frac{y^3}{3}$

Hence the coefficients of $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots .$

and, equating the coefficients of x on the two sides—

$$\mathbf{\log_e (1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots \dots} \quad (1)$$

which is known as the **logarithmic series.**

In the form shown, however, it is not convenient for purposes of calculation, because the right-hand side does not converge

rapidly enough; and a huge number of terms would need to be taken to ensure accurate results.

In the expansion for $\log_e(1+y)$ let y be replaced by $-y$; then—

$$\log_e(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \quad \dots\dots\dots \quad (2)$$

Subtracting the two series, *i. e.*, taking (2) from (1)—

$$\log_e(1+y) - \log_e(1-y) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \dots\right)$$

but $\quad \log_e(1+y) - \log_e(1-y) = \log_e \frac{(1+y)}{(1-y)}$

hence $\quad \log_e \frac{(1+y)}{(1-y)} = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \dots\right)$

Now let $\frac{(1+y)}{(1-y)}$ be denoted by $\frac{m}{n}$, i. e., $m - my = n + ny$

or $\quad y = \frac{m-n}{m+n}$

then $\quad \log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \dots\right\}$

which is a series well adapted for the calculation of logs.

Example 25.—To calculate $\log_e 2$.

Let $m = 2$, $n = 1$, and thus $y = \frac{1}{3}$

$$\text{then } \log_e 2 = 2\left\{\frac{1}{3} + \left(\frac{1}{3} \times \frac{1}{3^3}\right) + \left(\frac{1}{5} \times \frac{1}{3^5}\right) + \dots\right\}$$
$$= \cdot 6930$$

(which is one wrong in the 4th decimal place; and this error would have been remedied by taking one more term of the series).

An equally convenient series would be obtained by writing—

$$\frac{n+1}{n} \text{ for } \frac{1+y}{1-y}, \quad \text{i. e.,} \quad y = \frac{1}{2n+1}$$

Then—

$$\log_e \frac{n+1}{n} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\right\}$$

Thus if—

$$n = 1, \quad \log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3 \times 3^3} + \frac{1}{5 \times 3^5} + \dots\right\}$$
$$= \cdot 6930 \text{ as before.}$$

and this latter form is slightly easier to remember.

To obtain $\log_e 3$ let $n = 2$.

Then—

$$\log_e \frac{3}{2} = 2\left\{\frac{1}{5} + \frac{1}{3(5^3)} + \frac{1}{5(5^5)} + \ldots \right\}$$
$$= \cdot 40546$$

but $\log_e \frac{3}{2} = \log_e 3 - \log_e 2$

$$\cdot 40546 = \log_e 3 - \cdot 6931$$
$$\therefore \quad \log_e 3 = 1\cdot 0986.$$

Again, $\log 4 = 2 \times \log_e 2$ and $\log_e 5$ can be obtained by using the series for $\log_e \frac{n+1}{n}$ when $n = 4$, and the value of $\log_e 4$, so that a table of natural logs could be compiled: in fact, this is the way log tables are made.

The corresponding common logs are found by multiplying the natural logs by $\cdot 4343$.

Example 26.—The "modified area" A, a term occurring in connection with the bending of curved beams, is given by—

$$A = Rb \log_e \frac{2R + d}{2R - d}$$

for a rectangular section of breadth b and depth d.

Show that this can be written as—

$$A = bd\left[1 + \frac{1}{12}\left(\frac{d}{R}\right)^2 + \frac{1}{80}\left(\frac{d}{R}\right)^4 + \ldots\right]$$

$\frac{2R + d}{2R - d}$ might be written as $\dfrac{1 + \frac{d}{2R}}{1 - \frac{d}{2R}}$ and is therefore of the form,

$$\frac{1+y}{1-y}, \text{ where } y = \frac{d}{2R}$$

Hence—

$$A = Rb \log_e \frac{2R + d}{2R - d} = Rb \log_e \frac{1 + \frac{d}{2R}}{1 - \frac{d}{2R}}$$
$$= 2Rb\left[\left(\frac{d}{2R}\right) + \frac{1}{3}\left(\frac{d}{2R}\right)^3 + \frac{1}{5}\left(\frac{d}{2R}\right)^5 + \ldots\ldots\right]$$
$$= 2Rb\left[\frac{d}{2R} + \frac{d^3}{24R^3} + \frac{d^5}{160R^5} + \ldots\ldots\ldots\right]$$
$$= bd\left[1 + \frac{1}{12}\left(\frac{d}{R}\right)^2 + \frac{1}{80}\left(\frac{d}{R}\right)^4 + \ldots\ldots\ldots\right]$$

R in this formula is the radius of curvature of the beam, and hence if the beam is originally straight $R = \infty$, so that $\frac{d}{R} = 0$ and the expression for A reduces to bd, i. e., the area of the section.

Exercises 45.—On the Binomial Theorem, etc.

1. Write down the 5th term in the expansion of $(a - b)^7$.

2. Expand $(2a + 5c)^{11}$ to 4 terms.

3. Find the 20th term of the expansion of $(3x - y)^{23}$.

4. Expand $\left(\frac{m}{2} - \frac{2n}{5}\right)^8$ to 4 terms.

5. Write down the first 5 terms of the expansion of $(a - 2)^{-2}$.

6. Find the 7th term of $\left(1 - \frac{1}{x}\right)^{10}$

7. Expand to 3 terms $(2 - x^2)^{\frac{1}{2}}$.

8. Expand to 4 terms $(3a + 4c)^{-\frac{1}{2}}$

9. Write down the 3rd term of $(a - 2b)^{-\frac{3}{5}}$

10. Expand $\sqrt{1 - \frac{a^2}{l^2} \sin^2\theta}$ to 4 terms, and hence state its approximate value when $\frac{l}{a}\left(\frac{\text{length of connecting-rod}}{\text{length of crank}}\right)$ is large.

On Permutations and Combinations.

11. In the Morse alphabet each of our ordinary letters is represented by a character composed of dots and dashes.

Show that 30 distinct characters are possible if the characters are to contain not more than 4 dots and dashes, a single dot or dash being an admissible character.

12. Find the number of ways in which a squad of 12 can be chosen from 20 men.

(*a*) When the squad is numbered off (*i. e.*, each man is distinguished by his number).

(*b*) When no regard is paid to position in the line.

13. Find the values of ${}^{15}C_{13}$, ${}^{12}P_4$, ${}^{5}P_5$.

On Approximations.

14. Use the method of p. 467 to obtain the value of $(\cdot996)^4$.

15. Evaluate $\frac{1\cdot0015 \times 2\cdot063 \times \cdot998}{(\cdot997)^2}$ by the same method.

16. State the approximate values of—

(*a*) $(1002)^5$; (*b*) $(\cdot9935)^7$; (*c*) $(1 - \cdot006)^{45}$,
(*d*) $(10 + \cdot17) \times \cdot995 \times 4\cdot044$.

17. The maximum efficiency of a screw $= \left(\frac{1 - \tan \frac{1}{2}\phi}{1 + \tan \frac{1}{2}\phi}\right)^2$, where ϕ is the angle of friction, *i. e.*, $\tan \phi = \mu$. Show that this may be written in the form $\frac{1 - \mu}{1 + \mu}$ if μ is small.

On Series.

18. Find series for the expression $\cosh x$, *i.e.*, $\left(\frac{e^x + e^{-x}}{2}\right)$ and for $\sinh x$, *i.e.*, $\left(\frac{e^x - e^{-x}}{2}\right)$

19. Find, by means of a series, the value of $\log_e 4$ correct to 3 places of decimals.

20. Express $\frac{R}{R+y}$ as a series. What is the approximate value of this fraction when y is small compared with R?

21. A cable hanging freely under its own weight takes the form of a catenary, the equation of which curve is $y = c \cosh \frac{x}{c}$, c being the value of the ratio $\frac{\text{horizontal tension}}{\text{weight per foot run}}$

Express y as a series, and thence show that if the curve is flat it may be considered as a parabola, having the equation $y = \frac{H}{w} + \frac{wx^2}{2H}$

22. By substituting $\cdot 5$ for x in Newton's series—

$$\sin^{-1}x = x + \frac{1.x^3}{2.3} + \frac{1.3x^5}{2.4.5} + \frac{1.3.5x^7}{2.4.6.7} + \ldots;$$

calculate the value of π correct to 3 places of decimals.

Determinants.—When a long mathematical argument is being developed, as occurs for example when certain aspects of the stability of an aeroplane are being considered, it frequently happens that the coefficients of the variable quantities become very involved; and in such cases it is often convenient to express the coefficients in "determinant" form. This mode of expression is also utilised for the statement of some types of equations, for by its use the form of equation and its solution are suggested concisely and the attention is not distracted from the main theme of the working.

Thus when dealing with the lateral stability of an aeroplane in horizontal flight the equation occurred—

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

where A, B, C, etc., were all solutions of other equations and in some cases rather long expressions. For example, A had the form $a^2b^2 - c^4$ and E was equal to $g \sin \theta(l_1n_1 - l_2n_2) - g \cos \theta(l_1n_3 - n_2l_3)$. To avoid writing these expressions in their expanded form, they were expressed thus—

$$A = \begin{vmatrix} a^2 & -c^2 \\ -c^2 & b^2 \end{vmatrix} \text{ and } E = g \sin \theta \begin{vmatrix} l_1 & l_2 \\ n_2 & n_1 \end{vmatrix} - g \cos \theta \begin{vmatrix} l_1 & l_3 \\ n_2 & n_3 \end{vmatrix}$$

and it will be shown that from these "determinant" forms the expansions may easily be obtained.

Before proceeding to illustrate the use of determinants it is necessary to define them and to show how they may be evaluated.

Let—
$$D = \begin{vmatrix} a & b & c \\ d & f & g \\ h & k & l \end{vmatrix}$$

then D is called the determinant of the quantities $a\ b\ c\ .\ .\ .\ l$, and a determinant of the third order since there are three columns and three rows; its value being found according to the following plan :—

The letter a occurs both in the first row and in the first column : take this letter and associate it with the remaining columns and rows, thus—

$$a \begin{vmatrix} f & g \\ k & l \end{vmatrix}$$

[It will be observed that $\begin{vmatrix} f & g \\ k & l \end{vmatrix}$ is a determinant of the second order and it is termed the minor determinant of a.]

Then the value of the minor of a is found by multiplying f by l and subtracting from it the product k by g.

Thus $\begin{vmatrix} f & g \\ k & l \end{vmatrix} = fl - gk$ and $a \begin{vmatrix} f & g \\ k & l \end{vmatrix} = a(fl - gk) = \mathrm{A}.$

In like manner the minor containing the products of b is—

$$b \begin{vmatrix} d & g \\ h & l \end{vmatrix} = b(dl - gh) = \mathrm{B}$$

and for c
$$c \begin{vmatrix} d & f \\ h & k \end{vmatrix} = c(dk - fh) = \mathrm{C}.$$

Then the value of the full determinant

$$= \mathrm{D} = \mathrm{A} - \mathrm{B} + \mathrm{C} = a(fl - gk) - b(dl - gh) + c(dk - fh).$$

To avoid the minus sign before the second term the letters might be written out as follows—

$$\begin{matrix} a & b & c & a & b \\ d & f & g & d & f \\ h & k & l & h & k \end{matrix}$$

and the one sequence could be maintained, thus—

$$\mathrm{D} = a(fl - gk) + b(gh - dl) + c(dk - hf)$$

Similarly, for a determinant of the fourth order—

$$\mathrm{D} = \begin{vmatrix} a & b & c & d \\ f & g & h & k \\ l & m & n & p \\ q & r & s & t \end{vmatrix}$$

$$D = a\begin{vmatrix} g & h & k \\ m & n & p \\ r & s & t \end{vmatrix} - b\begin{vmatrix} f & h & k \\ l & n & p \\ q & s & t \end{vmatrix} + c\begin{vmatrix} f & g & k \\ l & m & p \\ q & r & t \end{vmatrix} - d\begin{vmatrix} f & g & h \\ l & m & n \\ q & r & s \end{vmatrix}$$

each of these determinants of the third order being evaluated in the manner previously explained.

Example 27.—Evaluate the determinant—

$$D = \begin{vmatrix} 2 & 3 & 5 \\ -6 & 4 & -2 \\ 3 & 1 & 9 \end{vmatrix}$$

$$D = 2[36-(-2)]-3[-54-(-6)] + 5[-6-12]$$
$$= 76 + 144 - 90 = \underline{130.}$$

Example 28.—Evaluate the determinant—

$$D = \begin{vmatrix} 2 & 4 & 1 & -2 \\ 3 & 6 & 5 & 3 \\ -1 & -2 & 2 & 3 \\ 4 & 8 & 2 & 4 \end{vmatrix}$$

$$D = 2\begin{vmatrix} 6 & 5 & 3 \\ -2 & 2 & 3 \\ 8 & 2 & 4 \end{vmatrix} - 4\begin{vmatrix} 3 & 5 & 3 \\ -1 & 2 & 3 \\ 4 & 2 & 4 \end{vmatrix} + 1\begin{vmatrix} 3 & 6 & 3 \\ -1 & -2 & 3 \\ 4 & 8 & 4 \end{vmatrix} + 2\begin{vmatrix} 3 & 6 & 5 \\ -1 & -2 & 2 \\ 4 & 8 & 2 \end{vmatrix}$$

$$= 2\{6(8-6)-5(-8-24)+3(-4-16)\}-4\{3(8-6)-5(-4-12) + 3(-2-8)\}$$
$$+ 1\{3(-8-24)-6(-4-12) + 3(-8+8)\} + 2\{3(-4-16)-6(-2-8) + 5(-8+8)\}$$
$$= 224-224+0+0 = \underline{0.}$$

It will be observed that all the numbers in the second column are the same multiple of the corresponding numbers in the first column; and it can be proved that when this is the case the determinant is equal to zero.

Example 29.—A number of equations in a long investigation reduced to the determinant form—

$$\begin{vmatrix} x + \cdot 15 & -\cdot 3 & -30 \\ \cdot 6 & x + 5 & 100x \\ 0 & -\cdot 1 & x^2 + 9x \end{vmatrix} = 0$$

Express this in the form necessary for the solution of the equation.

The determinant $= (x + \cdot 15)\{(x+5)(x^2+9x) + 10x\} + \cdot 3\{\cdot 6x^2 + 5\cdot 4x\} - 30\{-\cdot 06\}$

$$= x^4 + 14\cdot 15x^3 + 57\cdot 28x^2 + 9\cdot 87x + 1\cdot 8$$

and thus the equation is—

$$\underline{x^4 + 14\cdot 15x^3 + 57\cdot 28x^2 + 9\cdot 87x + 1\cdot 8 = 0.}$$

Solution of Simultaneous Equations of the first degree by the determinant method.—Equations containing two or more unknowns may be readily solved by setting them in a determinant form and proceeding according to the following scheme :—

To solve the equations
$$5x - 4y = 23$$
$$3x + 7y = -5.$$
Write the equations as
$$5x - 4y - 23 = 0$$
$$3x + 7y + 5 = 0$$

and set out in the determinant form—

$$\left|\begin{array}{ccc} x & y & 1 \\ 5 & -4 & -23 \\ 3 & 7 & 5 \end{array}\right| \quad \begin{array}{cc} x & y \\ 5 & -4 \\ & 7 \end{array}$$

the last column containing the constants.

Then— $$\frac{x}{x \text{ minor}} = \frac{-y}{y \text{ minor}} = \frac{1}{1 \text{ minor}}$$

i. e., $$\frac{x}{-20 + 161} = \frac{1}{35 + 12} \text{ and } \frac{-y}{25 + 69} = \frac{1}{35 + 12}$$

whence $$x = 3 \text{ and } y = -2.$$

Example 30.—Solve the equations—

$$4ax - cy = b^2$$
$$3bx + 2ay = a^2$$

x and y being the unknown quantities.

Set out thus—

$$\left|\begin{array}{ccc} x & y & 1 \\ 4a & -c & -b^2 \\ 3b & 2a & -a^2 \end{array}\right| \quad \begin{array}{cc} x & y \\ 4 & -c \\ b & \end{array}$$

Then—

$$\frac{x}{ca^2 + 2ab^2} = \frac{1}{8a^2 + 3bc} \text{ and } \frac{-y}{-4a^3 + 3b^3} = \frac{1}{8a^2 + 3bc}$$

whence
$$\left.\begin{aligned} x &= \frac{a^2c + 2ab^2}{8a^2 + 3bc} \\ y &= \frac{4a^3 - 3b^3}{8a^2 + 3bc.} \end{aligned}\right\}$$

Example 31.—Solve the equations—

$$\begin{aligned} 2a - 5b + 4c &= 28 \\ a + 11b - 5c &= -41 \\ 3a - 2b - c &= 3 \end{aligned}$$

Set out thus—

$$\begin{vmatrix} a & b & c & 1 \\ 2 & -5 & 4 & -28 \\ 1 & 11 & -5 & 41 \\ 3 & -2 & -1 & -3 \end{vmatrix}$$

Then—
$$\frac{a}{a \text{ minor}} = \frac{-b}{b \text{ minor}} = \frac{c}{c \text{ minor}} = \frac{-1}{1 \text{ minor}}.$$

Thus—

$$\frac{a}{-5(15+41)-4(-33+82)-28(-11-10)}$$
$$= \frac{-1}{2(-11-10)+5(-1+15)+4(-2-33)} = \frac{1}{112}$$
$$\frac{-b}{2(15+41)-4(-3-123)-28(-1+15)} = \frac{1}{112}$$

and
$$\frac{c}{2(-33+82)+5(-3-123)-28(-2-33)} = \frac{1}{112}$$

whence
$$\left.\begin{array}{l} a = 1 \\ b = -2 \\ c = 4 \end{array}\right\}$$

Exercises 46—On Determinants.

Evaluate the determinants in Nos. 1–4.

1. $\begin{vmatrix} 5\cdot4 & -6 \\ -3 & -5 \end{vmatrix}$ **2.** $\begin{vmatrix} R_1 & R_3 \\ R_0 & R_2 \end{vmatrix}$ when $R_0 = -3\cdot6$, $R_1 = 7\cdot2$, $R_2 = 710$, $R_3 = 220$

3. $\begin{vmatrix} 3 & 5 & 4 \\ 1\cdot5 & 2\cdot5 & 2 \\ -3 & 7 & 5 \end{vmatrix}$ **4.** $\begin{vmatrix} 2 & 3 & 5 & 1 \\ 3 & 2 & 4 & 6 \\ 8 & -4 & 3 & -5 \\ -2 & -1 & 6 & 2 \end{vmatrix}$

5. A coefficient C in an equation was expressed as—

$$C = \begin{vmatrix} Z_w & U_o + Z_q \\ M_w & M_q \end{vmatrix} + \begin{vmatrix} X_u & X_q \\ M_u & M_q \end{vmatrix} + K_B^2 \begin{vmatrix} X_u & X_w \\ Z_u & Z_w \end{vmatrix}$$

Evaluate this when $Z_w = -3$, $M_w = 2\cdot5$, $M_q = -200$, $Z_q = 9$, $U_o = 1$, $X_u = -\cdot1$, $X_q = \cdot5$, $M_u = 0$, $K_B = 20$, $X_w = \cdot2$, $Z_u = -1$.

6. Solve the equation $\begin{vmatrix} a & \cdot5 & 3 \\ 2 & a+2 & 1 \\ 3a & 1\cdot6 & \cdot4 \end{vmatrix} = 0$

Using the determinant method solve the equations in Nos. 7–10.

7. $11x - 4y = 31$
$2x + 3y = 28$

8. $8a - b = -20$
$-10a + 7b = 71$

9. $4x - 5y + 7z = -14$
$9x + 2y + 3z = 47$
$x - y - 5z = 11$

10. $2a + 3b + 5c = -4\cdot5$
$3c - 7a + 15b = 62\cdot7$
$9b - 10a = 39\cdot3$

ANSWERS TO EXERCISES

Exercises 1

1. 1·2	**2.** ·0009	**3.** 150	**4.** 60
5. 10·8	**6.** 1·2	**7.** ·000225	**8.** 28·5
9. ·009	**10.** ·052	**11.** 93	**12.** ·161
13. 2·7	**14.** ·9	**15.** 22	**16.** ·56
17. ·031	**18.** 7·4	**19.** 2·5	**20.** 6·3

Exercises 2

1. $\frac{1}{b^8}$; $\frac{4}{b^7}$; $\frac{1}{5a^2}$; $9^{\frac{1}{6}}c^{\frac{2}{3}}$; $\frac{x^{\frac{2}{7}}}{y^{\frac{4}{7}}}$ **2.** 2; $\frac{1}{32}$; $\frac{27}{8}$; 384; $958\frac{1}{3}$

3. $2187\sqrt{\frac{a^7b^{13}}{c^{25}}}$ **4.** $\frac{7}{81x^{\frac{1}{2}}y^{\frac{31}{3}}z^{\frac{23}{18}}}$ **5.** $\frac{11a^4c^{\frac{3}{4}}d^{\frac{41}{4}}}{59b^{\frac{2}{3}}}$

6. $-\frac{\cdot 3425}{v^{9\cdot 48}}$ **7.** $-1\cdot 41p$ **8.** $a^{\frac{2}{n}} - a^{\frac{n+1}{n}}$ **9.** 1·6

10. $\frac{8b^{\frac{7}{6}}c^{\frac{4}{3}}}{5a^{\frac{13}{8}}d}$ **11.** $\frac{p_2v_2 - p_1v_1}{1-n}$ $\left\{\begin{array}{c}Cv_2^{1-n} \text{ might be written} \\ p_2v_2^{\,n} \times v_2^{1-n}, \text{ etc.}\end{array}\right\}$

Exercises 3

1. 589·5	**2.** 246·5	**3.** ·02138	**4.** 57·03
5. ·0005423	**6.** 116700	**7.** 12·34	**8.** 19·63
9. ·06664	**10.** 1·924	**11.** 244·4	**12.** 29·14
13. 1618	**14.** ·00009506	**15.** 49·64	**16.** 3·114
17. ·0001382	**18.** ·6874	**19.** ·02231	**20.** ·2777
21. 3642	**22.** 1·669	**23.** ·00001509	**24.** ·3352
25. ·001155	**26.** $3\cdot 841 \times 10^6$	**27.** 20·17	**28.** ·2421
29. 4·814	**30.** $7\cdot 211 \times 10^{-14}$	**31.** ·07041	**32.** 971·8
33. 10·02	**34.** 85·8	**35.** 5·418	**36.** 32·75
37. 220·4; 1369	**38.** 3·29	**39.** 1·9839	**40.** 5400
41. 4·6	**42.** 500·77	**43.** 1·315; ·5918	**44.** 355·6
45. 20·13, 23·68	**46.** ·0935	**47.** 22 21	**48.** 400
49. 80·072	**50.** 1·434	**51.** ·506	**52.** ·479
53. 5121	**54.** 15·70	**55.** 46460	**56.** 2·815
57. $3\cdot 5 \times 10^5$	**58.** 1·392	**59.** $1\cdot 016 \times 10^6$	**60.** ·284
61. ·128	**62.** $t = \frac{7}{16}$, $d = 1\frac{1}{16}$, $B = 1\frac{9}{16}$, $T = 1\frac{3}{16}$, $n = 7$		

Exercises 4

1. lbs. per sq. in. **2.** 1·68 **3.** 2·79 **4.** 5·44

5. 2·785 **6.** lbs. **7.** Stress $= \frac{v^2}{75}$ **9.** $f = \frac{12wv^2}{g}$

10. Incorrect as written; H.P. $= \frac{pR^3N\pi^2}{396000}$ **11.** cu. ft. per sec.

12. 746 **13.** 7·731 inches.

Exercises 5

1. $-\frac{7}{16}$ **2.** $-2{\cdot}285$ **3.** $3{\cdot}62$ **4.** $7{\cdot}54$

5. $6{\cdot}93$ **6.** $\cdot75$ **7.** $\frac{7d}{10a^2b}$ **8.** $\frac{H}{ws}+t$

9. $30{\cdot}1 \times 10^6$ **10.** (a) $L=\frac{w}{w_1}(T-t)+T-t_1$; (b) 933

11. $\cdot1205$ **12.** 1 **13.** $5{\cdot}01$ **14.** $E=2{\cdot}5C$

15. $1{\cdot}97$ **16.** $R=\frac{AD}{D-d}+a$; $126{\cdot}5$ **17.** $1{\cdot}67$

18. $q=\frac{1}{L}\left\{\frac{W}{w}(h_2-h_1)+h_2-h\right\}$ **19.** 11 cells.

20. $22{\cdot}85$ **21.** $15{\cdot}43$ **22.** $d=\sqrt[3]{\frac{1{\cdot}274Pl}{f}}$

23. $6{\cdot}9$ **24.** $\left.\begin{array}{l}(a)\ \cdot4915\\(b)\ \cdot467\end{array}\right\}$ **25.** $\left.\begin{array}{l}(a)\ \Delta=\frac{2Wh}{Ea\delta-2W}\\(b)\ \cdot294\end{array}\right\}$

26. 3 **27.** $2{\cdot}4$ **28.** 80 **29.** 1493

30. $3{\cdot}3$ **31.** lbs. ft. must be first brought to lbs. ins.; $6{\cdot}46$ ins.

32. $200{,}000$ **33.** $2{\cdot}57$ **34.** $27{\cdot}3$ ft.

35. $\left.\begin{array}{l}(a)\ \frac{d_m}{d_s}=26{\cdot}67\\(b)\ \frac{d_m}{d_s}=2{\cdot}4\end{array}\right\}$ **36.** $\left.\begin{array}{l}(a)\ r=\frac{k^2}{2m(1-k)}\\(b)\ \cdot00675\end{array}\right\}$ **37.** $\left.\begin{array}{l}(a)\ C=\frac{3K(m-2)}{2(m+1)}\\(b)\ E=\frac{9CK}{3K+C}\end{array}\right\}$

38. $\frac{r}{2}+\frac{d}{16}$ **39.** 700 lbs./□″ **40.** $1{\cdot}37$

41. A's speed $=10$ m.p.h.; B's speed $=15$ m.p.h. **42.** $6{\cdot}31$

43. $M=221{\cdot}4$; $H=\cdot17$
{Multiply equations together; thus, $\frac{M}{H}\times MH=M^2$}

44. $h=3{\cdot}4$ ft.
$k=35{\cdot}8$ ins. **45.** $57{\cdot}7$ **46.** $D=d_1d_2\sqrt[5]{\frac{L}{l_1d_2^5+l_2d_1^5}}$

47. $y=\frac{4xr(l-x)}{l^2}$ **48.** $y=\frac{c}{2}\left(\frac{2h+c}{h+2c}\right)$

Exercises 6

1. $x=\frac{8}{7}$, $y=\frac{7}{3}$ **2.** $a=7$, $b=-2$ **3.** $m=1{\cdot}8$, $n=2{\cdot}6$ **4.** $x=1$, $y=0$

5. $x=1{\cdot}24$, $y=3{\cdot}59$ **6.** $x=12$, $y=20$ **7.** $x=5$, $y=3$ **8.** $a=1{\cdot}2$, $b=5{\cdot}7$, $c=-4{\cdot}8$

9. $p=9$, $s=7$, $t=-4$ **10.** $E=4+\cdot7t-\cdot2t^2$ **11.** $a=1160$, $b=69000$

12. $E=\cdot12W+4{\cdot}6$ **13.** $V=42+\frac{27}{A}$

14. $I=\cdot953B+1{\cdot}284$: $4{\cdot}81$ **15.** $a=\cdot916$, $b=\cdot191$ **16.** $R_o=4{\cdot}714$, $a=\cdot00707$

17. $L=1115-\cdot7t$; 967 **18.** $e=\frac{70\sqrt{\text{area}}}{\text{length}}+18$

19. $e = \dfrac{101{\cdot}6\sqrt{\text{area}}}{\text{length}} + 9{\cdot}7$ **20.** $\mathrm{E} = -164{\cdot}1 + 7{\cdot}309\mathrm{T} + {\cdot}000326\mathrm{T}^2$

21. $f = 14{\cdot}8 - {\cdot}0000138(t-60)^2$ **22.** $f = 16{\cdot}1 - {\cdot}000026(t-60)^2$

23. $\mathrm{W} = 8{\cdot}28 + \dfrac{1077}{p+4}$

24. $w = 10{\cdot}3 + \dfrac{1700}{\mathrm{I}}\left(w = \dfrac{\text{total steam per hour}}{\text{I.H.P.}} \text{ and must be first calculated}\right)$

25. 8610 of iron; 7000 of copper **26.** $\mathrm{C} = 99{\cdot}8$; $\mathrm{K} = 1$

27. $\mathrm{A} = 120$, $\mathrm{B} = 140$, $\mathrm{C} = 160$ **28.** $m_1 = 44{\cdot}71$, $m_2 = 31{\cdot}54$

29. 311·1 tons of saltpetre; 388 9 tons of ginger.

Exercises 7

1. $(x+22)(x-4)$ **2.** $(x-11)(x-8)$

3. $(x-5)(x-21)$ **4.** $(2a-5b^2)(4a^2+25b^4+10ab^2)$

5. $(8x-11)(3x+4)$ **6.** $(5a-3b)(5b-a)$

7. $(a+9b)(a-5b)$ **8.** $(3x-7y)(4x-15y)$

9. $(8+x)(11-3x)$ **10.** $2n(5m-2n)(2m-5n)$

11. $\dfrac{wx}{384}(96xl^2-16l^3+5lx^2-24x^3)$ **12.** $\dfrac{4}{3}\pi(\mathrm{R}-r)(\mathrm{R}^2+r\mathrm{R}+r^2)$

13. $(94x+321)(x-3)$ **14.** $\dfrac{wx}{24\mathrm{EI}}(2lx^2-x^3-l^3)$

15. $6a^2b(a+2c)(9a-25c)$ **16.** $(2a-3b+4c)(2a-3b-4c)$

17. $8(2c^2+\frac{1}{3}a^3b)(4c^4+\frac{1}{9}a^6b^2-\frac{2}{3}a^3bc^2)$ **18.** $\dfrac{\pi h}{3}\{\mathrm{R}^2+\mathrm{R}r+r^2\}$

19. $(x+7)(x-1)(2x-5)$ **20.** $(p-1)(3p+7)(2p+5)$

21. $199 \times 23(2+6-4) = 18308$ **22.** 14,150

23. $12(x-3)(x-4)(x+2)$ **24.** $\dfrac{a^{\frac{29}{6}}}{6b^{\frac{4}{3}}c^{\frac{19}{6}}}$

25. $\dfrac{4(x+2)}{5(x-2)}$ **26.** $\dfrac{3(3x^2-4x-6)}{2(x^2-2x-8)}$

27. $\dfrac{21(a-b)}{4(9a-14b)}$ **28.** $\dfrac{560-327x+99x^2-120x^3}{20(3x-5)(3x+10)(2x-7)}$

29. $\dfrac{11}{27}$ or ·407 **30.** 37·8 **31.** $\dfrac{4mdl}{(d^2-l^2)^2}$; $\dfrac{4ml}{d^3}$

32. $3x(x+9)(x-7)$; $(8-9x)(3+8x)$, $(5x+4y)(x+10y)$

33. $(x+8)(x-1)(x^2+7x+26)$. {Hint.—Let $\mathrm{X} = x^2+7x+6$.}

34. $\dfrac{Pas\,(18s+25)}{24(6s+1)(s+2)}$ **35.** $\dfrac{n}{n-1}(p_1v_1-p_2v_2)$ **36.** $\dfrac{2(b^2+ab+a^2)}{3(b+a)}$

Exercises 8

1. -4 or -1 **2.** 2·5 or 3 **3.** 4·13 or $-1{\cdot}13$

4. 4 or -1 **5.** 2 83 or $-{\cdot}83$ **6.** $-{\cdot}278 \pm {\cdot}381j$

7. 3 **8.** 4·23 or $-2{\cdot}43$ **9.** $-{\cdot}125 \pm 1{\cdot}219j$

10. $2{\cdot}421 \times 10^5$ or $-2{\cdot}379 \times 10^5$ **11.** 2·75 or 457

12. 28·98 or 1·03 **13.** $3{\cdot}89 \times 10^4$ or $-2{\cdot}97 \times 10^4$

14. 57·5 or $-56{\cdot}5$ **15.** 23 (-18 has no meaning here)

11

16. $f_s = \dfrac{nr \pm \sqrt{n^2r^2 + 4\left(f_1 - \frac{r}{2}\right)^2}}{2}$

17. $u = \dfrac{ab \pm \sqrt{a^2b^2 - 24t^2v^2 + 8atgv}}{4t}$

$v = \dfrac{ag \pm \sqrt{a^2g^2 - 24t^2u^2 + 12atbu}}{6t}$

18. 6·07

19. 120 or 13·3. (Divide all through by 75×10^6 first)

20. 155 or 32

21. ·845 (−2·845 has no meaning here)

22. $v = 95\sqrt{mi}$

23. 5 or −7·5

24. 80 or −90

25. ·211 and ·789 of span from one end

26. 13 or 10·42

27. 1·475

28. 6·55 or −3·05

29. 100 ft.

Exercises 9

1. $\left.\begin{array}{l} x = 7 \text{ or } \frac{3}{2} \\ y = -3 \text{ or } -\frac{20}{6} \end{array}\right\}$

2. $\left.\begin{array}{l} a = \frac{10}{3} \text{ or } 3 \\ y = -\frac{17}{6} \text{ or } \frac{1}{2} \end{array}\right\}$

3. $\left.\begin{array}{l} p = \mp 3 \\ q = \pm 2 \end{array}\right\}$

4. $\left.\begin{array}{l} x = 2 \\ y = 5 \end{array}\right\}$

5. $\left.\begin{array}{l} a = \cdot 6 \text{ or } 1 \\ b = 2{\cdot}8 \text{ or } 2 \end{array}\right\}$

6. 3·63 or −2·3

7. ·68″

8. 5

9. 765 (the work is shortened by dividing by 25 6 straight away)

10. 19·7

11. 27·4

12. $\frac{2}{3}$

13. $-\frac{3}{28}$

14. 9·22

15 7·37

16. $m = \cdot 01277$; $n = \cdot 0026$

17. 3·9 or 15·8

18. 6·763

Exercises 10

1. 91·1 nautical miles

2. 22·63″

3. 70·7

5. 31′-9″

6. (*a*) 1 in 12·5; (*b*) 1 in 12·46

7. 15 ft

8. AB = 12·37′; BC = 12·15′; AC = 21·63′

9. 72·4

10. 26

11. 405 grey; 340 red

12. 480

13. 51·2

14. 132

15. 2·36″

16. 6110 lbs.

17. 8·66″, ·307

18. 5·11 tons

19. 65·8□″

20. £3 17*s.* 0*d*

21. 7·24 sq. ins.

22. (*b*) 12·25; (*c*) 4·0, (*d*) 5 86

Exercises 11

1. 3·44 sq. ins.

2. 10·2″; 52·1 sq. ins.

3. 6 sq ins.; 2·4 ins.

4. ·2374

5. 12·82 sq ins.

6. ·536

7. 58·5 sq. ft.; 18·07 ft.

8. 137·5 sq. ft.

9. 3 61 ft.

10. 301·5 sq. ft.

11. 13·25 sq. ins.

12. 47·7 sq. ins.

13. 69·5

14. ·52 amp.

15. 6·42 ins.

16. 9·13 sq. ins.

Exercises 12

1. 22·4″

2. 29·1 ft.

3. 161 sq. ft.

4. 1·46; 2·14; 3·33; ·35; ·06

5. 2·51″

6. 1·571″

7. 2·9 ft.

8. 18″

9. 1380 ft. per sec.

10. 31·23″

11. 8·92″

12. 66

13. 970

14. 5935

15. 27·13 sq. ins.

16. 3′-2¼″

17. ·196 sq. in.

18. 12080 lbs.

19. 4816 lbs.

20. 3·76 miles

21. 12·8″

22. 16·65 sq. ins.

23. 1010 sq ft.

24. ·0294 ohm

25. 2·68

26. 24·378″; 66 and 36

27. 214 sq. ins.

28. 60 ohms

Exercises 13

1. $c_1 = 4{\cdot}75''$; $h = {\cdot}36''$ **2.** $c_2 = 44{\cdot}72''$, $h = 20''$
3. $r = 62{\cdot}8''$; $h = 4{\cdot}97''$ **4.** 6·12″; 2·07 sq. ins.
5. 9·06 sq. ins.; 60° **6.** 1·11′; 3·67′
7. 2·11″; 3·33″; 1·29 sq. ins.; ·56″ **8.** 13·1
9. 1686 **10.** 29·8 sq. ins. **11.** 6076 **12.** 5·8″
13. $B = \sqrt{\frac{8}{3}RT}$ **14.** 141° **15.** 1′-26″ **16.** ·375″

Exercises 14

1. 4·5″; 36□″; 8 48″ **2.** 1·8″; 1″; 7·2□″; ·55″ from base
3. 2·48″; 12″ **4.** 47□″; 8″; 1·8″; 25 14″; 25·91″; 25·57″
5. 66·3□″; 3·88″; 29·6″; 29·75″; 29·3″ **6.** 334□″
7. 30·4 ft. tons; 3·38 ft. tons **8.** 623 yds.
9. 3″; $1\frac{1}{2}''$ **10.** 7000

Exercises 15

1. 689 cu. ft. **2.** ·28; 7·75 **3.** ·3125″ **4.** 144
5. ·833 **6.** 5·13 cu. ft. **7.** 6 91; 25900
8. 41·1 **9.** 52600 **10.** 47·94 cu. ft.; 6712 lbs.
11. 23·85 sq. ft.; 40·17 sq. ft., 19·3 cu ft. **12.** 70
13. ·53 (watt = volts × amps) **14.** 31·85 lbs **15.** 245 lbs.
16. 9 **17.** 851·5 sq. ft **18.** 14″ **19.** 15′-6″
20. 508000 **21.** 0006 **22.** 1 74″
23. $1{\cdot}83 \times 10^{-10}$ ohms **24.** 137 **25.** 12·55″ **26.** 3·503

Exercises 16

1. 593 cu. ins.; 321 sq. ins.; 13·55 ins. **2.** 20·4″
3. 200 sq ft. **4.** 13·4 cu. ins. **5.** 1592 lbs.
6. 26·1 ft.; 581 sq ft. **7.** 4 03″; 10 69″ **8.** 36·7 cu ins.
9. 14520 sq ft.; 70420 cu. ft. **10.** 773·8; 967·4 lbs
11. 173·8 sq. ins.; 234·3° **12.** 213 5 cu. ft.; 29,890 lbs.
13. 241 tons **14.** ·389″
15. 155 cu ins., 40·2 lbs. **16.** 559 cu ins., 243 sq. ins.
17. 4·63″ **18.** 2″, 5″; 6 12″
19. 105 sq. ins.; 138 sq. ins. **20.** 1159 sq. ins.; 2530 cu ins.

Exercises 17

1. 160 sq ins; 190·5 cu ins **2.** 8 3; 518 lbs. **3.** 9·057 cu ins
4. 8590 lbs. **5.** ·1033 **6.** 5·44 cms
7. 100·4 sq. ins.; 151 cu ins. **8.** 4 2″ **9.** 636 **10.** 558·5 sq. ft.
11. ·0941″ **12.** 1439 sq. yds. **13.** 1·082″ **14.** 15,520 sq. ft.
15. 7 59″ from vertex **16.** 104 6 sq ins. **17.** 14·7; 2·45
18. 53·51 acres **19.** 77″ **20.** 16·1, 47·3, 27·6, 27·6 sq. ins.
21. 406 lbs. **22.** 72 ft. **23.** 1·3″

Exercises 18

1. 175 sq. ft.; 99·8 cu. ft. **2.** 326 sq. ins.; 244 cu. ins.
3. 373 sq. ins.; 231 cu. ins. **4.** 136 sq. ins., 98·2 cu. ins.
5. Paraboloid = $\frac{1}{2}$ cylinder **6.** 90·2 cu. ins. **7.** 2·02 lbs.

Exercises 19

1. 6·44 lbs. **2.** 2630 lbs. **3.** 1278 lbs. **4.** 960 lbs.
5. 272 lbs. **6.** 372 **7.** 1·16 tons **8.** 171 lbs.
9. 1·84 lbs. **10.** 761 lbs. **11.** 45·5 lbs. **12.** 10·25 tons
13. 5·08 lbs. **14.** 19·55 lbs. **15.** 28·2 lbs. **16.** 93·5 lbs.
17. 3·59 lbs. **18.** 258 lbs. **19.** 6·47 lbs.

Exercises 20

1. ·7″; 74·5 tons **2.** 30800; 650; 25000 **3.** 420 lbs. per min
4. 3400; $1\frac{3}{8}''$ **5.** 1440 **8.** 339; 55 **11.** 11850 tons
12. ·317 **13.** 55 mins.; 45° **14.** 63 mins.; 42°
22. 54000 lbs./□″ **24.** ·8% low **28.** 4·10 o'c.

Exercises 21

2. 250 **4.** Slope = ·375; intercept = −2·375 **8.** 5·78″
11. Slope = 2·5 if V is plotted along horizontal
12. $m = 1{\cdot}8$, $n = 2{\cdot}6$ **13.** $x = 1$, $y = 0$ **14.** $x = 1{\cdot}24$, $y = 3{\cdot}59$
15. $x = {\cdot}43$, $y = 2{\cdot}33$ **16.** $x = 3{\cdot}18$, $y = -4{\cdot}75$ **17.** 55°

Exercises 22

1. ·392 **2.** ·31 **3.** $30{\cdot}2 \times 10^6$ **4.** $17{\cdot}9 \times 10^6$
5. $I = {\cdot}862B + 4{\cdot}53$ **6.** $d_1 = {\cdot}84d - {\cdot}03$ **7.** $d_1 = {\cdot}95d - {\cdot}07$
8. $T = 51{\cdot}7\theta + 7$ **9.** $T = 3530\theta$ **10.** $R = {\cdot}78V + 86$
11. $R = {\cdot}784V + 63{\cdot}8$ **12.** $R = 2{\cdot}5V + 75$ **13.** $R_o = 1$, $a = {\cdot}004$
14. $R_o = 1{\cdot}125$, $a = {\cdot}00452$ **15.** $I = {\cdot}00232x - 96$
16. 32 **17.** $29{\cdot}25 \times 10^6$

Exercises 23

1. Vertex downward **2.** Vertex upward
7. Total weight $= 50l + 5l^2$ **14.** 6 or −1 **15.** −2·67 or 3 5
16. 1·44 or −7·65 **17.** Divide throughout by 10^4: − 9·22 or ·12
18. 17·1 **19.** (a) 4·9″, (b) 5·5″ **20.** $x = 3{\cdot}64'$; $h = 1{\cdot}83'$
21. 7·7 air to 1 of gas **22.** 5·5 **23.** $e = 55$, effy. = ·5
24. 15·22 knots; £946·4 **25.** 40°; ·69 **26.** 2·1
27. Assume some value for v: $u = \frac{v}{2}$; 1 **28.** 8·33; $\frac{8}{27}$
29. 2″ **30.** 2 rows of 8 **31.** 6 **34.** 1, − 2 or 1·5
35. ·2, 2·25 or −3 **36.** 1·2, −4·6 or −1·6
37. ·2, ·5 or −·8 **38.** max^m at $x = -3$, min^m at $x = +2$
39. $x = {\cdot}211l$ **40.** 1·475 **41.** 5·6° **42.** 1·3

Exercises 24

1. 88·1 lbs. **2.** 2·37 **3.** $v = 66{\cdot}3\sqrt{r}$; 1195
4. 10·89 tons **5.** 10970 lbs. **6.** 5″ **7.** ·028 cm.
8. 2·3 pence **9.** 13·75 ohms **10.** 246 **11.** 80
12. 22 knots **13.** 841 **14.** 4·27 **15.** 533
16. ·01088″ **17.** Cost $\propto \frac{M}{h}$

Exercises 25

1. 28; 72 **2.** 20th **3.** — 105 **4.** 3·7, 4·6 . . . 10
5. 12, 15, 18 or 9, 16·5, 24 **6.** 24 **7.** £1 15*s*. 6*d*.; £377 10*s*. 0*d*.
8. £10; 8160 **9.** 15·57 ft. **10.** 592 ft.; 16 secs.
11. 3·15 p.m. **12.** 2·074 **13.** 53·33 **14.** 2, 3, $4\frac{1}{2}$
15. 835·2 **16.** 5·5 **17.** 10·081; ·821 **18.** 20 lbs.
19. 7·52, 18, etc. **20.** $a = 2$, $b = 0$, $c = 1$; 8040
21. 25 days **22.** ·983 in.
23. 4·284; 6·116; 8·734; 12·48; 17·8; 25·43; 36·31; 51·84; 74·03; 105·7

Exercises 26

1. 3·0643; 1·1569; $\bar{2}$·1921 **2.** $\bar{5}$·0738; 4·2842; 1.4008
3. 0; ·0812; ·1946; ·285; ·5512, ·7744 **4.** 1 301 **5.** 924·3
6. ·09877 **7.** ·00005445 **8.** 4·612 **9.** 264
10. 1·086 **11.** 1546 **12.** 11·03 **13.** ·07784
14. $3·3 \times 10^{-26}$ **15.** 26, 560 **16.** 47·2 **17.** 75·4
18. 370 **19.** 38·2 **20.** ·123; ·109 **21.** ·401
22. ·1518 **23.** ·0336 **24.** ·475 **25.** ·0391
26. ·528 **27.** $\phi_w = ·391$, $\phi_s = 1·796$ **28.** 4000 **29.** ·638
30. 24·3 % **31.** ·296 **32.** ·325 **33.** ·103
34. 357 **35.** 4·48 **36.** ·0003336 **37.** 38 4
38. 8·51 **39.** 4·44 **40.** 71·5 **41.** ·65
42. $\frac{T}{t} = 1·875$; $T = 445$, $t = 237$ **43.** $pv^{1·06} = 392$ **44.** 1·47
45. 0 or 7·28 **46.** 2·16 **47.** 0 or 1·368 **48.** ·0955
49. 481·5 **50.** ·033 **51.** $v = 222\sqrt{H}$ **52.** ·01895
53. 4·6 **54.** 5380 **55.** $1·48 \times 10^8$ **56.** ·605
57. 1·44 **58.** 7965 **59.** 61·6 **60.** ·2
61. $n = 1·405$; $C = 502$ **62.** 81300 **63.** 47610 **64.** 1 115

65.

Thl Disch.	C_d
53	·66
118	·672
171	·658

66.

Thl Disch	C_d
122	·728
154·7	·7086
183	·727
220·4	·711

67. $r = ·00356v^{1·95}$ **68.** $h = ·1538v^{1·945}$
69. 3·07 {Take logs of both equations and solve as a pair of simultaneous equations.}
70. 8·41 **71.** $F = ·00277V^{1·9}$

Exercises 27

1. ·8746; ·3443; ·0523; ·2309
2. ·9756; 9641; 1 6139; ·2837 **3.** 3·817 **4.** ·3966
5. 17°15′ **6.** 252·3; 0 (the case of wattless current)
7. 33° 42′ **8.** 143 3 **9.** 20° **10.** 28° 6′ **11.** 181
12. 52·8 **13.** 1340 **14.** 6764 **15.** 3150 **16.** 823 3
17. ·00305 **18.** 61,200 ft. **19.** 23·4 **20.** 26°8′
21. 455 **22.** ·8923 **23.** 16850 **24.** 25°1′

Exercises 28

1. $a = 11{\cdot}65''$; $b = 43{\cdot}47''$
2. $b = 8{\cdot}72''$; $c = 14{\cdot}83''$
3. $a = 48{\cdot}3''$; $b = 43{\cdot}5''$
4. $a = 66{\cdot}73''$; $c = 74{\cdot}88''$
5. $a = 22{\cdot}14''$; $b = 16{\cdot}08''$
6. $a = 57{\cdot}66''$; $c = 92{\cdot}63''$
7. $a = 20{\cdot}8''$; $b = 10{\cdot}72''$
8. 6037 ft.; 2927 ft.
9. $8° 8'$
10. 30·6 ft.
11. $78°$
12. $14° 41'$
13. $2''$
14. (a) $14° 16'$; (b) $20° 53'$
15. $2{\cdot}38''$
16. $28°56'$
17. AB = 50×AD
18. A = 0, 50; B = 33·9, 77·8; C = 74·6, 20·5; D = 15·2, 13
    R.B. of BC = 35·5° S.E.
    R.B. of CD = 82·5° S.W.
    R.B. of DA = 23° N.W.
    Area = 2700 □′
19. A = 10, 20; B = 19·05, 14·8; C = 12·58, 12·06
    R.B. of BC = 67° S W.
    R.B. of CA = 18° N.W.
    Area = 29
20. 3 chns. 49 links
21. $235° 1'$
22. 73·6 ft.
23. $\cdot 2901''$
24. $\cdot 121''$

Exercises 29

1. ·8988, −·4384, −2·0503; −·6157, +·7880, −·7813; −·6157, −·7880, +·7813
2. −·9903, +·1392, −7·1154; −·6157, +·7880, −·7813; +·8480, −·5299, −1·6003
3. −·3289, −·9444, +·3482; −·3242, +·9460, −·3427; −·9953, −·0979, +10·17
4. −·7570, −·6534, +1·1585; −·8111, +·5850, −1·3865; ·7513, ·6600, 1·1383; ·9171, ·3987, 2·2998
5. −·9135, −·4067, +2·2460; −·6374, −·7705, +·8273; ·9218, ·3877, 2·3772
6. −·7265; ∞; ·1625
7. $124°54'$
8. $120°55'$
9. $119°30'$
10. $149°50'$ or $210°10'$

Exercises 30

1. $c = 4{\cdot}89''$, A = $34°25'$, C = $67°5'$
2. A = $80°52'$, $b = 59{\cdot}46''$, $c = 63{\cdot}04''$
3. B = $44°46'$ or $135°14'$, A = $108°24'$ or $17°56'$, $a = 11{\cdot}93''$ or $3{\cdot}87''$
4. B = $40°42'$, $a = 8{\cdot}84''$, $c = 8{\cdot}25''$
5. A = $53°43'$ or $126°17'$, B = $80°17'$ or $7°43'$, b = 12·61 or 1·72
6. a = 9·54 ft., B = $37°47'$, C = $68°57'$
7. A = $80°6'$, B = $48°18'$, C = $51°36'$
8. $c = 21{\cdot}97''$, B = $21°29'$, A = $28°31'$
9. B = $41°42'$ or $138°18'$, C = $109°18'$ or $12°42'$, $c = 8{\cdot}31''$ or $1{\cdot}94''$
10. C = 42° or 138°, A = 108° or 12°, a = 9779 or 2138
11. C = $69°40'$, B = $59°30'$, a = 830
12. A = $103°33'$ or $5°27'$, C = $40°57'$ or $139°3'$, a = 64·62 or 6·311

13. A = 86°31′, B = 45°57′, c = 16·11
14. C = 43°9′, A = 81°21′, a = 47·28; area = 637
15. 19·46 ft. **16.** 8° 30′
17. 55°, 87°, 38° **18.** 18·75 lbs.; 58°
19. OB = 1·224 chns., OC = ·3236 chn., BE = ·7673 chn., CF = 1·667 chns.
20. 56°36′ **21.** Jib = 26·2 ft.; tie = 17·4 ft. **22.** 1191 ft.
23. 647 ft.; 374 ft. **24.** 53·2 ft. **25.** 2083 ft.
26. AB = 2983 links; 767 links **27.** BG = 74·96 chns.; CH = 74·14
28. $\left.\begin{array}{l}\text{AB} = 527{\cdot}4\\ \text{DC} = 475{\cdot}3\\ \text{AC} = 767{\cdot}4\\ \text{BC} = 774{\cdot}6\end{array}\right\}$ links
29. $\left.\begin{array}{l}\text{AB (horiz.)} = 607{\cdot}5 \text{ yds.}\\ \text{Diff. of level} = 129{\cdot}3 \text{ yds.}\end{array}\right\}$
30. $\left.\begin{array}{l}r = 473{\cdot}3'\\ \text{BE} = \text{BF} = 126{\cdot}7'\\ \text{CF} = \text{CG} = 473{\cdot}3'\end{array}\right\}$
31. $\left.\begin{array}{l}\text{AP} = 983' \text{ (AC} = 1180')\\ \text{BP} = 967'\\ \text{CP} = 919'\end{array}\right\}$
32. 1233 f p.s.; 29° 36′ **33.** diam. = ·506 p **34.** 106·4°; 93′
35. 9·06 to 1 **36.** 60·3 sq. ft.; 422 lbs. **37.** 7·46″; 10·65″
38. 10·3″; 14·5″ **39.** 244 sq. ft.
40. 2286; 2912 **41.** 17·25 sq. ins.

Exercises 31

1. cos A = ·893, tan A = ·504 **2.** ·898
3. cos (A+B) = ·442, sin (A−B) = ·298
4. tan (A+B) = 36·7, tan (A−B) = ·536
5. $\dfrac{\mu - \tan a}{1 + \mu \tan a}$; 3·21 **6.** $P = \dfrac{W(p + 2\pi r\mu)}{2\pi r - p\mu}$
7. P = W (sin a + μ cos a) **8.** 1·162
9. 4·99 sin (5t + 1) **10.** 238·5 sin (50t − ·576)
11. $R = \dfrac{2V^2 \sin\theta \cos(\theta + A)}{g \cos^2 A}$ **12.** ·189; 10°42′
13. E = 121·6 sin (120πt + 1·022)

Exercises 32

1. cos 2A = ·566; tan 2A = 1·455
2. sin 2A = ·7962; cos 2A = ·605 **3.** $\frac{1}{2}(1 + \cos 28°)$
4 sin 2B = 2 cos B$\sqrt{1-\cos^2 B}$, ·731
5. $\sin\dfrac{A}{2}$ = ·161, $\cos\dfrac{A}{2}$ = ·987; sin 3A = ·8236
6. cos 4A = ·616; $\tan\dfrac{A}{2}$ = ·114 **7.** 2·5(1 − cos 4t)
8. 7·85 (sin 189° − sin 131°) = 7 85 (− sin 9° − sin 49°)
9. 2 sin 9t {cos 6t + sin 2t} **10.** $\left.\begin{array}{l}\sin A = {\cdot}261;\ \tan A = {\cdot}270\\ \cos\dfrac{A}{2} = {\cdot}991\end{array}\right\}$
11. 50s × sin 2a; 53
12. (a) 2 cos 32°30′ sin 15°30′; (b) − 2 cos 42°30′ cos 38°30′; (c) 24 sin 50° sin 45°
13. 25·91; 63·73 **14.** ·333 or − 1·25
15. 9·4 {·995 − cos (4πft − ·117)}

Exercises 33

1. $30°$ or $150°$ **2.** $45°, 135°, 225°$ or $315°$
3. $120°$ or $240°$ **4.** $8°8'$ or $188° 8'$, $153°26'$ or $333°26'$
5. $120°$ or $240°$ **6.** $53°8'$, $191°32'$, $126°52'$ or $348°28'$
7. $30°$ or $150°$, $210°$ or $330°$ **8.** $45°$, $71°34'$, $225°$ or $251°34'$
9. $30°$ or $150°$ **10.** $35°45'$ or $144°15'$
11. $45°$, $225°$, $161°34'$ or $341°34'$ **12.** $45°$ or $225°$
13. $45°$, $225°$, $18°26'$ or $198°26'$ **14.** $30°$, $150°$, $210°$ or $330°$
15. 0 or $45°$ **16.** 0 or $120°$ **17.** $27°4'$ or $243°30'$
18. $48°56'$ or $156°39'$ **19.** $146°19$ or $326°19'$
20. 0; 180; $20°56'$ or $339°4'$ **21.** $5°7'$

Exercises 34

1. 1·2552; 2·1293 **2.** 5·0018 **3.** 151 ft.
4. 40·54 ft.; 156·6 ft. **5.** $\mathrm{E} \cosh x \sqrt{\frac{r}{r_1}} + \sqrt[3]{r^2 r_1^4} \sinh x \sqrt[8]{\frac{r}{r_1}}$
6. $10°45'$ **7.** ·00383 **8.** ·93 **9.** $18°52'$ **11.** ·86 **12.** 19·4
14. 44·09 **15.** $1·928 + 2·298j$

Exercises 35

1. 318 **2.** 51·7 lbs. per sq. in. **3.** 38·35 sq. ft.; 575 cu. ft
4. 765 sq. ft. **5.** 7231 sq. ft. **6.** 269 ft.
7. 430 sq. ft.; 2190 cu. ft. per sec. **8.** 6850 sq. ft.
9. 730 **10.** 8·72 sq. ins. **11.** 60·5 lbs. per sq. in.

Exercises 36

1. 168750 cu. yds. **2.** 8350 tons **3.** 244000 tons
4. 44920 sq. yds. **5.** $5·21 \times 10^6$ galls. **6.** 40 ft.; 51·43 ft.
7. 12020 cu. yds. **8.** 96·6 ft.; 61·9 ft.; 11600 cu. yds.
9. 26·5 ft.; 17·66 ft.; 33·5 ft.; 22·35 ft.; 27·5 ft.; 18·33 ft.; 184, 325, 202 sq. ft.; 1375 cu. yds.
10. 28·25 and 43·3 ft. from the centre line

Exercises 37

14. The table of values would be arranged thus:—

θ	$\log \sin \theta$	$\cos \theta$	$1·84 \cos \theta - 1 = A$	$A \log \sin \theta + \log P$	$\log p$	p

15. Treat $\frac{1}{1100}\left(1 - \frac{1}{e^{\mu\theta}}\right)$ as a constant multiplier

16. Values of R and V are as follows:—

V	0	10	20	30	40	55
R	2·5	3·21	4·74	6·9	9·61	14·6

17. 2·9 **18.** Values of r and η are as follows:—

r	2	3	5	7	10	12
η	·962	·968	·961	·947	·934	·932

19. latus rectum $= 2\cdot5$; vertex is at $(2\cdot75, -8\cdot42)$
20. $4\cdot27$ tons per sq. in.; $23°$

Exercises 38

1. $-\cdot404$ **6.** $62\cdot2$ lbs.

8. Plot $E = \cosh x$ (x ranging from 0 to $500\sqrt{gr}$) and then alter both scales

Exercises 39

1. Amplitudes :— $8;\ \cdot2;\ 51\cdot8;\ \cdot116;\ \cdot91$

Periods :— $\frac{\pi}{2};\ \frac{2\pi}{3};\ \cdot02;\ \cdot0102;\ 36\cdot9$

2. Amplitude $= \cdot4$
Period $= \pi$

15. Period as for cosine curve : $\frac{2\pi}{3}$ or $120°$

Exercises 40

1. {Assume some convenient value for l}. $x = \cdot403l$
2. $z = 5\cdot3$ **3.** $1\cdot221$ **4.** $4\cdot58$ **5.** $\cdot36$ or $2\cdot17$
6. $1\cdot9$ or $-2\cdot45$ **7.** $2\cdot79$ **8.** $\cdot143$ or $\cdot333$
9. $2\cdot66$ **10.** $5\cdot37$ ($308°$) **11.** $l = \cdot35L$ **12.** $7\cdot876$
13. $4\cdot49$ rad. ($257°$) **14.** $10\cdot42$, 13 **15.** $5\cdot523'$
16. $1\cdot484'$ **17.** $2\cdot9$ **18.** $6\cdot005''$ **19.** $79°6'34''$

20. $1\cdot87$. (Plot the curves $y_1 = \cosh x$ and $y_2 = -\frac{1}{\cos x} = -\sec x$ and note the point of intersection.) **21.** $6\cdot34$ ft.

Exercises 41

2. $\cdot454$ **3.** $\cdot334$; 560 **7.** $1\cdot043$; $1\cdot077$

9. $\cdot22$. [*Hint.*—Let $\phi = a + b\tau$, also $\phi = \log_e \frac{\tau}{461} + \frac{q(1437-\tau)}{\tau}$; and solve for q.]

Exercises 42

1. $W = 47 + 60\ 5A$ **2.** $\mu = \cdot2 + \cdot004\sqrt{v}$ **3.** $W = 3\ 28d^2$
4. $m = \cdot41 + \frac{\cdot0066}{H}$ **5.** $l = \cdot0148a^2$ **6.** $W = 1\ 1d^3 + 18$
7. $S = 10\ 91t^{1\ 51}$ **8.** $H = \cdot0955v^{3\ 11}$ **9.** $T = 435\theta^{\cdot262}$
10. $d = 1\cdot2\sqrt{l}$ **11.** $\tau = 541p^{\ 079}$ **12.** $h = \cdot0724v^{1\ 8}$
13. $T = 1\cdot29 \times 10^{-7}\,.\,n^{2\ 46}$ **14.** $Q = 6\ 11H^{\cdot46}$ **15.** $v = 224\sqrt{H}$
16. $l = 10t^2$ **17.** $a = 1300$, $b = 52\cdot3$ **18.** $a = 1620$, $b = 50\cdot9$
19. $h = \frac{d^2}{3\cdot5 \times 10^6}$ **20.** $n = \ 87$, $C = 205$
21. $c = 14\cdot9$, $b = \cdot58$, $a = -\cdot02$
22. $E = \cdot1 + \cdot0132T - \cdot0000583T^2$
23. $E = -\cdot15 + \ 00795T - \cdot0000021T^2$
24. $A = 192\cdot8 - 4\cdot395V + \cdot027V^2$
25. $R = 160 - 16\cdot4V + \cdot4V^2$ **26.** $v = 3\ 195 + \cdot452D - \cdot77D^2$
27. $a = 10$, $b = \cdot277$ **28.** $\cdot2$ **29.** $\cdot3$ **30.** $\cdot4$
31. $y = 18e^{\cdot23x}$ **32.** $Q = 1\cdot5H^{2\ 5}$ **33.** $y = \frac{x}{2\cdot6 - \cdot3x}$

Exercises 43

3. $\cdot 115''$; $92\cdot 6$ lbs./□″
5. Write equation—
$\log d - \log 2\cdot 9 = \frac{1}{3}\log \text{H} - \frac{1}{3}\log \text{N}$ or $\overline{\text{D}} = \frac{1}{3}\overline{\text{H}} - \frac{1}{3}\overline{\text{N}}$

Exercises 44

1. $8,\ 8\frac{1}{10},\ 8\frac{1}{11},\ 8\frac{11}{120}$; $\cdot 009\%$ too low   2. $\frac{2947}{6930}$

3. $\frac{1}{10+}\ \frac{1}{2+}\ \frac{1}{15+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{7}$; $\frac{31}{325}$

4. $\frac{1}{12+}\ \frac{1}{4+}\ \frac{1}{4+}\ \frac{1}{1+}\ \dots$

5. $6\frac{32}{55}$, say 6 complete turns with 19 holes on 33 hole circle
6. 8 complete and 2 holes on 18 hole circle (approx.)
7. 10 holes on 17 hole circle   8. 50 to 127

9. $\frac{1}{x+1} + \frac{2}{x+6}$ 10. $\frac{4}{2x+3} - \frac{6}{3x+5}$ 11. $\frac{2}{x-11} + \frac{1}{x+8}$

12. $1 - \frac{2}{x-1} + \frac{6}{x-2}$ 13. $\frac{3}{x+4} + \frac{5}{x-5} - \frac{2}{x-2}$

14. $\frac{4}{2x+7} - \frac{5}{x-3} - \frac{2}{3(x+2)}$ 15. $\frac{5}{6(x-1)} + \frac{1}{2(x+1)} - \frac{4}{3(x+2)}$

16. $\frac{1}{7(x-3)} + \frac{8-x}{7(x^2+3x+3)}$ 17. $-\frac{6}{7}$ 18. $\frac{7}{6}$

20. 32   21. 48   22. 1   23. 1   24. $\frac{4}{5}b^2$

Exercises 45

1. $35a^3b^4$ 2. $2048a^{11} + 56320a^{10}c + 704000a^9c^2 + 5\cdot 29 \times 10^6a^8c^3$

3. $-717255x^4y^{19}$ 4. $\frac{m^8}{256} - \frac{m^7n}{40} + \frac{7m^6n^2}{100} - \frac{14m^5n^3}{125}$

5. $\frac{1}{a^2} + \frac{4}{a^3} + \frac{12}{a^4} + \frac{32}{a^5} + \frac{80}{a^6}$ 6. $\frac{210}{x^6}$ 7. $2^{\frac{1}{2}} - \frac{x^2}{2^{\frac{3}{2}}} - \frac{x^4}{2^{\frac{9}{2}}}$

8. $\frac{1}{3^{\frac{2}{3}}\cdot a^{\frac{2}{3}}} - \frac{8c}{3^{\frac{8}{3}}\cdot a^{\frac{5}{3}}} + \frac{80c^2}{3^{\frac{14}{3}}\cdot a^{\frac{8}{3}}} - \frac{2560c^3}{3^{\frac{20}{3}}\cdot a^{\frac{11}{3}}}$ 9. $\frac{48b^2}{25a^{\frac{11}{6}}}$

10. $1 - \frac{a^2\sin^2\theta}{2l^2} - \frac{a^4\sin^4\theta}{8l^4} - \frac{a^6\sin^6\theta}{16l^6}$; $1 - \frac{a^2}{2l^2}\sin^2\theta$

12. (*a*) 20.19.18 . . . 10.9, (*b*) 125970
13. 105; 11880; 120   14. $\cdot 984$ 15. $2\cdot 074$
16. (*a*) $1\cdot 01 \times 10^{15}$; (*b*) 9545; (*c*) $\cdot 73$; (*d*) 40 92

18. $\cosh x = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \dots$; $\sinh x = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \dots$

19. $1\cdot 386$ 20. $1 - \frac{y}{\text{R}} + \frac{y^2}{\text{R}^2} - \frac{y^3}{\text{R}^3}$; $1 - \frac{y}{\text{R}}$ 22. $3\cdot 142$

Exercises 46

1. -45 2. 5904   3. 0   4. -1728 5. 795
6. $\cdot 4372$ or $-2\cdot 449$ 7. $x = 5,\ y = 6$ 8. $a = -1\cdot 5,\ b = 8$
9. $x = 5,\ y = 4,\ z = -2$ 10. $a = 1\cdot 2,\ b = 5\cdot 7,\ c = -4\cdot 8$

MATHEMATICAL TABLES

TABLE I.—Trigonometrical Ratios

Angle. Degrees.	Angle. Radians.	Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
0°	0	0	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14 3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11 4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9 5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1 3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3 0777	·9511	1 176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1 161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1 147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1 133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1 1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1 075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1 060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1 045	1·0996	63
28	·4887	·484	·4695	·5317	1 8807	·8829	1·030	1 0821	62
29	·5061	·501	4848	·5543	1 8040	8746	1 015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1 6643	8572	·985	1 0297	59
32	·5585	·551	·5299	·6249	1 6003	8480	970	1 0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1 4281	8192	923	·9599	55
36	·6283	618	·5878	7265	1 3764	8090	·908	9425	54
37	·6458	·635	·6018	·7536	1 3270	·7986	·892	·9250	53
38	·6632	·651	·6157	7813	1 2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	7771	861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	8203	47
44	·7679	·749	·6947	·9657	1 0355	·7193	·781	8029	46
45°	·7854	·765	·7071	1·0000	1 0000	·7071	·765	7854	45°
			Cosine	Co-tangent	Tangent	Sine	Chord	Radians	Degrees
								Angle	

TABLE II.—Logarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 4	9 8	13 12	17 16	21 20	26 24	30 28	34 32	38 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 4	8 7	12 11	15 15	19 19	23 22	27 26	31 30	35 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 3	7 7	11 10	14 14	18 17	21 20	25 24	28 27	32 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 3	7 7	10 10	13 12	16 16	20 19	23 22	26 25	30 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 3	6 6	9 9	12 12	15 15	18 17	21 20	24 23	28 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 3	6 5	9 8	11 11	14 14	17 16	20 19	23 22	26 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 3	5 5	8 8	11 10	14 13	16 15	19 18	22 21	24 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 2	5 5	8 7	10 10	13 12	15 15	18 17	20 19	23 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 2	5 5	7 7	9 9	12 11	14 14	16 16	19 18	21 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 2	4 4	7 6	9 8	11 11	13 13	16 15	18 17	20 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

TABLE II. (*contd.*)

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE III.—Antilogarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	[illegible]	1	2	3	3	4	5	5	6
48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	[illegible]	1	2	3	4	4	5	6	6
49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

TABLE III. (*contd*).

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8172	8492	2	4	6	8	10	12	14	15	17
93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	[illegible]	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	[illegible]	9441	9462	9484	9506	[illegible]	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TABLE IV.—Napierian, Natural, or Hyperbolic Logarithms

Number.	0	1	2	3	4	5	6	7	8	9	Mean Differences. 1	2	3	4	5	6	7	8	9
0·1	$\bar{3}$·6974	7927	8797	9598	$\bar{0}$339	$\bar{1}$029	$\bar{1}$674	$\bar{2}$280	$\bar{2}$852	$\bar{3}$393									
0·2	$\bar{2}$·3906	4393	4859	5303	5729	6137	6529	6907	7270	7621									
0·3	·7960	8288	8606	8913	9212	9502	9783	$\bar{0}$057	$\bar{0}$324	$\bar{0}$584									
0·4	$\bar{1}$·0837	1084	1325	1560	1790	2015	2235	2450	2660	2866									
0·5	3068	3267	3461	3651	3838	4022	4202	4379	4553	4724									
0·6	4892	5057	5220	5380	5537	5692	5845	5995	6143	6289									
0·7	6433	6575	6715	6853	6989	7123	7256	7386	7515	7643									
0·8	7769	7893	8015	8137	8256	8375	8492	8607	8722	8835									
0·9	8946	9057	9166	9274	9381	9487	9592	9695	9798	9899									
1·0	0·0000	0100	0198	0296	0392	0488	0583	0677	0770	0862									
1·1	0953	1044	1133	1222	1310	1398	1484	1570	1655	1740	9	17	26	35	44	52	61	70	78
1·2	1823	1906	1989	2070	2151	2231	2311	2390	2469	2546	8	16	24	32	40	48	56	64	72
1·3	2624	2700	2776	2852	2927	3001	3075	3148	3221	3293	7	15	22	30	37	45	52	59	67
1·4	3365	3436	3507	3577	3646	3716	3784	3853	3920	3988	7	14	21	28	35	41	48	55	62
1·5	4055	4121	4187	4253	4318	4383	4447	4511	4574	4637	6	13	19	26	32	39	45	52	58
1·6	4700	4762	4824	4886	4947	5008	5068	5128	5188	5247	6	12	18	24	30	36	42	48	55
1·7	5306	5365	5423	5481	5539	5596	5653	5710	5766	5822	6	11	17	24	29	34	40	46	52
1·8	5878	5933	5988	6043	6098	6152	6206	6259	6313	6366	5	11	16	22	27	32	38	43	49
1·9	6419	6471	6523	6575	6627	6678	6729	6780	6831	6881	5	10	15	20	26	31	36	41	46
2·0	6931	6981	7031	7080	7129	7178	7227	7275	7324	7372	5	10	15	20	24	29	34	39	44
2·1	7419	7467	7514	7561	7608	7655	7701	7747	7793	7839	5	9	14	19	23	28	33	37	42
2·2	7885	7930	7975	8020	8065	8109	8154	8198	8242	8286	4	9	13	18	22	27	31	36	40
2·3	8329	8372	8416	8459	8502	8544	8587	8629	8671	8713	4	9	13	17	21	26	30	34	38
2·4	8755	8796	8838	8879	8920	8961	9002	9042	9083	9123	4	8	12	16	20	24	29	33	37
2·5	9163	9203	9243	9282	9322	9361	9400	9439	9478	9517	4	8	12	16	20	24	27	31	35
2·6	9555	9594	9632	9670	9708	9746	9783	9821	9858	9895	4	8	11	15	19	23	26	30	34
2·7	9933	9969	$\bar{0}$006	$\bar{0}$043	$\bar{0}$080	$\bar{0}$116	$\bar{0}$152	$\bar{0}$188	$\bar{0}$225	$\bar{0}$260	4	7	11	15	18	22	26	29	33
2·8	1·0296	0332	0367	0403	0438	0473	0508	0543	0578	0613	4	7	11	14	18	21	25	28	32
2·9	0647	0682	0716	0750	0784	0818	0852	0886	0919	0953	3	7	10	14	17	20	24	27	31
3·0	0986	1019	1053	1086	1119	1151	1184	1217	1249	1282	3	7	10	13	16	20	23	26	30
3·1	1314	1346	1378	1410	1442	1474	1506	1537	1569	1600	3	6	10	13	16	19	22	25	29
3·2	1632	1663	1694	1725	1756	1787	1817	1848	1878	1909	3	6	9	12	15	18	21	25	28
3·3	1939	1969	2000	2030	2060	2090	2119	2149	2179	2208	3	6	9	12	15	18	21	24	27
3·4	2238	2267	2296	2326	2355	2384	2413	2442	2470	2499	3	6	9	12	14	17	20	23	26
3·5	2528	2556	2585	2613	2641	2669	2698	2726	2754	2782	3	6	8	11	14	17	20	22	25
3·6	2809	2837	2865	2892	2920	2947	2975	3002	3029	3056	3	5	8	11	14	16	19	22	25
3·7	3083	3110	3137	3164	3191	3218	3244	3271	3297	3324	3	5	8	11	13	16	19	21	24
3·8	3350	3376	3403	3429	3455	3481	3507	3533	3558	3584	3	5	8	10	13	16	18	21	23
3·9	3610	3635	3661	3686	3712	3737	3762	3788	3813	3838	3	5	8	10	13	15	18	20	23
4·0	3863	3888	3913	3938	3962	3987	4012	4036	4061	4085	2	5	7	10	12	15	17	20	22
4·1	4110	4134	4159	4183	4207	4231	4255	4279	4303	4327	2	5	7	10	12	14	17	19	22
4·2	4351	4375	4398	4422	4446	4469	4493	4516	4540	4563	2	5	7	9	12	14	16	19	21
4·3	4586	4609	4633	4656	4679	4702	4725	4748	4770	4793	2	5	7	9	11	14	16	18	21
4·4	4816	4839	4861	4884	4907	4929	4951	4974	4996	5019	2	4	7	9	11	13	16	18	20
4·5	5041	5063	5085	5107	5129	5151	5173	5195	5217	5239	2	4	7	9	11	13	15	18	20
4·6	5261	5282	5304	5326	5347	5369	5390	5412	5433	5454	2	4	6	9	11	13	15	17	19
4·7	5476	5497	5518	5539	5560	5581	5602	5623	5644	5665	2	4	6	8	11	13	15	17	19
4·8	5686	5707	5728	5748	5769	5790	5810	5831	5851	5872	2	4	6	8	10	12	14	16	19
4·9	5892	5913	5933	5953	5974	5994	6014	6034	6054	6074	2	4	6	8	10	12	14	16	18
5·0	6094	6114	6134	6154	6174	6194	6214	6233	6253	6273	2	4	6	8	10	12	14	16	18

TABLE IV (*contd.*)

0	1	2	3	4	5	6	7	8	9	Mean Differences. 1	2	3	4	5	6	7	8	9
·6292	6312	6332	6351	6371	6390	6409	6429	6448	6467	2	4	6	8	10	12	14	16	18
6487	6506	6525	6544	6563	6582	6601	6620	6639	6658	2	4	6	8	10	12	13	15	17
6677	6696	6715	6734	6752	6771	6790	6808	6827	6846	2	4	6	8	9	11	13	15	17
6840	6883	6901	6919	6938	6956	6975	6993	7011	7029	2	4	6	7	9	11	13	15	17
7048	7066	7084	7102	7120	7138	7156	7174	7192	7210	2	4	5	7	9	11	13	14	16
7228	7246	7263	7281	7299	7317	7334	7352	7370	7387	2	4	5	7	9	11	12	14	16
7405	7422	7440	7457	7475	7492	7509	7527	7544	7561	2	4	5	7	9	10	12	14	16
7579	7596	7613	7630	7647	7664	7682	7699	7716	7733	2	3	5	7	9	10	12	14	15
7750	7767	7783	7800	7817	7834	7851	7868	7884	7901	2	3	5	7	8	10	12	13	15
7918	7934	7951	7968	7984	8001	8017	8034	8050	8067	2	3	5	7	8	10	12	13	15
8083	8099	8116	8132	8148	8165	8181	8197	8213	8229	2	3	5	7	8	10	11	13	15
8246	8262	8278	8294	8310	8326	8342	8358	8374	8390	2	3	5	6	8	10	11	13	14
8406	8421	8437	8453	8469	8485	8500	8516	8532	8547	2	3	5	6	8	10	11	13	14
8563	8579	8594	8610	8625	8641	8656	8672	8687	8703	2	3	5	6	8	9	11	12	14
8718	8733	8749	8764	8779	8795	8810	8825	8840	8856	2	3	5	6	8	9	11	12	14
8871	8886	8901	8916	8931	8946	8961	8976	8991	9006	2	3	5	6	8	9	11	12	14
9021	9036	9051	9066	9081	9095	9110	9125	9140	9155	2	3	4	6	7	9	10	12	13
9169	9184	9199	9213	9228	9243	9257	9272	9286	9301	2	3	4	6	7	9	10	12	13
9315	9330	9344	9359	9373	9387	9402	9416	9431	9445	1	3	4	6	7	9	10	12	13
9459	9473	9488	9502	9516	9530	9545	9559	9573	9587	1	3	4	6	7	9	10	11	13
9601	9615	9629	9643	9657	9671	9685	9699	9713	9727	1	3	4	6	7	8	10	11	13
9741	9755	9769	9782	9796	9810	9824	9838	9851	9865	1	3	4	6	7	8	10	11	12
9879	9892	9906	9920	9933	9947	9961	9974	9988	$\overline{0}001$	1	3	4	5	7	8	10	11	12
2·0015	0028	0042	0055	0069	0082	0096	0109	0122	0136	1	3	4	5	7	8	9	11	12
0149	0162	0176	0189	0202	0216	0229	0242	0255	0268	1	3	4	5	7	8	9	11	12
0281	0295	0308	0321	0334	0347	0360	0373	0386	0399	1	3	4	5	7	8	9	11	12
0412	0425	0438	0451	0464	0477	0490	0503	0516	0528	1	3	4	5	7	8	9	10	12
0541	0554	0567	0580	0592	0605	0618	0631	0643	0656	1	3	4	5	6	8	9	10	11
0669	0681	0694	0707	0719	0732	0744	0757	0769	0782	1	3	4	5	6	8	9	10	11
0794	0807	0819	0832	0844	0857	0869	0882	0894	0906	1	3	4	5	6	7	9	10	11
0919	0931	0943	0956	0968	0980	0992	1005	1017	1029	1	3	4	5	6	7	9	10	11
1041	1054	1066	1078	1090	1102	1114	1126	1138	1151	1	2	4	5	6	7	9	10	11
1163	1175	1187	1199	1211	1223	1235	1247	1259	1270	1	2	4	5	6	7	8	10	11
1282	1294	1306	1318	1330	1342	1354	1365	1377	1389	1	2	4	5	6	7	8	9	11
1401	1412	1424	1436	1448	1459	1471	1483	1494	1506	1	2	4	5	6	7	8	9	11
1518	1529	1541	1552	1564	1576	1587	1599	1610	1622	1	2	4	5	6	7	8	9	10
1633	1645	1656	1668	1679	1691	1702	1713	1725	1736	1	2	3	5	6	7	8	9	10
1748	1759	1770	1782	1793	1804	1816	1827	1838	1849	1	2	3	5	6	7	8	9	10
1861	1872	1883	1894	1905	1917	1928	1939	1950	1961	1	2	3	5	6	7	8	9	10
1972	1983	1994	2006	2017	2028	2039	2050	2061	2072	1	2	3	4	6	7	8	9	10
2083	2094	2105	2116	2127	2138	2149	2159	2170	2181	1	2	3	4	6	7	8	9	10
2192	2203	2214	2225	2235	2246	2257	2268	2279	2289	1	2	3	4	5	7	8	9	10
2300	2311	2322	2332	2343	2354	2365	2375	2386	2397	1	2	3	4	5	6	8	9	10
2407	2418	2428	2439	2450	2460	2471	2481	2492	2502	1	2	3	4	5	6	7	9	10
2513	2523	2534	2544	2555	2565	2576	2586	2597	2607	1	2	3	4	5	6	7	8	10
2618	2628	2638	2649	2659	2670	2680	2690	2701	2711	1	2	3	4	5	6	7	8	9
2721	2732	2742	2752	2762	2773	2783	2793	2803	2814	1	2	3	4	5	6	7	8	9
2824	2834	2844	2854	2865	2875	2885	2895	2905	2915	1	2	3	4	5	6	7	8	9
2925	2935	2946	2956	2966	2976	2986	2996	3006	3016	1	2	3	4	5	6	7	8	9
2·3026																		

TABLE V.—NATURAL SINES.

Degree.	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10
45	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10

TABLE V. (*contd.*)

Degree.	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
45	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9155	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0
90	1·000														

TABLE VI.—Natural Cosines

Degree.	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
0	1·000	1·000	1·000	1·000	1·000	1·000	·9999	9999	9999	9999	0	0	0	0	0
1	·9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	·9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	·9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	·9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	·9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	·9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	·9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	·9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	·9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	·9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	·8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	·8910	8902	8894	8886	8878	8870	8862	8854	9846	8838	1	3	4	5	7
28	·8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	·8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	·8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	·8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	·8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	·8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	·8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	·8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	·7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	·7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	·7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	·7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	·7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10
45	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10

TABLE VI (*contd.*)

Degree.	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1′	2′	3′	4′	5′
45	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	·6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	14
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	14
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	12	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	14
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15
90	·0000														

TABLE VII.—Natural Tangents.

Degree.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°·0	0°·1	0°·2	0°·3	0°·4	0°·5	0°·6	0°·7	0°·8	0°·9	1′	2′	3′	4′	5′
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30

TABLE VII. (*contd.*)

Degree.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°·0	0°·1	0°·2	0°·3	0°·4	0°·5	0°·6	0°·7	0°·8	0°·9	1′	2′	3′	4′	5′
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	$\bar{0}057$	$\bar{0}145$	$\bar{0}233$	$\bar{0}323$	$\bar{0}413$	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2·9042	9208	9375	9544	9714	9887	$\bar{0}061$	$\bar{0}237$	$\bar{0}415$	$\bar{0}595$	29	58	87	116	145
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972					
77	4·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646					
78	4·7046	7453	7867	8288	8716	9152	9594	$\bar{0}045$	$\bar{0}504$	$\bar{0}970$					
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80	5·6713	7297	7894	8502	9124	9758	$\bar{0}405$	$\bar{1}066$	$\bar{1}742$	$\bar{2}432$					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	$\bar{0}264$					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	$\bar{0}285$					
83	8·1443	2636	3863	5126	6427	7769	9152	$\bar{0}579$	$\bar{2}052$	$\bar{3}572$					
84	9·514	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					
90	∞														

Mean differences are no longer sufficiently accurate, since the differences vary considerably along each line.

TABLE VIII.—LOGARITHMIC SINES.

Degree.	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
0	$-\infty$	$\bar{3}$·2419	5429	7190	8439	9408	$\bar{0}$200	0870	1450	1961					
1	$\bar{2}$·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4	·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	·9403	9489	9573	9655	9736	9816	9894	9970	0046	0120	13	26	39	52	65
6	$\bar{1}$·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6
45	·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6

TABLE VIII. (*contd.*)

Degree.	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
45	$\bar{1}$·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	·8569	8577	8584	8591	8598	8606	9613	8620	8627	8634	1	2	4	5	6
47	·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	·9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74	·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80	·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84	·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	·9999	9999	$\bar{0}$000	$\bar{0}$000	$\bar{0}$000	$\bar{0}$000	$\bar{0}$000	$\bar{0}$000	$\bar{0}$000	$\bar{0}$000	0	0	0	0	0
90	0·0000														

TABLE IX.—Logarithmic Cosines

Degree.	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
0	0·0000	0000	0000	0000	0000	0000	0000	0000	0000	$\bar{9}$999	0	0	0	0	0
1	$\bar{1}$·9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2	·9997	9997	9997	9996	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3	·9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4	·9989	9989	9988	9988	9987	9987	9986	9985	9985	9984	0	0	0	0	0
5	·9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6	·9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7	·9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8	·9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9	·9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10	·9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11	·9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12	·9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13	·9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14	·9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	·9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	·9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	2	2
17	·9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	·9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	·9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20	·9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	·9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	·9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	·9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	·9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	·9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	·9537	9533	9529	9525	9522	9518	9514	9510	9507	9503	1	1	2	3	3
27	·9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	·9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	·9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30	·9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	·9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32	·9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	·9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	·9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	·9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	·9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	·9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	·8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	·8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40	·8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	·8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42	·8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	·8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44	·8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6
45	·8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6

TABLE IX (*contd.*)

Degree.	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
45	$\bar{1}$·8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46	·8418	8410	8402	8394	9386	8378	8370	8362	8354	8346	1	3	4	5	7
47	·8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48	·8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49	·8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50	·8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51	·7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52	·7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53	·7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54	·7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55	·7586	7575	7564	7553	7542	7531	7520	7509	7498	7187	2	4	6	7	9
56	·7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57	·7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58	·7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59	·7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60	·6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61	·6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62	·6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63	·6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64	·6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65	·6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66	·6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67	·5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68	·5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69	·5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70	·5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71	·5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72	·4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73	·4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74	·4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75	·4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76	·3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77	·3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78	·3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79	·2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80	·2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81	·1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82	·1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83	·0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84	·0192	0120	0046	9970	9894	9816	9736	9655	9573	9489	13	26	39	52	65
85	$\bar{2}$·9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	64	80
86	·8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87	·7188	7041	6889	6731	6567	6397	6220	6035	5842	5640					
88	·5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89	·2419	1961	1450	0870	0200	9408	8439	7190	5429	2419					
90	$-\infty$														

TABLE X.—Logarithmic Tangents.

Degree.	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
0	−∞	$\bar{3}$·2419	5429	7190	8439	9409	$\bar{0}$200	$\bar{0}$870	$\bar{1}$450	$\bar{1}$962					
1	$\bar{2}$·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4	·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	·9420	9506	9591	9674	9756	9836	9915	9992	0068	0143	13	26	40	53	66
6	$\bar{1}$·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	·9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44	·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13
45	0·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13

TABLE X. (*contd.*)

Degree.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
	0°·0	0°·1	0°·2	0°·3	0°·4	0°·5	0°·6	0°·7	0°·8	0°·9	1′	2′	3′	4′	5′
45	·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50	·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	·1387	1403	1419	1435	1451	1467	1483	1490	1516	1532	3	5	8	11	13
55	·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60	·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	6	10	13	16
65	·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69	·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70	·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80	·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81	·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83	·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84	·9784	9857	9932	0008	0085	$\bar{0}164$	$\bar{0}241$	0326	$\bar{0}409$	$\bar{0}494$	13	26	40	53	66
85	1·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	64	81
86	1·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663					
87	1·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357					
88	1·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	1·7581	8038	8550	9130	9800	$\bar{0}591$	$\bar{1}561$	2810	$\bar{4}571$	$\bar{7}581$					
90	$-\infty$														

TABLE XI.—Exponential and Hyperbolic Functions

x	e^x	e^{-x}	cosh x $=\dfrac{e^x+e^{-x}}{2}$	sinh x $=\dfrac{e^x-e^{-x}}{2}$	tanh x $=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$
·1	1·1052	·9048	1·0050	·1002	·0997
·2	1·2214	·8187	1·0201	·2013	·1974
·3	1·3499	·7408	1·0453	·3045	·2913
·4	1·4918	·6703	1·0811	·4108	·3799
·5	1·6487	·6065	1·1276	·5211	·4621
·6	1·8221	·5488	1·1855	·6367	·5370
·7	2·0138	·4966	1·2552	·7586	·6044
·8	2·2255	·4493	1·3374	·8881	·6640
·9	2·4596	·4066	1·4331	1·0265	·7163
1·0	2·7183	·3679	1·5431	1·1752	·7616
1·1	3·0042	·3329	1·6685	1·3357	·8005
1·2	3·3201	·3012	1·8107	1·5095	·8337
1·3	3·6693	·2725	1·9709	1·6984	·8617
1·4	4·0552	·2466	2·1509	1·9043	·8854
1·5	4·4817	·2231	2·3524	2·1293	·9051
1·6	4·9530	·2019	2·5775	2·3756	·9217
1·7	5·4739	·1827	2·8283	2·6456	·9354
1·8	6·0496	·1653	3·1075	2·9422	·9468
1·9	6·6859	·1496	3·4177	3·2682	·9562
2·0	7·3891	·1353	3·7622	3·6269	·9640
2·1	8·1662	·1225	4·1443	4·0219	·9704
2·2	9·0251	·1108	4·5679	4·4571	·9757
2·3	9·9742	·1003	5·0372	4·9370	·9801
2·4	11·0232	·0907	5·5570	5·4662	·9837
2·5	12·1825	·0821	6·1323	6·0502	·9866
2·6	13·4638	·0743	6·7690	6·6947	·9890
2·7	14·8797	·0672	7·4735	7·4063	·9910
2·8	16·4446	·0608	8·2527	8·1919	·9926
2·9	18·1741	·0550	9·1146	9·0596	·9940
3·0	20·0855	·0498	10·068	10·018	·9951
3·1	22·1980	·0450	11·122	11·076	·9959
3·2	24·5325	·0408	12·287	12·246	·9967
3·3	27·1126	·0369	13·575	13·538	·9973
3·4	29·9641	·0334	14·999	14·965	·9978
3·5	33·1155	·0302	16·573	16·543	·9982
3·6	36·5982	·0273	18·313	18·285	·9985
3·7	40·4473	·0247	20·236	20·211	·9988
3·8	44·7012	·0224	22·362	22·339	·9990
3·9	49·4024	·0202	24·711	24·691	·9992
4·0	54·5982	·0183	27·308	27·290	·9993
4·1	60·3403	·0166	30·178	30·162	·9995
4·2	66·6863	·0150	33·351	33·330	·9996
4·3	73·6998	·0136	36·857	36·843	·9996
4·4	81·4509	·0123	40·732	40·719	·9997
4·5	90·0171	·0111	45·014	45·003	·9997
4·6	99·4843	·0101	49·747	49·737	·9998
4·7	109·9472	·0091	54·978	54·969	·9998
4·8	121·5104	·0082	60·759	60·751	·9999
4·9	134·2898	·0074	67·149	67·141	·9999
5·0	148·4132	·0067	74·210	74·203	·9999

APPENDIX

THE FULLER SLIDE RULE

The 10-inch straight slide-rule is the type most widely used because it is easy of transport, and it gives, quickly, results which are sufficiently accurate for most practical purposes. It is obvious that the number of graduations that can be marked upon a scale will be increased as the length of the scale is increased; and consequently it is possible to be certain of one more figure on a 20-inch rule than on a 10-inch rule. Beyond 20 inches for the length of the rule it is not desirable to go, if the rule is to be straight, since it becomes cumbersome. The increased length necessary for the greater degree of accuracy may, however, be obtained by marking the graduations round a cylinder; this being the scheme upon which Prof Fuller, of Belfast, worked in designing his cylindrical rule in 1878

In the model illustrated in Fig 258 the spiral is marked on the outside of the cylinder B, the diameter of which is 3·18″, and there are 50 turns of the spiral, the pitch being ·1112″, so that the axial length of the spiral is 5·56″.

The circumference of the cylinder is $\pi \times 3{\cdot}18$ and the length of one turn of the spiral is $\sqrt{(\pi \times 3{\cdot}18)^2 + ({\cdot}1112)^2}$. hence the total length of the spiral is the length of 50 turns, *i e.* 500 inches. This great length of scale, combined with the ease of manipulation, makes the Fuller rule a very valuable asset in the drawing-office It cannot be used for such a variety of different operations as can the ordinary straight rule, but for multiplication and division its merit is undoubted. Logarithms can be read directly and correctly to four figures, and by the aid of the logarithms, powers and roots can be found.

Description of the Rule.—Reference has already been made to the cylinder B upon which is inscribed the spiral which is graduated

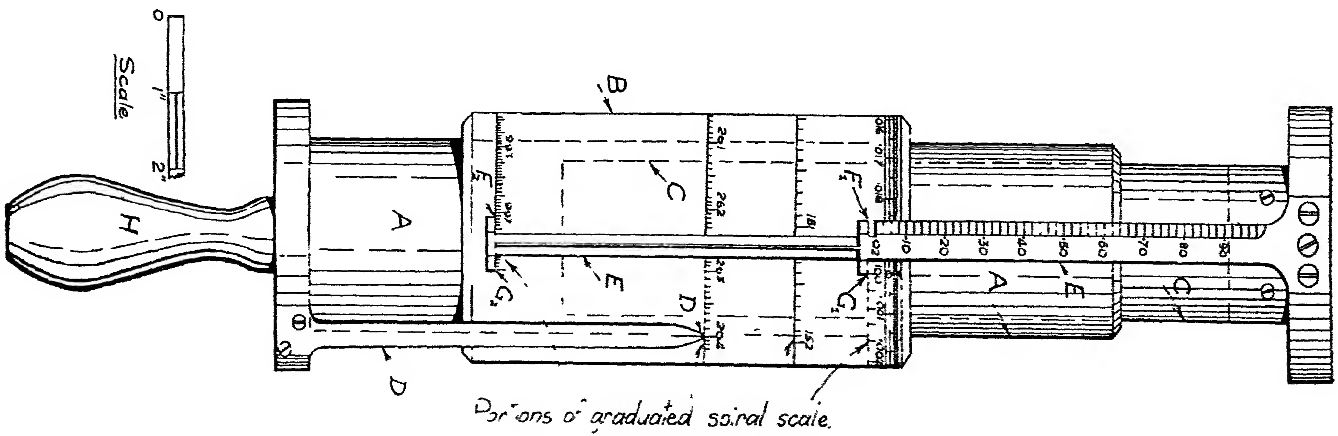


Fig. 258.—The Fuller Slide Rule.

logarithmically. Round the top of the cylinder is marked an evenly divided scale which is subsidiary to the scale on E. Between B and A, as also between A and C, are bushes of felt, so that the cylinders can slide or rotate upon one another with ease, there being just sufficient friction to keep the cylinders in the positions in which they may be placed.

Cylinder A is attached to the handle H and carries, screwed to its flange, the brass pointer D, at the point of which a line is shown, about half an inch long. There is clearance between D and B, but the pointer can be lightly sprung in to touch the scale when needed.

The third distinct part is the thin brass cylinder C which carries the brass indicator E, with its vertical graduated scale and its eight "index corners," like F_1. The distance from F_1 to F_2, or from G_1 to G_2 is the axial length of the spiral, so that if G_1 is at one end of the graduated scale, G_2 is at the other. G_1 may be used in place of G_2 if it is found necessary, or F_1 and F_2 may be interchanged, but G_1 must not be replaced by F_2, for example.

Use of the Rule.

(a) *For multiplication.* Let it be required to find the product 264×479 :—Holding the handle in the left hand, rotate B with the right hand until the pointer D is exactly at 264 Now, keeping B fixed, slide C about until the corner G_1 is at 100, the appearance of the rule at this stage being as shown in Fig. 258. With the right hand move B until G_2 is at 479 : then the reading opposite the pointer D is the required product. This reading is 12646 (the last figure being estimated) and since the answer must be in the neighbourhood of 250×500, *i. e.* 125,000, we can state the answer as 126,460. Actually the product is 126,456, so that the result as obtained from the rule is correct to 5 significant figures.

To multiply 264 by 14·53, the first setting would be as before, but cylinder B would now be moved until G_1 was at 1453, the reading at the pointer D being 3836. The approximation gives 250×15, *i. e.* 3750 and thus the result is stated as 3836, the result obtained by actual multiplication being 3835·92.

For continued multiplication the process is repeated. Thus to find the value of $\cdot 4013 \times 166{\cdot}2 \times \cdot 007614$.—

The approximation gives $4 \times 1{\cdot}5 \times 8$ $\qquad \dfrac{\text{II}}{\text{I}\quad\text{III}}$

i. e. ·48

Set pointer D at 4013 and move C until G_1 is at 100. Keeping E and D fixed in the same relative positions move B until G_1 is at 1662. Now keep B fixed whilst G_1 is again moved to 100 and then rotate B until G_2 is at 7614: the reading opposite D is 50783 and consequently the product is ·50783.

(b) *For division.* To divide 984·7 by 26·183 (the approximation gives 1000 ÷ 25, *i.e.* 40): Set pointer D at 9847 and G_1 at 26183. Now move B until G_1 is at 100, and then read off 3761 at the pointer D: the quotient is thus 37·61.

(c) *For proportion and percentage.* For sets of numbers in the same proportion the distances on a log scale between the two numbers making the ratio will be the same. Thus the distance from 2 to 4 on a log scale is the same as that from 3·7 to 7·4, for example. Consequently if E is kept in the same position relative to D while B is rotated, the readings at G_1 and D always form the same ratio. Thus if D is set at 327 and G_1 at 191 and then B is moved until G_1 is at 1392, the reading at D is 2383, and $\frac{2383}{1392} = \frac{327}{191}$.

For percentages, G_1 is set at 100 and D at the maximum. Thus to convert 337, 498 and 127 to percentages of 665, G_1 is set at 100 and D at 665. Then B is rotated until D is at 337, 498 and 127 in turn, the readings at G_2 being respectively 5067, 7489, and 19094, and the required percentages 50·67, 74·89 and 19·09.

(d) *For combined multiplication and division.* It is when dealing with examples of this type that the utility of the rule is most evident.

Thus to evaluate $\frac{\cdot 9647 \times 1183}{\cdot 05057 \times 68\cdot 16}$.

The approximation is $\frac{1 \times 1\cdot 2}{5 \times 7} \qquad \frac{11111}{1}$

i. e. $\cdot 03 \times 10000$

or 300.

As with the straight rule the procedure must be division and multiplication alternately, finishing always with multiplication, even if the last multiplication is only by unity.

Place the pointer D at 9647, and G_1 at 5057: next move B until G_1 is at 1183, and, keeping B fixed, place G_2 at 6816. Finally rotate B until G_1 is at 100 and the result 331·1 is read off at D.

It will be noticed that the process is one of alternate movements of E and B, the first and last readings only being determined by the position of D. Thus in this example the movements are of E, B, E and B.

As a further example: To evaluate $\dfrac{\cdot 2174 \times \cdot 0908 \times 1543}{87\cdot 64 \times \cdot 237}$

The approximation gives $\dfrac{2 \times 9 \times 1\cdot 5}{9 \times 2\cdot}$ $\dfrac{111\ 1}{1\ 11\ 1}$

i. e. 1·5

and the settings are as follows :—

Pointer D at 2174, G_2 at 8764 : move B until G_2 is at 908, now

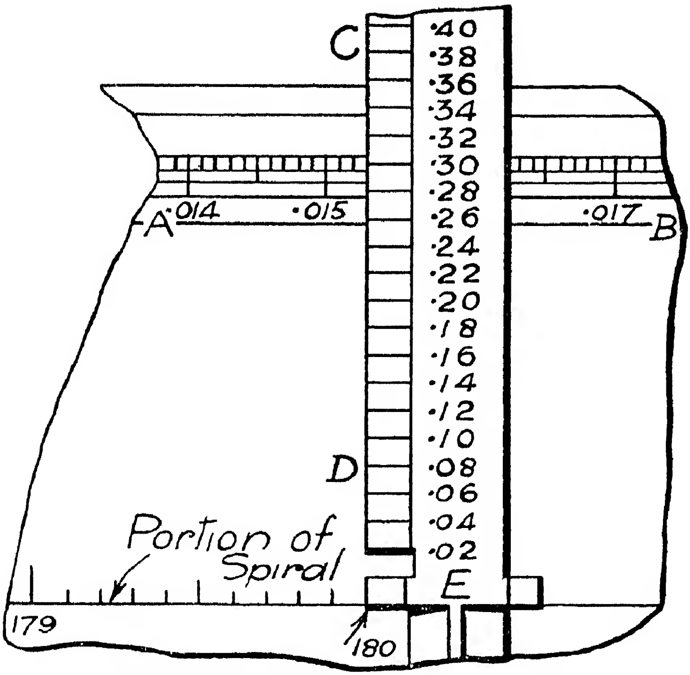


Fig 259.—Use of the Fuller Rule for the determination of logarithms.

move G_2 to 237, then move B until G_2 is at 1543. D is now at 14666, so that the result is 1·4666.

(c) *To read logarithms.* For this purpose the graduated scale on E (Fig. 258) is used in conjunction with the scale at the top of B; the numbers of which the logarithms are to be found being on the spiral.

Fig. 259 shows the setting for the determination of the mantissa of the log of 180. The lower left hand index corner of E is placed at the number, viz. 180 and the intersection of the line AB on the cylinder with the edge CD is seen to be between ·24 and ·26, nearer

the latter.. Thus the mantissa is ·24 + and the remaining portion of the decimal is found by noting the reading at the intersection of the edge CD with the horizontal scale, viz. ·0153. Hence log 180 = 2·24 + ·0153 = 2·2553, the characteristic being settled in the usual way.

Involution and evolution may be performed by multiplying or dividing the logs and then using the rule in the reverse way to determine the antilogarithms, but it is questionable whether this method is to be preferred to the use of log tables.

Exercises 47. Miscellaneous.

1. Find the weight of a plank of Honduras mahogany, 11′–6″ by 8″ by $\frac{3}{8}$″, at 35 lbs. per cu. ft. (8·387 lbs.).

2. An oil tank on a spring-buckling press had dimensions 3′-9″, 1′–2″ and 1′–9″. Find its capacity in gallons, and also the weight of oil contained when full, if 1 ton of oil occupies 38 cu. ft. (47·77 : 451·2 lbs.).

3. Draw, on the same diagram, magnetisation curves from the following data :—

(*a*) For cast steel

Ampere-turns per centimetre length	3·5	5	6	7·5	10	15	22	30	40	55
Flux-density (1000 lines per sq. cm.)	2	4	6	8	10·4	11·9	13·1	14	15·2	16

(*b*) For wrought iron

Ampere-turns, etc.	1	2	3	5	10	16	20	35	55
Flux-density.	2	6	8	10·7	13·4	14·7	15·5	16·5	17

The boss of an airscrew has an outside dia. of 260 mms. and an inside dia. of 73 mms. and it is 180 mms. long. Find its weight in mahogany, at 35 lbs. per cu. ft. (10·88 lbs.).

5. One end of a gravity conveyor (as used in workshops for transporting tools, etc.) is 5 ft. 3 ins. above the ground and the other end is 3 ft. high. If the length along the slope is 55 ft., find the inclination to the horizontal. (2° 19·35′).

6. The major and minor axes of the section of a " rafwire " (a wire used for bracing the wing structure of an aeroplane) are ·80 inch and ·18 inch respectively. Find the area of the section. (·1131 sq. in.).

7. For a four-cylinder petrol engine the total unbalanced secondary (a part of the expression for the crank-effort) is given by

$$E_2 = 2(A_2 \cos 2\theta + B_2 \sin 2\theta).$$

Put this in the form $E_2 = M \sin(2\theta + c)$ if $A_2 = -10{\cdot}5$ and $B_2 = 46{\cdot}4$. $[E_2 = 95{\cdot}13 \sin(2\theta - {\cdot}2225)]$

8. The fifth harmonic in the expansion of the series for the crank-effort of a certain ten-cylinder Anzani engine was

$$E_5 = 3{\cdot}82 \sin 5\theta - 11{\cdot}43 \cos 5\theta.$$

Express this in the form $E_5 = M \sin(5\theta - c)$

$[E_5 = 12{\cdot}06 \sin(5\theta - 1{\cdot}247)]$.

9. The horse-power absorbed by an airscrew varies directly as the cube of the R.P.M., and as the fifth power of the diameter. If the H.P. is 200 when the dia. is 10 ft. 6 ins. and the screw rotates at 1200 R.P.M., find the dia. when the H.P. is 180 and the R.P.M. = 1000. (11·48 ft.).

10. The following determinant occurred in connection with the balancing of a four-crank system :—

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & a^3 & a^4 \\ 1 & b & b^3 & b^4 \\ 1 & c & c^3 & c^4 \end{vmatrix} = 0$$

Show that this can be written as

$$(a-b)(b-c)(c-a)(a-1)(b-1)(c-1)(a+b+c+ab+bc+ca) = 0$$

[Hint. Expand and use the Factor Theorem]

11. The time t (in minutes) for a certain aeroplane to climb to a height h is given by

$$t = \frac{h_1}{c} \log_e\left(\frac{h_1}{h_1 - h}\right)$$

If $h_1 = 13600$ and $c = 600$ plot the values of t against h for values of h from 3000 to 10000 feet.

12. A column loaded with 80 tons is to be carried on a foundation 4′-6″ square. Find the minimum depth d of the foundation from the conditions

$$\frac{\text{load} + \text{weight of concrete}}{\text{area}} = wd\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2$$

if 1 cu. ft. of concrete weighs 150 lbs , w = weight of 1 cu. ft. of earth = 125 lbs and ϕ = angle of repose of earth = 30°. (9·074 ft).

13. The inner pan of a steam-jacketed vessel used for melting tallow consists of a cylindrical portion of length 12″, together with a hemispherical base. The inner diameter is 28″ and the thickness of the metal is $\frac{3}{4}$″. The flange at the top has outside diameter 37″ and is 1″ thick. Find the weight of this pan in [illegible]

(Vol = 2742 cu. ins.; weight = 582·9 lbs.

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